EIGENVALUE MODELS OF THE MARXIAN VALUE SYSTEM

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FOREWORD

Many of today's most significant socioeconomic problems, such as slower economic growth, the decline of some established industries, and shifts in patterns of foreign trade, are inter- or transnational in nature. But these problems manifest themselves in a variety of ways; both the intensities and the perceptions of the problems differ from one country to another, so that intercountry comparative analyses of recent historical developments are necessary. Through these analyses we attempt to identify the underlying processes of economic structural change and formulate useful hypotheses concerning future developments. The understanding of these processes and future prospects provides the focus for IIASA's project on Comparative Analysis of Economic Structure and Growth.

Our research concentrates primarily on the empirical analysis of interregional and intertemporal economic structural change, on the sources of and constraints on economic growth, on problems of adaptation to sudden changes, and especially on problems arising from changing patterns of international trade, resource availability, and technology. The project relies on IIASA's accumulated expertise in related fields and, in particular, on the data bases and systems of models that have been developed in the recent past.

In this paper, Ernő Zalai addresses problems related to the formal analysis and computational aspects of the Marxian labor value theory. Since the labor theory of value is customarily burdened with considerable ideological overtones, one of the aims of the paper is to disperse, through a formal analysis, some of the associated myths. Labor values constitute one possible set of accounting prices that could be used in international comparisons, in order to reduce the inconsistencies that arise from differences in price systems.

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"Comparative Analysis of
Economic Structure and Growth"

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1. INTRODUCTION

Formal discussion of Marxian labor values has customarily taken place in the framework of an open Leontief system (see for example, Brody 1974 or Morishima 1973). For quite some time only Brody (1974) attempted to pose the determination of the labor values as an eigenvalue problem. As will be seen, his solution is valid only for the rather special case of simple reproduction.

Reich (1979) and Zalai (1980) have independently shown that if labor is heterogeneous and the rate of surplus is uniform, then the labor values and the surplus rate can, in general, be determined only in the form of an eigenvalue problem. Both developed their model as a critique of Morishima's (1973) earlier solution and neither presented any proof of existence and uniqueness.

This paper is concerned with the alternative eigenvalue formulations of the labor value system and with the conditions guaranteeing the existence of positive and uniquely determined values. Although the values will be defined as eigenvectors, we will *not* assume the irreducibility of the input-output matrix in question, as is usually done in such exercises. Instead, we will base our proof on weaker assumptions and on the existence of an irreducible *basic* economy, defined by basic commodities similar to those discussed by Sraffa (1960).

2. REVIEW AND CRITIQUE OF PREVIOUS RESULTS

The standard model of Marxian labor values assumes that there are n commodities, each produced by one and only one single-product technology. Let $A = (a_{ij})$ be the input-output matrix, where a_{ij} indicates the quantity of commodity i required as input for the production of one unit of commodity j. Besides these *common** commodities, amounts m_j of labor are also required for the same output. Let us denote the unit labor input requirement by vector m and the vector of labor values by p. The latter is defined as

$$p = pA + m \qquad (1)$$

i.e. total value equals the value of the means of production plus new value added by labor (both A and m are assumed to be nonnegative).

Let us further suppose that the reproduction of one unit, say one hour, of labor power requires the consumption of a nonnegative quantity $f = (f_i)$ of various commodities and denote by

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* We use here Marx's distinction between *common* commodities (those other than labor) and the *peculiar* commodity, i.e. labor power. To stress the difference between labor power, i.e. the commodity being exchanged, and its useful service, i.e. labor, we refer to the commodity itself sometimes as labor power, rather than simply labor. The peculiarity of labor power arises from several features, but mainly from the fact that its production is not a value producing process and it is governed by other laws as well as by the law of value. If, however, we assume, as usual, that the reproduction of labor power does not directly require labor itself, this peculiarity does not show up in the formal analysis.

p_o the value of labor power, which is defined as*

$$p_o = pf \tag{2}$$

If A is productive (i.e. its dominant eigenvalue is less than 1), then from (1) we can get nonnegative solutions for p, as

$$p = m(I-A)^{-1} \tag{3}$$

and substituting p in (2) by the above expression we obtain

$$p_o = m(I-A)^{-1} f \tag{4}$$

From (4) it can be readily seen that the value of labor power is nothing other than the amount of labor necessary for its reproduction, i.e. the necessary labor. Thus, the surplus labor is $1 - p_o$ (provided this expression is positive). From this we can define the rate of surplus as

$$r = \frac{1 - p_o}{p_o} = \frac{1 - pf}{pf} \tag{5}$$

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* If m_o were the direct labor input required by the reproduction of labor power, the peculiar nature of labor power would mean that the labor used in its production adds just its value to the means of production and creates no surplus. Thus, we would have instead of (2)

$$p_o = pf + p_o m_o \tag{2'}$$

But from this, we could again get, after simple rearrangement, a form similar to (2):

$$p_o = p\left(\frac{1}{1-m_o}\right) f$$

where $\left(\frac{1}{1-m_o}\right) f$ represents the necessary consumption of both the productively employed labor and that of the (unproductive) labor used in the reproduction of labor. Thus, we may consider that this transformation has already been done above.

As we can see, it is possible to separate the determination of the value of common commodities and labor power, as well as the surplus rate. Brody (1974), however, realized that in the case of simple reproduction (i.e. when there is no surplus and thus $p_o = 1$), the values can be defined jointly by the solution of a closed system, as follows

$$(1,p) = (1,p) \begin{pmatrix} 0 & m \\ f & A \end{pmatrix} \tag{6}$$

From this he proceeded to the case of positive surplus and defined the value proportions as the solution to an equation system like (6) with the following coefficient matrix (Brody 1974, pp. 31,32)

$$\begin{pmatrix} 0 & 0 & m_s \\ 0 & 0 & m_n \\ f_s & f_m & A \end{pmatrix} \tag{7}$$

where m_s and m_n are the surplus and necessary labor, and f_s and f_m are the surplus product and consumption by laborers, respectively (our notation).

Closer examination of Brody's proposition reveals, however, that it does not provide the correct solution. Let us decompose the equations defined by matrix (7)

$$p_s = pf_s \tag{8}$$

$$p_m = pf_m \tag{9}$$

$$p = pA + p_m m_n + p_s m_s \tag{10}$$

Suppose p is indeed the vector of values of the common commodities; then p_m is the value of labor power (since f_m is the necessary consumption), i.e. the necessary labor. Similarly, p_s is the amount of surplus labor. Thus, we must have $p_s = rp_m$, which means that unless $r = 1$, p_s and p_m will be different.

Let us denote total labor added to the value of the means of production by $m = m_n + m_s$, as earlier, and choose $p_m = 1$.* It is clear that the "value added" in eqn. (10), i.e. $m_n + rm_s$, cannot be equal to the true one $(m_n + m_s)$, except for the special cases of $r = 1$ or $r = 0$ $(m_s = 0, f = 0)$.

For this reason, Brody's solution cannot be considered as the correct eigenvalue form of the value system. Another problem with his solution is that he assumes knowledge of the rate of surplus prior to the determination of the values. In the next section we will show several possible ways in which one can obtain the correct eigenvalue formulation of the labor values in the general case.

3. ALTERNATIVE EIGENVALUE FORMS OF THE VALUE SYSTEM

In what follows we will use the term *value system* to refer to the values of all commodities, including labor power and, implicitly, the rate of surplus as well. This latter can be simply determined if we know the values (see eqn. 5, earlier). Rearranging eqn. (5) we get

$$p_o + rp_o = 1 \tag{11}$$

which simply expresses the basic fact that the new value added can be divided into necessary (p_o) and surplus (rp_o) labor. Making this division in the determination of values, eqn. (1) can be rewritten as

$$p = pA + p_o m + rp_o m \tag{12}$$

Observe, however, that the above transformation does not leave the determination of labor values qualitatively unchanged. Equation (1) determines only the values of the common commodities and it is an inhomogeneous system. Equation (12) is, in contrast, homogeneous (free choice of value level) and has an additional degree of freedom compared to eqn. (1). Thus, it cannot in itself be used for determining the values of the common commodities. The full system that exhaustively characterizes labor values is made up of eqns. (2), (11) (or 5), and (12).

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* For the moment, and for the sake of simplicity, we assume that all values are positive.

This system can be put in various forms, but here we will consider only three of them. Let us introduce first the following matrices

$$B = \begin{pmatrix} 0 & m \\ f & a \end{pmatrix} , \qquad B_m = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$$

and $\hat{p}$ to denote the complete value sector, i.e. $\hat{p} = (p_o, p)$. With the help of this notation we can combine eqns. (2) and (12) into the following

$$\hat{p} = \hat{p}(B + rB_m) \tag{13}$$

This is a special (parametric or nonlinear) eigenvalue equation, which reduces to the form discussed by Brody in the case of no surplus ($r=0$). At the same time it also gives a correct eigenvalue formulation of the complete value system for the case of positive (or indeed negative, as a matter of fact) surplus. If labor values exist, then the surplus rate (r) must be such that 1 is an eigenvalue of the matrix $B + rB_m$. Moreover, if B is irreducible or all values are strictly positive, then 1 must be the dominant eigenvalue. These latter statements are direct consequences of the well-known Perron-Frobenius theorems (see, for example, Nikaido 1968).

Alternatively, we could have followed another route. Instead of using the above *extended* form, we can *reduce* the system by eliminating the value of the labor power and the corresponding equations. Let us again start by introducing a little new notation:

$$F = fm', \qquad M = A + F$$

where F is the diadic product of the vectors f and m, and therefore its elements denote the consumption requirements of the laborers engaged in producing the various commodities. Consequently, matrix M contains both the means of production (A) and the consumption (F) that are needed as inputs for the reproduction of the different commodities.

As will be shown later, either B or M can be viewed as the *complete* input-output coefficient matrix of the given commodity production system. Matrix B will be called its *extended* form and M its *reduced* form. Both have been used in the literature concerned with mathematical models of Marx's economics; therefore it will be of some interest to show that their essential mathematical properties are really the same.

Before doing so, let us see what we get if we eliminate p_o from eqn. (12) by substituting it with the right-hand side of eqn. (2)

$$p = pA + pF + rpF \tag{14}$$

From this we can obtain the reduced version of the eigenvalue equation (12)

$$p = p(M + rF) \tag{15}$$

The same analysis applies here as above, *mutatis mutandis*. Neither eqn. (12) nor eqn. (15) are proper eigenvalue forms, since r is also variable. If we wanted to use the above forms to calculate labor values, we should first discover a value for r such that 1 will be an eigenvalue of the respective matrix sums. Equation (14) can, however, be rearranged under suitable conditions into a proper eigenvalue equation. If the matrix I-A has a nonnegative inverse, then eqn. (17) yields a normal eigenvalue equation with nonnegative coefficient matrix

$$p = (1+r)\, pF(I-A)^{-1} = (1+r)pF_n \tag{16}$$

where we denote $F(I-A)^{-1}$ as F_n. Matrix F_n contains the quantities of necessary consumption for one unit of final output. In other words, it measures the consumption by the laborers that is required directly or indirectly to produce one unit of the various commodities. Therefore, eqn. (16) expresses in matrix form the well-known Marxian relation that exists between the value of the net product and that of the associated necessary consumption.

We also know that $p_o = 1/(1+r)$; thus, eqn. (16) can be further rearranged to give

$$p_o p = pF_n \tag{17}$$

This form illustrates an interesting property of the value system. The value of the labor power is an eigenvalue of the necessary consumption coefficient matrix, and the values of the other commodities are given by the left-hand eigenvector associated with it.

In what follows we will focus our attention on two possible characterizations of the value system through eigenvalue forms. These are eqns. (12) and (15), which must be supplemented with appropriate scaling conditions (eqn. 11) in order to get a full definition of values. In the next section we will prove some important common characteristics of the coefficient matrices in the extended and the reduced forms.

4. THE EQUIVALENCE OF THE EXTENDED AND REDUCED FORMS

We have seen that one can obtain two very similar eigenvalue forms of the value system, depending on the treatment of the "peculiar" commodity, labor power. In the two forms, different coefficient matrices B and M appear, which exhibit, however, essentially identical mathematical properties. We state these similarities more precisely in the following theorem.

THEOREM 1. *Let A be a nonnegative quadratic matrix, and f and m be semipositive vectors of the same dimension. Construct B and M in the following way:*

$$B = \begin{pmatrix} 0 & m \\ f & A \end{pmatrix} \qquad \text{and } M = A + fm'$$

For B and M the following properties hold:

(i) the dominant eigenvalue of B is less than, greater than, or equal to 1 if and only if the same is true for M;

(ii) the submatrix of $(I-B)^{-1}$ *defined by the common commodities is equal to* $(I-M)^{-1}$*;*

(iii) matrix B is irreducible if and only if the same is true for M.

Proof.

(i) From the Perron-Frobenius theorems we know that, for a nonnegative quadratic matrix A, the inequality $x > Ax$ has a positive solution if and only if the dominant eigenvalue of A is smaller than 1. Suppose the dominant eigenvalue of B is less than 1. Thus we can find a strictly positive vector $\hat{x} = (x_o, x)$ such that $\hat{x} > B\hat{x}$, i.e.

$$x_o > mx \tag{18}$$

$$x > Ax + fx_o \tag{19}$$

Replacing x_o by m_x in (19) yields

$$x > Ax + fm'x = Mx \tag{20}$$

from which it follows that M too must have a dominant eigenvalue of less than 1. Conversely, suppose there is $x > 0$ such that $x > Mx$. Define $\hat{x}$ as (kmx, x), where k is larger than but close enough to 1. Such an $\hat{x}$ clearly satisfies inequality (18) (with $x_o = kmx$). It is also clear that, because inequality (20) holds, one can choose k close enough to 1 such that the inequality

$$x > Ax + kfm'x$$

would also be true. Thus, the dominant eigenvalues of B and M can only be less than 1 *simultaneously*.

From the same theorems we also know that the inequality $x \leqq Ax$ has a semipositive solution if and only if the dominant eigenvalue of A is greater than or equal to 1. Using similar arguments to those above, we can show that if $\hat{x} = (x_o, x)$ is semipositive and $\hat{x} \leqq B\hat{x}$ then $x \leqq Mx$ and x is necessarily semipositive; and conversely, if x is semipositive and $x \leqq Mx$ then $\hat{x} = (mx, x)$ satisfies $\hat{x} \leqq B\hat{x}$. Thus, the dominant eigenvalues of B and M can, once again, only be greater than or equal to 1 *simultaneously*.

(ii) By calculating the inverse of (I-B), choosing first the upper left-hand element as the pivotal one, we can immediately get the inverse in the desired form

$$\begin{pmatrix} 1 + m(I-M)^{-1}f & m(I-M)^{-1} \\ (I-M)^{-1}f & (I-M)^{-1} \end{pmatrix}$$

(iii) Irreducibility is a structural property of the given matrices; therefore, without loss of generality, we may assume that the dominant eigenvalues of B and M are less than 1. In such a case irreducibility is equivalent to stating that the Leontief-inverse is strictly positive. From the above form of the Leontief-inverse of matrix B the truth of our statement immediately follows. If $(I-B)^{-1}$ is strictly positive, so is $(I-M)^{-1}$, and *vice versa*.

Q.E.D.

The common properties of matrices B and M proved above form the mathematical basis for the equivalence of the extended and reduced eigenvalue models of labor values. From the corresponding analysis of Brody (1974) we also know that the size of their dominant eigenvalues, relative to 1, will play a crucial role in determining the existence or otherwise of positive values. If the matrices are irreducible we can easily prove this; but, as will be seen below, we do not need the rather rigid assumption of irreducibility.

THEOREM 2. *Suppose the complete input-output coefficient matrices (B and M) are semipositive and irreducible. Positive values and positive surplus exist if and only if their dominant eigenvalue is less than 1.*

Proof. In view of Theorem 1 it suffices to prove the theorem for only one form. Let us choose the reduced form. Suppose there exists a positive vector p and a positive scalar r that satisfy the value equation $p = p(M+rF)$. Since r is positive and M is irreducible, $M + rF$ is also irreducible (and, of course, semipositive). Therefore the Perron-Frobenius theorems ensure that p (strictly positive) belongs to the dominant eigenvalue of M+rF (which is 1). We also know that the dominant eigenvalue is a strictly monotonic increasing function of any of the positive elements of a semipositive matrix (see, for example, Nikaido 1968). This means that the dominant eigenvalue of M must be less than 1.

Conversely, suppose the dominant eigenvalue of M is less than 1. M is equal to A + F and is irreducible; therefore $A + (1+r)F$ has the same property for any $r > -1$. Moreover, its dominant eigenvalue is a strictly monotonic increasing and unbounded function of r. This ensures that at some positive value of r the dominant eigenvalue of $A + (1+r)F = M + rF$ becomes equal to 1. Associated with it there is a unique (up to a scalar multiplication) and strictly positive left-hand eigenvector, which gives us the vector of values.

Q.E.D.

The proof of Theorem 2 heavily depends on the assumed irreducibility of the complete input-output coefficient matrix and the corresponding eigenvalue theorems. One may question whether this is a tenable assumption or not. Brody (1974, p.25) attempts to defend this assumption on economic grounds, but he can do so only at the cost of disregarding military and government expenditures as well as luxury commodities (which do not enter into the consumption of the laborers). Therefore, it remains to be determined whether the existence, positivity, and uniqueness of labor values can be guaranteed without making use of the irreducibility assumption.

5. EXISTENCE AND UNIQUENESS OF POSITIVE VALUES: THE REDUCIBLE CASE

Proofs of the existence and positivity of Marxian values are usually based on some mathematical properties of the coefficient matrices. In an earlier paper (Zalai 1983) we have tried to show that these required properties can be deduced from two rather sound economic hypotheses in the context of the Marxian analysis. These are the *impossibility of complete automation* of production and the principle of *pure commodity production*. The first means that with a given technology it is impossible to produce a nonnegative net output without using labor. The second is simply stating the fact that the data are derived from an economy where, at the prevailing prices and wage rates (all assumed to be positive), no commodity is produced at a loss.

In the following we will show that the same assumptions enable us to rigorously prove that the eigenvalue equations introduced earlier have unique positive solutions even if the coefficient matrices are reducible. The underlying economic reason is that there always exists a group of basic commodities*, defining a subsystem within the economy, whose complete coefficient matrix is irreducible. These and only these coefficients determine the surplus rate of the whole economy. Moreover, since their value is necessarily positive and they are directly or indirectly used in the production of all other commodities, this ensures the positivity of the value of the nonbasic commodities as well.

According to our definition, the set of *basic commodities* (I_b) includes all those commodities that are directly or indirectly required for the reproduction of labor power. By *basic economy* we mean that part of the economy that is defined solely by the basic commodities.

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* The concept of basic commodities used here is related to but different from that used by Sraffa (1960). The main difference is that he did not treat labor power as a commodity, whereas here it is consistently considered as a basic commodity with a crucial role in determining the whole set of basic commodities.

THEOREM 3. *If in a given economy labor is used directly or indirectly in the production of every commodity then:*

(i) the value of basic commodities depends only on the conditions of their production;
(ii) the surplus rate of the basic economy alone determines that of the original economy;
(iii) the complete coefficient matrix of the basic economy is irreducible;
(iv) the basic commodities are directly or indirectly required for the production of every commodity.

Proof. The proof will be based on the extended form.

(i) Suppose I_b is a proper subset of the set of all commodities. Without loss of generality, we may assume the commodities are arranged in some order, such that the basic commodities precede the nonbasic ones. Since labor is required to produce any commodity, labor power will also belong to the set of basic commodities. Let us decompose matrix B according to basics and nonbasics:

$$B = \begin{pmatrix} 0 & m_1 & m_2 \\ f_1 & A_{11} & A_{12} \\ f_2 & A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ 0 & A_{22} \end{pmatrix}$$

It is clear that both f_2 and A_{21} must be zero, since otherwise some "nonbasic" commodities would also be needed directly ($f_2 \neq 0$) or indirectly ($A_{21} \neq 0$) for reproducing labor power, and this would contradict our definition of basic commodities. Thus, the value of the basic commodities, $p_b = (p_o, p_1)$, is determined as

$$p_o = p_1 f_1$$

$$p_1 = p_1 A_{11} + m_1$$

(ii) We know that consumption by laborers consists only of basic commodities. Therefore, the value of labor power is the same in the original and in the basic economy. The value of labor power at the same time determines uniquely the rate of surplus, $r = (1-p_o)/p_o$.

(iii) Without loss of generality, we may assume that the complete coefficient matrix of the basic economy is productive, i.e. it has a nonnegative Leontief-inverse. If we can show that it has, in fact, to be strictly positive, then we will have proved that the coefficient matrix is irreducible.

Let us calculate the Leontief-inverse of B_{11} by choosing first the block $I - A_{11}$ as the pivotal element. We thus obtain

$$(I-B_{11})^{-1} = \begin{pmatrix} s & s\, m_1 (I-A_{11})^{-1} \\ s(I-A_{11})^{-1} f_1 & (I-A_{11})^{-1} + s(I-A_{11})^{-1} f_1 m_1' (I-A_{11})^{-1} \end{pmatrix}$$

where

$$s = \frac{1}{1 - m_1' (I-A_{11})^{-1} f_1}$$

and s is positive by assumption (productivity). Since labor is indispensable $m_1 (I - A_{11})^{-1}$ must be positive. From the very definition of basic commodities it immediately follows that $(I - A_{11})^{-1} f$ must be positive as well. Thus we can easily establish the strict positivity of the Leontief-inverse.

(iv) The basic commodities are directly or indirectly required for the reproduction of labor power and labor is, in the same way, required for the production of every commodity; this means that the basic commodities themselves are indispensable for the production of each commodity.

Q.E.D.

Now we have paved the way for the main theorem in this paper, in which we provide sufficient conditions for uniquely determined, positive-valued (eigen)vectors.

THEOREM 4. *Suppose the input coefficients (nonnegative A, and semipositive m and f) of an economy are such that*

(a) *complete automation is impossible, i.e. there is no such* $x \geqq 0$ *that* $x \geqq Ax$ *and* $mx = 0$,

(b) *no commodity is produced at a loss*, i.e. if* p_a *and* w_a *are the prevailing prices and wage rate, assumed to be all positive, then*

$$p_a \geq p_a A + w_a m \tag{21}$$

In such a case the special eigenvalue equations that simultaneously define labor values and the rate of return have a unique nonnegative solution in terms of values. In this solution all values are positive and 1 + r is also positive.

If, in addition, at least one commodity is produced at a profit (i.e. inequality holds in at least one component of (21)), and wages cover the cost of necessary consumption ($w_a \geqq p_a f$)*, then the surplus rate is positive, too.*

Proof. In Zalai (1983) it was proved that conditions (a) and (b) imply on the one hand that the dominant eigenvalue of A is less than 1, and on the other hand, that labor is directly or indirectly required for the production of every commodity. Since f is semipositive, the set of basic commodities is not empty.

Let us turn our attention to the basic economy. The dominant eigenvalue of A_{11}, being a minor of A, is also less than 1. As is well known, the dominant eigenvalue of a matrix A is less than 1 if and only if there exists a

* In an earlier paper (Zalai 1983) we have shown that assumption (b) could be replaced by an alternative one, which is based on the concept of "self-sufficient without self-serving production".

strictly positive vector x such that $x > Ax$. This property ensures that if r is larger than, but still close to -1, the dominant eigenvalue of matrix $A_{11} + (1+r)F_{11}$ is still smaller than 1. In Theorem 3 we have shown that $A_{11} + F_{11} = M_{11}$ is irreducible, and so is $A_{11} + (1+r)F_{11}$ for $r > -1$. This ensures that for some $r > -1$ the dominant eigenvalue of $A_{11} + (1+r)F_{11}$ reaches 1. This value of r is the surplus rate and the unique (up to a scalar) and positive left-hand eigenvector associated with it gives us the values of the basic commodities(after proper scaling).

If the additional assumptions (profit exists and wages allow for necessary consumption) are also fulfilled, then $p_{a1} \geq p_{a1}M_{11}$ (i.e. there is strict inequality in at least one component). This inequality and the irreducibility of M_{11} implies that the dominant eigenvalue of M_{11} is less than 1. From this it already follows that the above value of r will be positive.

We can now turn to the equations determining the values of the nonbasic commodities:

$$p_2 = p_1A_{12} + p_2A_{22} + (1+r)p_1F_{12}$$

In this equation p_1 and r are already determined, and the dominant eigenvalue of A_{22} is smaller than 1. Thus, we can uniquely determine the remaining values as

$$p_2 = p_1(A_{12} + (1+r)F_{12}) (I-A_{22})^{-1}$$

where every term is nonnegative, and thus p_2 is also nonnegative. Since we know that each commodity requires all basic commodities (directly or indirectly), this ensures the strict positivity of p_2.

Q.E.D.

5. CONCLUSIONS

We have shown that Marxian labor values can be adequately analyzed in the framework of eigenvalue equations. Such a formulation reveals the intrinsic interdependence of the values of the common commodities, labor power, and the rate of surplus. We have also compared alternative eigenvalue models of the value system and shown their essential identity. Finally, we have demonstrated that the usual assumption of the irreducibility of the complete input coefficient matrix can be replaced by weaker and more plausible ones. The positivity and uniqueness of the labor values can be traced back to a set of basic commodities, which form an irreducible core of an otherwise possibly reducible economic system.

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