THE COMBINED EQUILIBRIUM OF TRAVEL NETWORKS AND RESIDENTIAL LOCATION MARKETS*

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FOREWORD

Contribution to the Metropolitan Study: 11

The project "Nested Dynamics of Metropolitan Processes and Policies" started as a collaborative study in 1983. The Series of contributions is a means of conveying information between the collaborators in the network of the project.

This paper demonstrates the existence and uniqueness of a simultaneous equilibrium of household's choices of commuting networks and residential locations. The analysis contributes to the Metropolitan Study by considering the interaction between several markets and behavior of subsystems. It also contains a preliminary discussion of the stability properties of the equilibrium solution.

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ABSTRACT

The combined "user" equilibrium of travel networks and residential location markets is shown to exist and to be unique in the expected allocation of households to residential locations and to the routes and links of the network, in the vacancies and rents of residential locations and in the congested travel time and cost of each network link. The formulation combines a multinomial logit model of households' location and route choices derived from utility maximization, a binary logit model of house owners' offer decisions derived from profit maximization and the standard model of network congestion. A travel disutility measure (consistent with utility maximization) replaces the standard "generalized cost function". The proof utilizes a non-linear programming formulation which reproduces the simultaneous equilibrium conditions of the behavioral formulation. The stability of the unique equilibrium position is briefly discussed, a computational algorithm is proposed and hints for generalized formulations are provided.

CONTENTS

THE COMBINED EQUILIBRIUM OF
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LOCATION MARKETS

Alex Anas

1. Introduction

This paper concerns the simultaneous formulation and solution of two equilibrium problems each of which has attracted a great deal of attention.

The first of these problems is the equilibrium assignment of commuters to the links of a congestible link-node travel network. This problem has attracted the attention of transportation planners at least since 1952 and is of central importance in the formulation of "traffic assignment models", a key step in practical transportation planning procedures.

The second problem is the equilibrium assignment of households to geographic housing submarkets. This problem has attracted the attention of urban economists at least since the early sixties. It is of central importance in the formulation of "residential location models" which are crucial to housing market analysis and also to transportation planning, since the locations of families is a first step in any calculation of travel demands.

Although each of these two equilibrium problems has been studied rather extensively, there is no treatment in the literature of their simultaneous solution.

In treating the problem of travel network equilibrium, it is normally assumed that residential and employment locations are predetermined and that the number of trips (or flows) originating at a workplace and destined to a home location are known and fixed for all pairs of the origin-destination matrix. These flows are obtained from the trip generation, trip distribution and mode split procedures which normally precede the network assignment problem. In the network equilibrium problem, the flows are assigned among the routes available for travel between each origin to destination pair. Routes consist of a sequence of links on the network, and a link is normally shared by several routes connecting various origin-destination pairs.

A function of average travel time and cost (dubbed "generalized cost") incurred in traveling along a link is assumed to be an increasing function of the number of trips simultaneously traveling on that link. The network problem is then to find the equilibrium flow and generalized cost on each link (and, by summation, on each route). Two equilibrium concepts have been developed and applied.

The first equilibrium concept may be labelled "deterministic user equilibrim". It was stated by Wardrop (1952) and analyzed in the formulation of the network problem by Beckmann et al. (1956). This equilibration principle assumes that each trip takes the least costly route between the origin and destination points, and that all travelers perceive costs identically. Consequently, all routes connecting an origin-destination pair and carrying some traffic at equilibrium have equal costs at equilibrium, and all competing routes which remain unused have higher costs. These conditions are also known as those of "user optimal equilibrium" because they incorporate the notion that each traveler is in equilibrium and, once at equilibrium cannot improve his cost by changing route. Beckmann et al. showed that Wardrop's user optimal equilibrium conditions can be obtained as the unique solution of an optimization problem.

The second equilibrium concept may be called "stochastic equilibrium" and is developed in a formulation by Dial (1971) and more recently by Daganzo and

Sheffi (1977). Underlying the stochastic equilibrium formulation is that not all travelers perceive the same travel cost structure. Thus, while each traveler may still behave as a deterministic cost minimizer, a population of such travelers who are identical in all aspects will disperse over the available routes because of unobserved (to the analyst) probabilistic variations in their perceived travel costs. At equilibrium, each of the available routes between an origin-destination pair will carry some traffic, even though the observed component of these route costs can vary greatly among them. At equilibrium, the expected number of travelers (expected demand) choosing each route will create those congested costs which give rise to precisely the same expected number of travelers. Probabilistic network assignment has had substantial appeal because the preceding steps in transportation planning, for example, trip distribution and mode split were already conceived in probabilistic terms and systematically formulated as such by Wilson (1970). The stochastic assignment models thus made possible the application of the trip dispersion concept at all levels of the transportation planning process.

More recently, Florian et al. (1975, 1978) and Evans (1976) developed models which combined Wilson-type trip distribution with stochastic user equilibrium. Boyce (1980) and Boyce et al. (1981, 1983) extended the scope of these combined models to incorporate the choice of route, mode, destination and location[1]. A major characteristic of these combined models (those dealing with destination and location choices) is that they do not consider the geographic distribution of the housing stock and the equilibrium assignment of households to residential locations via the adjustment of housing prices (or rents). To gain a better understanding of this equilibrium assignment we turn to the second equilibrium problem: the assignment of households to residential location submarkets.

In treating the second problem, it is normally assumed that network travel times and costs are fixed and that they enter the utility functions of households alongside with housing prices and housing and location attributes. Housing prices, however, are not fixed. In the short run, during which the housing stock distribution remains unchanged, prices adjust to balance the expected number of households wishing to locate in each housing submarket with the expected number of sale and rental of dwellings in that submarket. This assignment determines equilibrium prices as well as vacancies in each submarket. At equilibrium, each household locates in the submarket which maximizes the household's utility and each dwelling goes to the highest profit use (occupied or vacant). In the long run, the housing stock can change and land prices adjust to match the expected demand for housing with the expected supply of it in each zone.

Location models faithful to the above principles have been examined since the pioneering work of Alonso (1964) and the linear programming model by Herbert and Stevens (1960). The latter model deals with the allocation problem deterministically but a probabilistic version of it, incorporating dispersion in residential locations, was proposed by Senior and Wilson (1974). A disequilibrium model which incorporated dispersion was proposed by Anas (1973), and a model by Los (1979) proposed another disequilibrium formulation incorporating the concept of bid rent in a model with a travel network. More recently, McFadden (1978) examined the demand for residential location using multinomial logit and related generalized extreme value models. Anas (1982, 1983) developed a large scale econometric model of the residential location market in the Chicago SMSA, employing logit and nested logit models for household and house owner behavior, showing how equilibrium rent distributions can be computed given exogenous changes in the travel time and travel cost structure[2].

The present paper extends Anas's model by incorporating route choice simultaneously with location choice. Housing rent, travel time and travel cost appear in the utility function simultaneously and the concept of generalized cost is discarded in favor of an endogenously determined travel disutility measure. The model employs probabilistic network assignment whereby the equilibrium link travel times and costs are determined simultaneously with the equilibrium location rents and the physical allocation of households to the network and the housing stock.

This paper achieves an overdue closure by providing a rigorous mathematical treatment of a problem which up to the present time received only ad hoc treatment. There have been several practical attempts, reported in the literature, to combine a network equilibrium model with a location model taking land (or housing stock) constraints into account. All of these attempts are in the tradition established by Lowry (1964). Most notably Putman (1974) attempted to combine Goldner's (1964) PLUM model with a network assignment model. A similar attempt is the work of Peskin (1977). In a series of articles, Berechman (1980, 1981) examined the structure of such "integrated" models but did not propose a consistent improved formulation. The central deficiency with all of these attempts, as well as with Berechman's investigation, is that the equations do not incorporate location prices. Thus, equilibrium is achieved by artificially forcing demands to match supplies by means of arbitrary adjustment factors or unrealistic reallocations of excess demands as in the original Lowry model. The current paper shows how these deficiencies can be eliminated.

The assumptions and notation are described in section 2, the combined equilibrium is formulated in section 3, existence and uniqueness of equilibrium are discussed and proved in sections 4 and 5, alternative computational algorithms are proposed in section 6 and several extended formulations are discussed in the final section which also outlines a future research agenda for the further development of combined models.

2. Assumptions and Notation

Most of my assumptions are standard in the contexts of the residential market and travel network problems. A few simplifying assumptions are inessential and do not affect the basic conclusions. Their relaxation will be discussed in the final section.

The assumptions are as follows:

A1: Each household has one working member who makes one commuting trip (inessential).

A2: There is only one mode of travel for commuting and this is a congestible network (for example, a highway network). The assumption of a single mode is inessential.

A3: Workplaces and dwellings are aggregated into mutually exclusive geographic zones, hereafter called "zone of residence". Each zone is assumed to be internally homogeneous. For example, all dwellings located in a zone are assumed to be identical (inessential).

A4: The travel network consists of a number of links and nodes. A node is a point where two or more links meet. Zones of work and residence are identified with the nodes of the network and all trips originating in such zones are loaded onto or unloaded from the network at these nodes. Travel within zones is assumed to be free of congestion and is neglected. Given any pair of work-residence zones there is a finite (and realistically, "small") number of routes for travel between the two zones. Each route is a sequence of links to be traversed in that order. Links are normally shared by more than one route and each link belongs to at least one route (standard).

A5: All travel is assumed to begin simultaneously in a rush-hour type of behavior. Due to congestion, the travel cost and travel time along a link are functions of the number of commuters traveling on that link (standard).

A6: All households (commuters) are assumed to be homogeneous in preferences except for random disturbance terms (inessential). Their utility is a function of travel time, travel cost, the rent for housing and other attributes of the route of travel and the zone of residence. Given the zone of work, each household chooses a zone of residence (where a dwelling is selected) and a route of travel from the zone of work to the zone of residence. These choices are made simultaneously and by maximizing utility over all available zones of residence and associated routes of travel.

A7: The owner of each dwelling (or landlord) decides whether the dwelling should be let (or sold) or kept vacant. This offer decision depends on the rent, the differential costs of maintenance for occupied and vacant dwellings and other factors.

Our notation is as follows:

N_i : number of households (= commuters) employed at i,

S_j : number of dwellings at j,

R_{ij}: the set of routes, feasible for travel, connecting zone of work i and zone of residence j. The feasibility rule can be used to exclude overly circuitous routes, routes which repeat the use of the same link and others which would not be used in reality. However, each link on the network must belong to at least one feasible route.

$\delta_{\ell ij\rho}$: a set of Kroenecker deltas such that with ℓ denoting a link and R_{ij} denoting a route, $\delta_{\ell ij\rho}=1$ if $\ell\epsilon\rho\epsilon R_{ij}$, and $\delta_{\ell ij\rho}=0$ otherwise.

f_ℓ : the number of trips on network link ℓ.

τ_ℓ : travel time on link ℓ.

c_ℓ: travel cost on link ℓ.

$g_\ell(f_\ell)$: link congestion function for link ℓ, measuring link travel time as a function of travel volume.

$h_\ell(f_\ell)$: link congestion function for link ℓ, measuring link travel cost as a function of link travel volume.

r_j : residential rent in zone of residence j.

$p_{ij\rho}$: probability that a commuter employed at i will choose residence at j and route of travel $\rho \varepsilon R_{ij}$.

q_{jo}: probability that a dwelling owner at j will keep his dwelling unit vacant.

q_{j1}: probability that a dwelling owner at j will offer his dwelling for occupancy.

$x_{ij\rho}$: the expected number of trips from zone of work i to zone of residence j and via route $\rho \varepsilon R_{ij}$.

y_{jo}: expected number of vacancies at zone of residence j.

y_{j1}: expected number of occupied units at zone of residence j.

$u_{ij\rho}$: the fixed part of the utility of a household (employed at i, residing at j and choosing route $\rho \varepsilon R_{ij}$), which depends on fixed factors other than rent, travel time and travel cost.

v_{jo}: the maintenance cost of a vacant dwelling in zone of residence j.

v_{j1}: the maintenance cost of an occupied dwelling in zone of residence j.

$\varepsilon_{ij\rho}$: the part of a household's utility which is treated as a random variable and varies across households for each (i,j,ρ) due to unobserved attributes.

η_{jo}: the part of the cost of a vacant dwelling which is random and varies across dwellings.

η_{j1}: the part of the cost of an occupied dwelling which is random and varies across dwellings.

$\alpha,\gamma_\tau,\gamma_c<0$: the marginal disutilities of rent, travel time and travel cost respectively.

$\beta>0$: the marginal profitability of rent.

$\hat{u}_{ij\rho}$: the total utility of choosing zone of residence j and route of travel ρ for a household employed at i.

π_{jo}: the total profit of a vacant dwelling in zone j.

π_{j1}: the total profit of an occupied dwelling in zone j.

3. The Combined Equilibrium Problem

We first discuss the utility and profit maximizing submodels and then formulate the combined travel network and residential location equilibrium problem as a simultaneous equations problem.

3.1 Utility maximization: choice of residential location and travel route

Suppose the utility function is given by

$$\hat{u}_{ij\rho} = \alpha r_j + \gamma_\tau \sum_\ell \delta_{\ell ij\rho} \tau_\ell + \gamma_c \sum_\ell \delta_{\ell ij\rho} c_\ell + u_{ij\rho} + \varepsilon_{ij\rho} \ . \tag{1}$$

Then,

$$p_{ij\rho} = \text{Prob}\ [\hat{u}_{ij\rho} > \hat{u}_{kms},\ \forall (k,m,s) \neq (i,j,\rho)] \ . \tag{2}$$

If we assume that the $\varepsilon_{ij\rho}$'s are identically and independently distributed according to the extreme value distribution then (2) becomes the multinomial logit model,

$$p_{ij\rho} = \frac{\exp\{\alpha r_j + \gamma_\tau \sum_\ell \delta_{\ell ij\rho} \tau_\ell + \gamma_c \sum_\ell \delta_{\ell ij\rho} c_\ell + u_{ij\rho}\}}{\sum_m \sum_{s \in R_{im}} \exp\{\alpha r_m + \gamma_\tau \sum_\ell \delta_{\ell ims} \tau_\ell + \gamma_c \sum_\ell \delta_{\ell ims} c_\ell + u_{ims}\}} \ , \tag{3}$$

and the expected number choosing zone of residence j and route ρ is,

$$x_{ij\rho} = N_i \, p_{ij\rho} \ . \tag{4}$$

Of course, $\sum_j \sum_{\rho \in R_{ij}} p_{ij\rho} = 1$.

Since α, γ_τ, γ_c and the $u_{ij\rho}$'s are constants, equation (3) can hereafter be denoted as $p_{ij\rho} = p_{ij\rho}(\bar{r}, \bar{\tau}, \bar{c})$. Each choice probability is a strictly decreasing function of the rent of its own zone but a strictly increasing function of the rent of other (substitute) zones. In particular,

$$\partial p_{ij\rho}/\partial r_k = \begin{cases} \alpha p_{ij\rho}(1 - \sum_{s\epsilon R_{ij}} p_{iks}) < 0 & \text{for } k = j \\ -\alpha p_{ij\rho} \sum_{s\epsilon R_{ik}} p_{iks} > 0 & \text{for } k \neq j, \end{cases} \tag{5}$$

where $s \epsilon R_{ik}$.

3.2 Profit maximization: the decision to let a dwelling

Let a landlord's profits for a vacant and occupied dwelling respectively be given as,

$$\pi_{jo} = -v_{jo} + \eta_{jo} \tag{6}$$

$$\pi_{j1} = \beta r_j - v_{j1} + \eta_{j1} \ . \tag{7}$$

Then,

$$q_{jo} = \text{Prob } [\pi_{jo} > \pi_{j1}] \tag{8}$$

and,

$$q_{j1} = 1 - q_{jo} \ . \tag{9}$$

Assuming that η_{jo} and η_{j1} are identically and independently distributed according to the extreme value distribution, we derive the binary logit probabilities,

$$q_{jo} = \frac{\exp\{-v_{jo}\}}{\exp\{\beta r_j - v_{j1}\} + \exp\{-v_{jo}\}} \tag{10}$$

$$q_{j1} = \frac{\exp\{\beta r_j - v_{j1}\}}{\exp\{\beta r_j - v_{j1}\} + \exp\{-v_{jo}\}} \ , \tag{11}$$

and the expected choices are,

$$y_{jo} = S_j q_{jo} \tag{12}$$

$$y_{j1} = S_j q_{j1} \tag{13}$$

The probability of occupancy is an increasing function of the rent. In particular,

$$\partial q_{j1}/\partial r_j = \beta q_{j1} q_{jo} > 0 \tag{14}$$

3.3 Network congestion

The flow on any link ℓ is computed as,

$$f_\ell = \sum_i \sum_j \sum_{\rho \in R_{ij}} \delta_{\ell ij\rho} \, x_{ij\rho} \tag{15}$$

and the travel time and cost of the link are,

$$\tau_\ell = g_\ell(f_\ell) \tag{16}$$

and

$$c_\ell = h_\ell(f_\ell). \tag{17}$$

Some comments about the properties of these functions are needed. The assumption that travel time increases as a function of travel volume, i.e. $\partial g_\ell(f_\ell)/\partial f_\ell > 0$, is of course valid, but it is reasonable to assume further that $\partial^2 g_\ell(f_\ell)/\partial f_\ell^2 > 0$, and that $\lim_{f_\ell \to K_\ell} g_\ell(f_\ell) = \infty$ where K_ℓ is the physical capacity of link ℓ. The travel cost function is known empirically to exhibit a minimum. This occurs because as traffic volume falls away from the physical capacity K_ℓ, travel speed increases and this improves fuel consumption efficiency initially.

In practical applications of network models, analysts commonly deal not with travel time and travel cost separately but with a weighted combination of the two, dubbed "generalized cost". The generalized cost function is assumed to have the shape indicated in figure 1(c) with the concept of a "design capacity", D_ℓ, replacing the physical capacity, K_ℓ. For example in the well known Bureau of Public Roads function b = 1.15 and d = 4.0 with a_ℓ and D_ℓ link-specific parameters.

The problem with generalized cost measures is that the weighting of time and cost is not objective, as presumed, but subjective and occurs in the utility function. Thus, instead of generalized cost functions one should refer directly to the "disutility of travel function" which is constructed by weighting the $g_\ell(\cdot)$ and $h_\ell(\cdot)$ functions with their respective utility coefficients. The rest of this paper will rely on this more sensible procedure. Thus, the disutility of travel on link ℓ is,

$$\Delta_\ell = -\gamma_\tau g_\ell(f_\ell) - \gamma_c h_\ell(f_\ell) > 0. \tag{18}$$

Since $h_\ell(\cdot)$ is not everywhere increasing, (18) need not be everywhere increasing either. If not, this introduces a nonconvexity which may lead to the presence of multiple equilibria in the network equilibrium problem. To avoid this possibility, it is sufficient to assume that the disutility function is everywhere strictly increasing because the strictly increasing $g_\ell(\cdot)$ dominates the $h_\ell(\cdot)$. Thus, to obtain all the proofs of this paper I will assume that,

$$-\gamma_\tau \frac{\partial g_\ell(\cdot)}{\partial f_\ell} - \gamma_c \frac{\partial h_\ell(\cdot)}{\partial f_\ell} > 0 . \tag{19}$$

Of course, this assumption is technically no more restrictive than the assumption of increasing generalized cost functions commonly employed in the literature.

3.4 Combined equilibrium

Let there be $j = 1 \ldots J$ zones of residence and $\ell = 1 \ldots L$ links on the travel network, then the combined equilibrium problem can be written as a system of $J + L$ simultaneous equations in as many unknowns which are: the vector of zonal rents $\bar{r} = [r_1, r_2, \ldots, r_J]$ and the vector of link disutilities $\bar{\Delta} = [\Delta_1, \Delta_2, \ldots, \Delta_L]$. Once these disutilities are obtained, link flows can be obtained from (18) and link times and costs computed from the $g_\ell(\cdot)$ and $h_\ell(\cdot)$ functions.

The combined equilibrium conditions are:

$$\sum_i N_i \sum_{\rho \in R_{ij}} p_{ij\rho}(\overline{r}, \overline{\Delta}) - S_j q_{j1}(r_j) = 0 \; ; \quad j = 1 \ldots J \tag{20}$$

$$\Delta_\ell + \gamma_\tau \, g_\ell(f_\ell) + \gamma_c h_\ell(f_\ell) = 0 \; ; \; \ell = 1 \ldots L \tag{21}$$

where,

$$f_\ell \overset{\Delta}{=} \sum_i N_i \sum_j \sum_{\rho \in R_{ij}} \delta_{\ell ij\rho} \, p_{ij\rho}(\overline{r}, \overline{\Delta}) \; .$$

Equations (20) state that the expected number of households choosing zone j are equal to the expected number of occupied dwellings in zone j. This is the condition of residential equilibrium. It is proven in Anas (1982) that given fixed values for the travel time and travel cost vectors, $\overline{\tau}$ and $\overline{c}$, equation (20) can be solved for a globally stable unique equilibrium rent vector $\overline{r}$. One existence and uniqueness proof can be obtained by showing that the Jacobian matrix of (20) has a negative dominant diagonal. However, since we will prove existence and uniqueness for the combined problem we will not dwell on the details of this proof.

Equation (21), given the rent vector $\overline{r}$, represents the traditional network equilibrium problem. The uniqueness and stability of this problem hinges on the assumption of an increasing travel disutility (19). For a paper focusing on the existence, uniqueness and stability of traffic equilibria see Smith (1979).

4. Existence: A Graphical Illustration

Establishing existence of an equilibrium for the combined problem is straightforward. The proofs of the next section establish both existence and uniqueness. Therefore, the purpose of this section is not only to demonstrate existence but to provide the useful graphical illustration of the solution to the combined equilibrium problem.

First, we consider the residential equilibrium solution given arbitrary values for the vectors of link times and costs, $\bar{\tau}$ and $\bar{c}$ and thus for the link disutilities, $\bar{\Delta}$. From (5), the expected demand for zone j is everywhere a downward sloping function of rent, r_j. Furthermore, given fixed and arbitrary values of the rents of zones other than j, the expected demand for zone j can be made to get arbitrarily close to zero by increasing the rent, r_j. This establishes the fact that the expected demand function is always asymptotic to the rent axis. Similarly, from (14) the expected supply of dwellings in zone j is a strictly increasing function of the zone's rent. Furthermore, as the zonal rent, r_j, approaches infinity the expected supply approaches the existing supply since q_{j1} approaches unity. This establishes that the expected supply function is asymptotic to the vertical line at S_j (see figure 2). It follows then that the expected supply and expected demand functions intersect only once in the intervals $(0,S_j)$ for each zone j. Such an intersection occurs regardless of the values of $\bar{\Delta}$ (or $\bar{\tau}$, $\bar{c}$) and the rents of the other zones.

Note that there is nothing in the specification of the choice probability functions to prevent negative zone rents from occuring. A negative rent will occur in a zone if the expected demand function intersects the horizontal axis at some point z which falls between zero and S_j and the supply function intersects the same axis in (z, S_j). Then the two functions will intersect each other below the horizontal axis.

The possibility of negative rents may appear troublesome but, of course, this is not the case. First, the possibility of negative rents is theoretically valid in a short run model with the housing stock in place and with vacancies possible. If the cost of maintaining vacant units is sufficiently higher than the cost of maintaining occupied units, then landlords will find it preferrable to subsidize occupancy (for example, by providing services and privileges to tenants greater in value than a nominal rent) rather than keep dwellings vacant. Of course, it is easy to rule out negative rents by making some simple changes in the specification of demand side choice probabilities. For example, suppose that r_j in the utility function (1) is replaced throughout by $r_j' \overset{\Delta}{=} \log r_j$. Then each zonal demand function is asymptotic to the horizontal axis from above, and negative rents are not possible regardless of the precise specification of the supply side choice probabilities (as long as these increase with rent).

The existence of an equilibrium for the network problem can be seen by examining demand and supply for each link on the network as a function of travel disutility (18). When this disutility is zero, then regardless of the values of $\bar{r}$, and times and costs on other links, there is a finite volume of traffic on link ℓ. As the disutility approaches infinity the flow diminishes asymptotically toward zero (see figure 3).

The above arguments prove that there is a unique intersection point for each network link and each residential zone regardless of the values of the unknowns for other zones and links. Thus, at least one equilibrium point exists for the entire system of zones and links.

5. Uniqueness: A Nonlinear Programming Formulation

In this section I will derive the nonlinear simultaneous equations (20) - (21) as the solution of a nonlinear programming problem. Since this programming problem incorporates the concept of entropy introduced by Wilson (1967) we also obtain a macrobehavioral interpretation of the combined model which, in section 3, was derived from utility and profit maximization. At the same time, the programming formulation allows us to prove the uniqueness of the equilibrium solution.[3]

The programming problem is as follows:

Minimize $\{x_{ij\rho}, y_{j1}, y_{jo}, f_\ell$ given $\alpha, \gamma_\tau, \gamma_C < 0, \beta > 0\}$

$$S \triangleq \frac{\gamma_\tau}{\alpha}\sum_\ell \int_o^{f_\ell} g_\ell(s)ds + \frac{\gamma_C}{\alpha}\sum_\ell \int_o^{f_\ell} h_\ell(s)ds - \frac{1}{\alpha}\sum_i\sum_j\sum_{\rho\varepsilon R_{ij}} x_{ij\rho}\log x_{ij\rho} + \frac{1}{\alpha}\sum_i\sum_j\sum_{\rho\varepsilon R_{ij}} x_{ij\rho}u_{ij\rho}$$
$$+ \frac{1}{\beta}\sum_j y_{j1}\log y_{j1} + \frac{1}{\beta}\sum_j v_{j1}y_{j1} + \frac{1}{\beta}\sum_j y_{jo}\log y_{jo} + \frac{1}{\beta}\sum_j v_{jo}y_{jo} \qquad (22)$$

subject to:

$$\sum_j\sum_{\rho\varepsilon R_{ij}} x_{ij\rho} - N_i = 0 \; ; \; i = 1 \ldots J, \qquad (23)$$

$$\sum_i\sum_{\rho\varepsilon R_{ij}} x_{ij\rho} - y_{j1} = 0 \; ; \; j = 1 \ldots J, \qquad (24)$$

$$y_{jo} + y_{j1} - S_j = 0 \; ; \; j = 1 \ldots J, \qquad (25)$$

$$\frac{1}{\alpha} f_\ell - \frac{1}{\alpha}\sum_i\sum_j\sum_{\rho\varepsilon R_{ij}} \delta_{\ell ij\rho}x_{ij\rho} = 0 \; ; \; \ell = 1 \ldots L, \qquad (26)$$

$$x_{ij\rho} \geqq 0 \quad \text{all } (i,j,\rho),\; y_{j1} \geqq 0,\; y_{jo} \geqq 0 \text{ all } j \text{ and } f_\ell \geqq 0 \text{ all } \ell . \qquad (27)$$

In forming the Lagrangian of the above problem, we assign Lagrangian multipliers σ_i to (23), r_j to (24), λ_j to (25) and Δ_ℓ to (26). Forming the Lagrangian, differentiating with respect to x_{ij} , y_{j1}, y_{jo} and f_ℓ, setting the resulting equations to zero and rearranging terms we get the following conditions necessary for an interior solution:

$$x_{ij\rho} = \exp\,(-1 + \alpha\sigma_i)\,\exp\,(\alpha r_j - \sum_\ell \delta_{\ell ij\rho}\Delta_\ell + u_{ij\rho}), \tag{28}$$

$$y_{j1} = \exp\,(-1 - \beta\lambda_j)\,\exp\,(\beta r_j - v_{j1}), \tag{29}$$

$$y_{jo} = \exp\,(-1 - \beta\lambda_j)\,\exp\,(-v_{jo}), \tag{30}$$

$$\Delta_\ell = -\gamma_\tau g_\ell(f_\ell) - \gamma_c h_\ell(f_\ell)\ . \tag{31}$$

Differentiating with respect to the Lagrange multpliers we recover the constraints (23) - (26). Substituting (28) into (23) we eliminate the $\alpha\sigma_i$'s and we recover the household expected choice relative frequencies

$$\frac{x_{ij\rho}}{N_i}\ (\equiv p_{ij\rho}) = \frac{\exp(\alpha r_j - \sum_\ell \delta_{\ell ij\rho}\Delta_\ell + u_{ij\rho})}{\sum_k \sum_{s\varepsilon R_{ik}} \exp(\alpha r_k - \sum_\ell \delta_{\ell iks}\Delta_\ell + u_{iks})}\ . \tag{32}$$

Substituting (29) and (30) into (25) we eliminate λ_j and we recover the supply side expected choice relative frequencies,

$$\frac{y_{j1}}{S_j}\ (\equiv q_{j1}) = \frac{\exp(\beta r_j - v_{j1})}{\exp(\beta r_j - v_{j1}) + \exp(-v_{jo})}\ , \tag{33}$$

$$\frac{y_{jo}}{S_j} \; (\equiv q_{jo}) = \frac{\exp(-v_{jo})}{\exp(\beta r_j - v_{j1}) + \exp(-v_{jo})} \; . \qquad (34)$$

Substituting (32) and (33) into (24) we recover the residential market equilibrium equations (20), and substituting (32) into (26), and f_ℓ from (26) into (31) we obtain the network equilibrium equations (21). The Lagrangian multipliers of (24) appear as the zone rents, r_j, and the multipliers of (26) as the link disutilities, Δ_ℓ.

Existence and uniqueness proofs can now be formulated:

Theorem 1: An equilibrium solution to (20), (21) exists if and only if

$$\sum_i N_i \leq \sum_j S_j \; .$$

Proof: Suppose $\sum_i N_i > \sum_j S_j$. Then, from (25), $y_{j1} > S_j$ for at least some j and $y^*_{jo} < 0$ for that j. Such a solution is not feasible and thus there is no solution to the optimization problem (22) subject to (23) - (27). However, if $\sum_i N_i \leq \sum_j S_j$ a nonempty, closed and bounded feasible set exists and thus an optimal solution which reproduces the equilibrium equations (20) and (21) exists. Theorem proved.

Theorem 2: An equilibrium solution to (20), (21) is unique under the assumption of an increasing travel disutility function (19).

Proof: The optimization problem (22) subject to (23) - (27) is a programming problem with an objective function which is strictly convex in the variables $\{x_{ijp}, y_{j1}, y_{jo}, f_\ell\}$. This strictly convex objective function is defined only for non-negative values of these variables. The feasible set defined by equations (23) - (27) is convex and bounded. It follows that there is a unique interior solution: the expected allocation of households to zones, routes and links and the expected allocation of dwellings to vacancies are unique. To see the uniqueness of the zonal rents, r_j, we can write (24) as,

$$\sum_i \sum_{\rho \varepsilon R_{ij}} x^*_{ij\rho} = S_j q_{j1}(r_j).$$

From (14), the right hand side increases monotonically with r_j. Thus, a unique solution r^*_j exists. From (18), since f^*_ℓ is unique, τ^*_ℓ , c^*_ℓ and Δ^*_ℓ computed from this f^*_ℓ are also unique because $g_\ell(\cdot)$ and $h_\ell(\cdot)$ are single valued. Therefore, the entire solution of allocations, rents and travel times and costs $\{x^*_{ij\rho} , y^*_{j1} , y^*_{jo} , f^*_\ell , r^*_j , \tau^*_\ell , c^*_\ell\}$ is unique. Theorem proved.

No results regarding the local or global stability of the unique combined equilibrium position are provided in this paper. The stability of traffic equilibria (i.e. equations (21), given $\bar{r}$) has been proven (see Smith (1979)). Similarly Anas (1982) proves the global stability of residential location equilibria (i.e. equations (20), given $\bar{\Delta}$).

6. Implementation: Estimation and a Proposed Algorithm

The implementation of the model requires two steps: (a) estimation using maximum likelihood and (b) an algorithm for obtaining the equilibrium solution given the estimated coefficients.

Estimations of the demand side choice model (3) and the supply side choice model (10), (11) are separate because these two models do not have any coefficients in common.

Estimation of the demand side model consists of finding α, γ_τ, γ_c and any coefficients included in $u_{ij\rho}$. This quantity would normally be specified as

$$u_{ij\rho} = \sum_{k=1}^{K} \gamma_k w_{ij\rho k} + \gamma_o \{\log \sum_{n=1}^{S_j} \exp(z_{jn})\} \qquad (35)$$

where the w's are attributes describing zone and neighborhood characteristics including zonal average dwelling characteristics and also attributes of the route other than time and cost. The quantity {·} is a precomputed inclusive value measuring the expected utility of choosing a dwelling within zone j as a function of the composite intrazonal utility z_{jn}. The conditional probability of choosing dwelling n having chosen the zone j can then be given as,

$$p_{n|ij\rho} = \exp(z_{jn}) / \sum_{m=1}^{S_j} \exp(z_{jm}) \quad . \tag{36}$$

The intrazonal utility z_{jn} should be a function of intrazonal, dwelling specific deviations in rent, time, cost and w's from the zonal mean values. For consistency with utility maximization $0 < \gamma_0 \leq 1$. The combined equations (3) and (36) yield the joint probability $p_{ij\rho n} = p_{ij\rho} p_{n|ij\rho}$ known as the nested logit model.

If aggregate choices of zone and route are observed as $n_{ij\rho}$, the log-likelihood function to be maximized is,

$$\text{Log-Likelihood} = \sum_i \sum_j \sum_{\rho \varepsilon R_{ij}} n_{ij\rho} \text{Log } p_{ij\rho} \quad . \tag{37}$$

If disaggregate choices are observed so that $\theta^h_{ij\rho} = 1$ if commuter h chooses (j,ρ) from workplace i and $\theta^h_{ij} = 0$ otherwise, then the log-likelihood function is,

$$\text{Log-Likelihood} = \sum_h \sum_i \sum_j \sum_{\rho \varepsilon R_{ij}} \theta^h_{ij\rho} \log p^h_{ij\rho} \quad , \tag{38}$$

where $p^h_{ij\rho}$ is equation (3) evaluated using the values of the attributes for commuter h. In each case we maximize the likelihood function with respect to $\alpha, \gamma_\tau, \gamma_C, \gamma_1, \ldots, \gamma_K, \gamma_0$ given observations on rent, time, cost, the w's and the inclusive value.

Estimation of the supply side model follows similar lines. In aggregate estimation we observe the number of vacant and occupied dwellings in each zone (m_{j1} and m_{jo}) and we maximize

$$\text{Log-Likelihood} = \sum_j (m_{j1} \log q_{j1} + m_{jo} \log q_{jo}) \tag{39}$$

with respect to β given the zonal average values of r_j, v_{j1}, v_{jo}.

In disaggregate estimation we maximize,

$$\text{Log-Likelihood} = \sum_k (\delta_1^k \log q_{k1} + \delta_o^k \log q_{ko}) \tag{40}$$

where if dwelling k is vacant then $\delta_o^k = 1$, $\delta_1^k = 0$ and if it is occupied then $\delta_o^k = 0$ and $\delta_1^k = 1$. In this case we must observe r_k, v_{k1}, and v_{ko}. For empirical estimates see chapter 4 in Anas (1982).

A computational algorithm to solve the combined equilibrium problem is easy to construct. Efficient algorithms which solve large network equilibrium problems exist. Anas (1982) has developed and tested a very efficient algorithm for solving the residential location equilibrium problem (equations (20)) and has implemented this algorithm to the Chicago SMSA where 1690 equations or zones were used (see chapter 5).

Interfacing Anas's algorithm with a network equilibrium algorithm would work as follows:

Step 1: Given the observed $\vec{\tau}^0$ and $\vec{c}^0$ use Anas's algorithm to find the first estimate of the rent vector $\vec{r}^1$.

Step 2: Given $\vec{r}^1$ use the network algorithm to find $\vec{\tau}^1$ and $\vec{c}^1$.

Step 3: Return to Step 1 and continue until $\bar{r}$ and $\bar{\tau}$, $\bar{c}$ converge arbitrarily closely to their equilibrium values $\vec{r}^*$, $\vec{\tau}^*$, $\vec{c}^*$. Other convergence criteria defined on the flows and occupancy levels can also be used.

7. Extensions

Many extensions of the model can be considered. Some hints and brief discussions are provided here.

First, the theoretical structure is not affected by the level of disaggregation. If the network is very detailed (large number of links and routes), residential zones can be made arbitrarily small for compatibility. Ultimately, it is possible to have each $N_i = 1$ and each $S_j = 1$ and represent each commuter and dwelling separately mapping these to appropriate nodes of the network. In this case we know from (4) and (12), (13) that $x_{ij\rho} = p_{ij\rho}$, $y_{jo} = q_{jo}$, $y_{j1} = q_{j1}$. At such a level of detail a microsimulation procedure may be a more desirable implementation method.

Second, the model can be easily extended to include many traveler types with distinct utility functions, many travel modes (each with a congestible network) and distinct dwelling types with distinct rents. Existence and uniqueness can be established when all these extensions are introduced simultaneously, by modifying the nonlinear programming formulation.

Third, choice of employment location can be introduced by making the demand for jobs a function of wages and other factors. Wages can be determined by location by balancing the demand for jobs with the supply of jobs determined by firms' employment and location decisions.

Fourth, two or more commuters per household can be introduced by classifying families by "workplace situations" (pairs or triples etc. of workplaces) and properly accounting for their trips over the network.

Fifth, congestion at the intrazonal level can be considered via a nested logit structure (see (35) and (36)). One can first do an intrazonal traffic equilibration, compute the inclusive values and enter these into the interzonal network equilibrium problem. A sequential nonlinear programming formulation may be used to prove existence and uniqueness. An intrazonal housing market equilibration could be introduced in a similar way.

It is hoped that these and other extensions will receive attention in future research.

FOOTNOTES

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1 In these models "destination" may refer to residential location and "location" may refer to workplace location. Alternatively, "destination" may refer to shopping destination and "location" to residential location.

2 This model known as the Chicago Area Transportation/Land Use Analysis System or CATLAS is dynamic with yearly periods. The residential market clears within each year and the housing stock adjusts with a one year lag.

3 In this paper entropy maximization is used only as a mathematical tool to prove uniqueness. Thus, there will be no discussion of the macrobehavioral interpretation of entropy. The equivalence between entropy formulations and multinomial logit models is by now well known. See my recent article, Anas (1983).

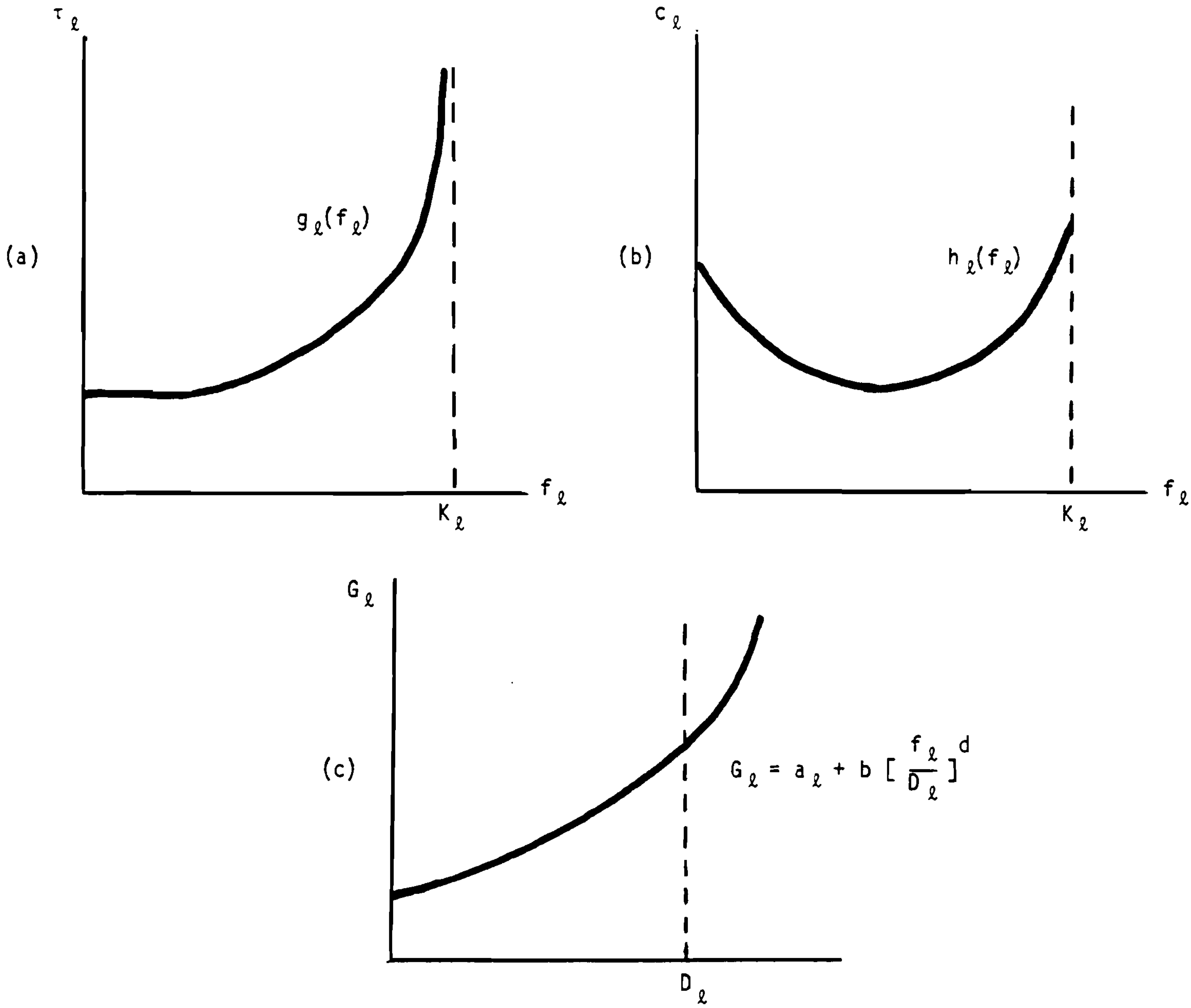

FIGURE 1: Realistic congested link travel time (a) and travel cost (b) functions and shape of "generalized cost" function (c) assumed in practice.

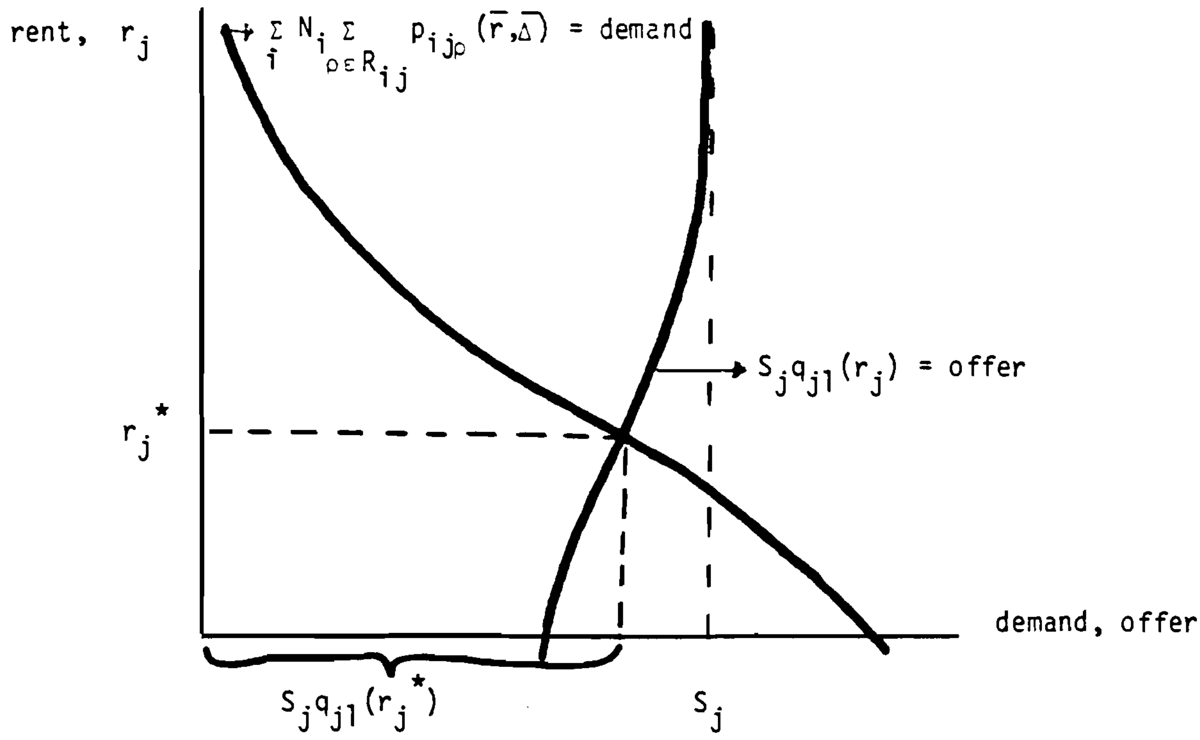

FIGURE 2: Equilibrium of the demand for and offer of dwellings in zone j.

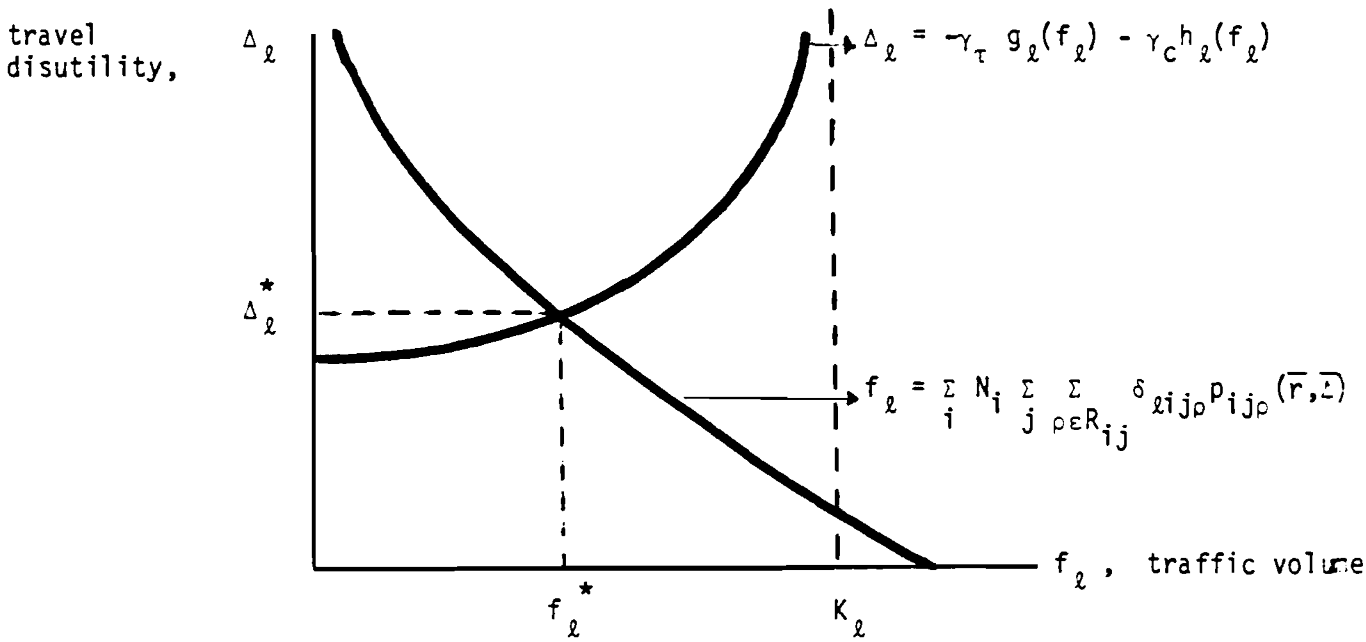

FIGURE 3: Equilibrium of traffic on link ℓ.

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