ON THE MINIMIZATION OF A QUASIDIFFERENTIABLE FUNCTION SUBJECT TO EQUALITY-TYPE QUASIDIFFERENTIABLE CONSTRAINTS

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PREFACE

The difficult problem of minimizing a function subject to equality-type constraints is of considerable importance in mathematical programming. In this paper, the author considers the case in which both the function to be minimized and the function describing the set over which minimization is to be performed are quasidifferentiable.

This paper is a contribution to research on nondifferentiable optimization currently underway within the System and Decision Sciences Program.

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This paper considers the problem of minimizing a quasidifferentiable function on a set described by equality-type quasidifferentiable constraints. Necessary conditions for a minimum are derived under regularity conditions which represent a generalization of the well-known Kuhn-Tucker regularity conditions.

Key words: Quasidifferentiable Functions, Quasidifferentiable Constraints, Regularity Conditions, Necessary and Sufficient Conditions for a Minimum.

1. Introduction

In this paper we consider the problem of minimizing a quasidifferentiable function [2,5] subject to equality-type constraints which may also be described by quasidifferentiable functions. A regularity condition is stated which in the smooth case is similar to the first-order Kuhn-Tucker regularity condition. Sufficient conditions for this regularity qualification to be satisfied are then formulated in terms of sub- and superdifferentials of the constraint function. We also consider cases where the quasidifferentiable constraint is given in the form of the union or intersection of a finite number of quasidifferentiable sets: analytical representations of the cone of feasible directions (in a broad sense) are obtained for such cases. Necessary and sufficient conditions for a minimum of a quasidifferentiable function on an equality-type quasidifferentiable set are proved, as are sufficient conditions for a strict local minimum. A method of finding steepest-descent directions in the case where the necessary conditions are not satisfied (but under some additional natural assumptions) is also given.

The theory is illustrated by means of examples, some of which cannot be studied using the Clarke subdifferential or other similar constructions.

Let h be a locally Lipschitzian function which is quasidifferentiable on E_n, and $Dh(x) = [\underline{\partial} h(x), \bar{\partial} h(x)]$ be its quasidifferential at $x \in E_n$. Then the directional derivative of h is given by

$$\frac{\partial h(x)}{\partial g} = \max_{v \in \underline{\partial} h(x)} (v,g) + \min_{w \in \bar{\partial} h(x)} (w,g) . \tag{1}$$

Let

$$\Omega = \{x \in E_n | h(x) = 0\} . \tag{2}$$

Assume that the set Ω is non-empty and contains no isolated points.

For every $x \in \Omega$ set

$$\gamma_0(x) = \left\{ g \in E_n \middle| \frac{\partial h(x)}{\partial g} = 0 \right\} .$$

It is clear that $\gamma_0(x)$ is a closed cone which depends on h . It is not difficult to check that

$$\gamma_0(x) = \bigcup_{\substack{v \in \underline{\partial} h(x) \\ w \in \overline{\partial} h(x)}} [\text{cone}^+(\overline{\partial} h(x)+v) \cap (-\text{cone})^+(\underline{\partial} h(x) + w)]. \quad (3)$$

Here and elsewhere cone A is understood to refer to the conical hull of set A , and cone^+ A to the cone conjugate to cone A .

Example 1. Let $\Omega = \{x \in E_2 | h(x) = 0\}$, where

$$x = (x^{(1)}, x^{(2)}) \in E_2$$

$$h(x) = \max \{0, h_1(x), h_2(x)\}$$

$$h_1(x) = (x^{(1)})^2 + (x^{(2)}-2)^2 - 4$$

$$h_2(x) = -(x^{(1)})^2 - (x^{(2)}-1)^2 + 1 .$$

Let $x_0 = (0,0) \in E_2$: it is clear that $x_0 \in \Omega$. It is not difficult to show that we can take the pairs of sets

$$Dh_1(x_0) = [\{(0,4)\} , \{(0,0)\}]$$

$$Dh_2(x_0) = [\{(0,2)\} , \{(0,0)\}]$$

$$Dh(x_0) = [\text{co}\{(0,4) , (0,2)\} , \{(0,0)\}]$$

as quasidifferentials of functions h_1, h_2 and h at x_0 . Here co A denotes the convex hull of set A . Then we have

$$\gamma_0(x_0) = \bigcup_{v\in\underline{\partial} h(x_0)} [\text{cone}^+(v) \cap (-\text{cone})^+(\bar{\partial} h(x_0))] =$$

$$= \bigcup_{\lambda\in E_1} (\lambda,0) = E_1 \times \{0\} .$$

For any $x \in \Omega$ introduce the closed cone

$$\Gamma(x) = \left\{ g \in E_n \mid \exists \; \lambda > 0 , \{x_i\} : \; x_i \to x , \; x_i \neq x , \; x_i \in \Omega \right. \tag{4}$$

$$\left. \frac{x_i - x}{\|x_i - x\|} \longrightarrow g_0 , \; g = \lambda g_0 \right\} .$$

The cone $\Gamma(x)$ is called the cone of feasible directions (in a broad sense) of set Ω at x .

We say that the regularity condition is satisfied for function h at $x \in \Omega$ if

$$\Gamma(x) = \gamma_0(x) . \tag{5}$$

Note that in Example 1 the regularity condition is satisfied at $x=x_0$.

2. Sufficient conditions for the regularity qualification to be satisfied

From the definition of a quasidifferentiable function it follows that the directional derivative is a continuous, positively homogeneous function of direction g and is defined on E_n . We shall use the following notation. Define

$$h_x(g) = \frac{\partial h(x)}{\partial g}\ ,\ h_x^+(g) = \max_{v\in\underline{\partial} h(x)} (v,g)\ ,\ h_x^-(g) = \min_{w\in\bar{\partial} h(x)} (w,g)\ ,$$

where $Dh(x) = [\underline{\partial}h(x)\ ,\ \bar{\partial}h(x)]$ is a quasidifferential of h at x . Then $h_x(g) = h_x^+(g) + h_x^-(g)$.

We shall now find a quasidifferential of function $h_x(g)$ at point $g \in E_n$. Since function $h_x^+(g)$ is finite and convex on E_n , and function $h_x^-(g)$ is finite and concave on E_n , the sets

$$\begin{aligned} &\underline{\partial}h_x^+(g) = co\{v|v \in R^+(x)\}\ ,\ \bar{\partial}h_x^+(g) = \{0\}\ , \\ &\underline{\partial}h_x^-(g) = \{0\}\ ,\ \bar{\partial}h_x^-(g) = co\{w|w \in R^-(x)\}\ , \end{aligned} \qquad (6)$$

can be taken as subdifferentials and superdifferentials of functions $h_x^+(g)$ and $h_x^-(g)$, where

$$R^+(x) = \{v \in \underline{\partial}h(x)\,|\,(v,g) = h_x^+(g)\}$$

$$R^-(x) = \{w \in \bar{\partial}h(x)\,|\,(w,g) = h_x^-(g)\}\ .$$

Therefore

$$Dh_x(g) = [\underline{\partial}h_x^+(g)\ ,\ \bar{\partial}h_x^-(g)]\ . \qquad (7)$$

Note that at point g=0 an arbitrary quasidifferential of function h at x can be taken as a quasidifferential of function

h_x . For all other points $g \in E_n$ we have

$$\underline{\partial} h_x(g) \subset \underline{\partial} h(x) , \qquad \bar{\partial} h_x(g) \subset \bar{\partial} h(x) . \tag{8}$$

The converse is also true: any quasidifferential of function h_x at point $g=0$ is a quasidifferential of function h at x .

Theorem 1. *If the function* $h_x(g)$ *has no strict local extrema on* $\gamma_0(x)$ *then the regularity condition is satisfied for function* h *at point* $x \in \Omega$.

Proof. Since the function $h(x)$ is assumed to be locally Lipschitzian, the following inclusion (see [3]) holds:

$$\Gamma(x) \subset \gamma_0(x) .$$

We shall now try to prove the opposite. Choose an arbitrary $\bar{g} \in \gamma_0(x)$ and assume that $\bar{g} \notin \Gamma(x)$. Since the function h is continuous (see [2]), there exists a positive number α_0 such that for every $\alpha \in (0,\alpha_0]$ and any $g \in S_{\alpha_0}(\bar{g})$, $g \neq \bar{g}$, the inequality $h(x+\alpha g) \neq 0$ holds and sign $h(x+\alpha g)$ = constant . (Here $S_r(z) = \{v \in E_n \mid \|v-z\| \leq r\}$.)

Let us first assume that for all $\alpha \in (0,\alpha_0]$ and $g \in S_{\alpha_0}(\bar{g})$, $g \neq \bar{g}$, the inequality

$$h(x+\alpha g) > 0 \tag{9}$$

holds. Since

$$h(x+\alpha g) = h(x) + \alpha h_x(g) + o(\alpha,g)$$

and

$$\frac{o(\alpha,g)}{\alpha} \xrightarrow[\alpha\to+0]{} 0 \quad ,$$

then without loss of generality we can assume that $h_x(g) \geq 0 = h_x(\bar{g}) \quad \forall g \in S_{\alpha_0}(\bar{g})$. From the assumptions of the theorem the function $h_x(g)$ has no strict local minimum at $\bar{g}$ and therefore inequality (9) is not satisfied.

In the same way it can be shown that there exists an $\alpha_1 > 0$ such that for every $\alpha \in (0,\alpha_1]$ and any $g \in S_{\alpha_1}(\bar{g})$, $g \neq \bar{g}$, the inequality $h(x+\alpha g) < 0$ is also not satisfied. The contradiction means that $\gamma_0(x) \subset \Gamma(x)$ and thus proves the theorem.

Theorem 2. *If the function* h *has a quasidifferential* $Dh(x) = [\underline{\partial}h(x) , \bar{\partial}h(x)]$ *at* $x \in \Omega$ *such that* $-\underline{\partial}h(x) \cap \bar{\partial}h(x) = \phi$, *then the function* h *satisfies the regularity condition at point* x .

Proof. Since $0 \in \gamma_0(x)$, it follows from the properties of a quasidifferential of function $h_x(g)$ at 0 (see (8)) and the assumptions of the theorem that neither the necessary condition for a minimum nor that for a maximum is satisfied for quasidifferentiable function h_x on E_n at any point $g \in E_n$. Thus, it follows from Theorem 1 that function h satisfies the regularity condition at point x , and Theorem 2 is proved.

This regularity condition is first-order and therefore it possesses all the deficiencies characteristic of first-order conditions.

We shall now consider an example in which this condition is not satisfied.

<u>Example 2</u>. Let

$$\Omega = \{x \in E_1 | h(x) = 0\} ,$$

where

$$h(x) = \begin{cases} (x-1)^2 , & x > 1 \\ 0 , & -1 \le x \le 1 \\ (x+1)^2 , & x < -1 . \end{cases}$$

Then $\Omega = \mathrm{co}\{-1,1\}$.

The function h is smooth and achieves its minimum value on E_1 at every point $x \in \Omega$. It is clear that the regularity condition is satisfied at every point of the set Ω except for points -1 and +1 . At these points $\gamma_0(x) = E_1$, $\Gamma(1) = -g$ and $\Gamma(-1) = g$, where $g \ge 0$.

Let h_i , $i \in I=1:N$, be locally Lipschitzian functions which are quasidifferentiable on E_n , and let $Dh_i(x)=[\underline{\partial}h_i(x), \bar{\partial}h_i(x)]$ be their quasidifferentials at $x \in E_1$. Set

$$\Omega_i = \{x \in E_n \mid h_i(x) = 0\} ,$$

$$\gamma_{i0} = \left\{ g \in E_n \;\middle|\; \frac{\partial h_i(x)}{\partial g} = 0 \right\} .$$

(a) Assume that $\Omega = \bigcap_{i \in I} \Omega_i$. Then

$$\Omega = \{x \in E_n \mid h(x) = 0\} , \tag{10}$$

where $h(x) = \max \{|h_i(x)| \mid i \in I\}$.

In the case where the set Ω is non-empty and function h satisfies the regularity condition at some point $x \in \Omega$, we have

$$\Gamma(x) = \gamma_0(x) = \bigcap_{i \in I} \gamma_{i0}(x) = \bigcap_{i \in I} \bigcup_{\substack{v_i \in \underline{\partial} h_i(x) \\ w_i \in \bar{\partial} h_i(x)}} T(v_i, w_i) =$$

$$= \bigcup_{\substack{v_1 \in \partial h_1(x);\, w_1 \in \bar{\partial} h_1(x) \\ \cdots\cdots \\ v_N \in \underline{\partial} h_N(x);\, w_N \in \bar{\partial} h_N(x)}} T(v_1, w_1, \ldots, v_N, w_N) ,$$

where

$$T(v,w) = \text{cone}^+ (\bar{\partial} h(x) + v) \cap [(-\text{cone})^+ (\underline{\partial} h(x) + w)]$$

$$T(v_1, w_1, \ldots, v_N, w_N) = \bigcap_{i \in I} T(v_i, w_i) .$$

(b) Let us now consider the case where Ω is the union of a finite number of quasidifferentiable sets:

$$\Omega = \bigcup_{i \in I} \Omega_i . \tag{11}$$

Then

$$\Omega = \{x \in E_n \mid h(x) = 0\} ,$$

where

$$h(x) = \min \{|h_i(x)| \mid i \in I\} .$$

If, in addition, the regularity condition is satisfied by function h at point $x \in \Omega$, then

$$\Gamma(x) = \gamma_0(x) = \bigcup_{i \in I(x)} \gamma_{i0}(x) = \bigcup_{i \in I(x)} T(v_i, w_i) ,$$

where $I(x) = \{i \in I \mid x \in \Omega\}$.

3. Necessary conditions for a minimum of a quasidifferentiable function on an equality-type quasidifferentiable set

Let quasidifferentiable functions f and h be locally Lipschitzian on E_n and let

$$Df(x) = [\underline{\partial} f(x) , \bar{\partial} f(x)] , Dh(x) = [\underline{\partial} h(x) , \bar{\partial} h(x)]$$

be their quasidifferentials at some point $x \in \Omega$. Assume also that the set Ω is described by relation (2) . We shall consider the following problem:

Find

$$\min_{x \in \Omega} f(x) . \qquad (12)$$

<u>Theorem 3</u> (see [6]). *If* x^* *is a solution to* (12) *and if* $\Gamma \subset \Gamma(x^*)$ *is a convex cone then*

$$-\bar{\partial} f(x^*) \subset \underline{\partial} f(x^*) - \Gamma^+ \ .$$

<u>Theorem 4</u>. *Assume that function* h *satisfies the regularity condition at some point* $x^* \in \Omega$. *Then for* x^* *to be a minimum point of* f *on* Ω *it is necessary that*

$$-\bar{\partial} f(x^*) \subset \bigcap_{\substack{v \in \underline{\partial} h(x^*) \\ w \in \bar{\partial} h(x^*)}} [\underline{\partial} f(x^*) - T^+(v,w)] \ . \tag{13}$$

<u>Proof</u>. Let x^* be a minimum point of function f on Ω . Then it follows from Theorem 3 and (3) that

$$-\bar{\partial} f(x^*) \subset \underline{\partial} f(x^*) - T^+(v,w) \ , \ v \in \underline{\partial} h(x^*) \ , \ w \in \bar{\partial} h(x^*) \ . \tag{14}$$

Note that

$$T^+(v,w) = \mathrm{cl}\left(\mathrm{cone}(\bar{\partial} h(x^*)+v) - \mathrm{cone}\ (\underline{\partial} h(x^*)+w) \right) \ .$$

Inclusion (14) holds for every $v \in \underline{\partial} h(x^*)$ and $w \in \bar{\partial} h(x^*)$, and therefore

$$-\bar{\partial} f(x^*) \subset \bigcap_{\substack{v \in \underline{\partial} h(x^*) \\ w \in \bar{\partial} h(x^*)}} [\underline{\partial} f(x^*) - T^+(v,w)] \ .$$

This completes the proof.

If set Ω is described by (10) and function h satisfies the regularity condition at some point $x^* \in \Omega$, then for x^* to be a minimum point of f on Ω it is necessary that

$$-\bar{\partial} f(x^*) \subset \bigcap_{\substack{v_1 \in \underline{\partial} h_1(x^*),\, w_1 \in \bar{\partial} h_1(x^*) \\ \cdots\cdots \\ v_N \in \underline{\partial} h_N(x^*),\, w_N \in \bar{\partial} h_N(x^*)}} [\underline{\partial} f(x^*) - \mathrm{cl}\,(\sum_{i \in I} T^+(v_i, w_i))] .$$

If set Ω is described by (11) and function h satisfies the regularity condition at some point $x^* \in \Omega$, then for x^* to be a minimum point of f on Ω it is necessary that

$$-\bar{\partial} f(x^*) \subset \bigcap_{i \in I(x^*)} \bigcap_{\substack{v_i \in \underline{\partial} h_i(x^*) \\ w_i \in \bar{\partial} h_i(x^*)}} [\underline{\partial} f(x^*) - T^+(v_i, w_i)] .$$

A point x^* for which condition (13) is satisfied will be called an *inf-stationary point* of function f on set Ω.

Example 3. Let function f be superdifferentiable on E_2 (i.e., such that it has a quasidifferential of the form $Df(x)=[\{0\},\bar{\partial} f(x)]$) at each $x \in E_2$. The set Ω is described by the relation $\Omega = \{x=(x^{(1)}, x^{(2)}) \in E_2 \mid h(x) = 0\}$, where $h(x)=\left|\,|x^{(1)}|+x^{(2)}\right|$.

Consider the point $x_0 = (0,0)$. We have

$$\underline{\partial} h(x_0) = \mathrm{co}\{(2,2) ,\ (-2,2) ,\ (0,0)\}$$

$$\bar{\partial} h(x_0) = \mathrm{co}\{(-1,-1) ,\ (1,-1)\} .$$

It is easy to check that

$$\gamma_0(x_0) = \bigcup_{\lambda \in E_1} (\lambda, -|\lambda|) ,$$

$$\bigcap_{\substack{v \in \underline{\partial} h(x_0) \\ w \in \overline{\partial} h(x_0)}} T^+(v,w) = \text{cone } (\text{co}\{(-1,-1) , (1,-1)\}) .$$

It is clear that function h satisfies the regularity condition at x_0 . Therefore for x_0 to be a minimum point of f on Ω it is necessary that

$$\overline{\partial} f(x_0) \subset \text{cone } (\text{co}\{(-1,-1) , (1,-1)\}) .$$

4. Steepest-descent directions

Assume that point x is not an inf-stationary point of quasidifferentiable function f on quasidifferentiable set Ω , and that f satisfies the regularity condition at x .

We shall now find a steepest-descent direction of function f on Ω at point x .

First compute

$$g_0 = \arg \min_{\substack{\|g\|=1 \\ g \in \Gamma(x)}} \frac{\partial f(x)}{\partial g} .$$

We have

$$0 > \min_{\substack{\|g\|=1 \\ g \in \Gamma(x)}} \frac{\partial f(x)}{\partial g} = \min_{\substack{\|g\|=1 \\ g \in \Gamma(x)}} \min_{w \in \overline{\partial} f(x)} \max_{z \in \underline{\partial} f(x)+w} (z,g) =$$

$$= \min_{w \in \overline{\partial} f(x)} \min_{\substack{\|g\|=1 \\ g \in \gamma_0(x)}} \max_{z \in \underline{\partial} f(x)+w} (z,g) =$$

$$= \min_{w \in \overline{\partial} f(x)} \; \min_{v' \in \underline{\partial} h(x)} \; \min_{w' \in \overline{\partial} h(x)} \left(- \min_{\substack{z \in \underline{\partial} f(x)+w \\ t \in T^{+}(v',w')}} \| z-t \| \right) =$$

$$= - \max_{\substack{w \in \overline{\partial} f(x) \\ v' \in \underline{\partial} h(x) \\ w' \in \overline{\partial} h(x)}} \; \min_{\substack{z \in \underline{\partial} f(x)+w \\ t \in T^{+}(v',w')}} \| z-t \| \; .$$

Let $z_0 \in \underline{\partial} f(x)+w_0$, $w_0 \in \overline{\partial} f(x_0)$, $v_0 \in \underline{\partial} h(x_0)$, $w_0' \in \overline{\partial} h(x)$, $t_0 \in T^{+}(v',w')$ be such that

$$\| z_0 - t_0 \| = \max_{\substack{w \in \overline{\partial} f(x) \\ w' \in \overline{\partial} h(x) \\ v' \in \underline{\partial} h(x)}} \; \min_{\substack{z \in \underline{\partial} f(x)+w \\ t \in T^{+}(v',w')}} \| z-t \| \; .$$

Then the direction $g_0 = -\left(\dfrac{z_0 - t_0}{\| z_0 - t_0 \|} \right)$ is a steepest-descent direction of quasidifferentiable function f on set Ω (described by (2)) at point x . This steepest-descent direction may not be unique.

5. Sufficient conditions for a strict local minimum

If quasidifferentiable functions f and h are directionally differentiable at $x \in E_n$, then

$$f(x+\alpha g) = f(x) + \alpha \frac{\partial f(x)}{\partial g} + o(\alpha,g)$$

$$h(x+\alpha g) = h(x) + \alpha \frac{\partial h(x)}{\partial g} + o_1(\alpha,g) ,$$

where

$$\frac{o(\alpha,g)}{\alpha} \xrightarrow[\alpha\to+0]{} 0 , \quad \frac{o_1(\alpha,g)}{\alpha} \xrightarrow[\alpha\to+0]{} 0 . \tag{15}$$

Assume that the convergence described by (15) is uniform with respect to $g \in E_n$, $\|g\|=1$.

Denote by $r(w,v',w')$ the radius of the largest ball centered at the origin which can be inscribed in the set

$$\underline{\partial} f(x) + w - T^+(v',w') ,$$

where $w \in \bar{\partial} f(x)$, $w' \in \bar{\partial} h(x)$, $v' \in \underline{\partial} h(x)$. Let

$$r(x) = \min_{\substack{w\in\bar{\partial} f(x)\\ v'\in\underline{\partial} h(x)\\ w'\in\bar{\partial} h(x)}} r(w,v',w') .$$

<u>Theorem 5</u>. *If set* Ω *is described by* (2), *point* $x_0 \in \Omega$ *and*

$$-\bar{\partial} f(x_0) \subset \operatorname{int} \bigcap_{\substack{v'\in\underline{\partial} h(x_0)\\ w'\in\bar{\partial} h(x_0)}} [\underline{\partial} f(x_0) - T^+(v',w')] , \tag{16}$$

then

$$\min_{\substack{g\in\gamma_0(x_0)\\ \|g\|=1}} \frac{\partial f(x_0)}{\partial g} = r(x_0) > 0 .$$

Proof. If inclusion (16) is satisfied at $x_0 \in \Omega$, then for every $w \in \bar{\partial} f(x_0)$, $v' \in \underline{\partial} h(x_0)$, $w' \in \bar{\partial} h(x_0)$ we have

$$\min_{\substack{g \in T(v',w') \\ \|g\|=1}} \max_{v \in \underline{\partial} f(x_0)+w} (v,g) = r(w,v',w') .$$

But since

$$\gamma_0(x_0) = \bigcup_{\substack{v' \in \underline{\partial} h(x_0) \\ w' \in \bar{\partial} h(x_0)}} T(v',w') ,$$

then

$$\min_{\substack{g \in \gamma_0(x_0) \\ \|g\|=1}} \frac{\partial f(x_0)}{\partial g} =$$

$$\min_{w \in \bar{\partial} f(x_0)} \min_{w' \in \bar{\partial} h(x_0)} \min_{v' \in \underline{\partial} h(x_0)} \min_{\substack{g \in T(v',w') \\ \|g\|=1}} \max_{v \in \underline{\partial} f(x_0)+w} (v,g) =$$

$$= \min_{w \in \bar{\partial} f(x_0)} \min_{v' \in \underline{\partial} h(x_0)} \min_{w' \in \bar{\partial} h(x_0)} r(w,v',w') = r(x_0) .$$

It is clear that $r(x_0) > 0$, thus proving the theorem.

Theorem 6. *If inclusion* (16) *is satisfied at* $x_0 \in \Omega$ *, then* x_0 *is a strict local minimum of* f *on* Ω *and there exist numbers* $\varepsilon > 0$ *and* $\delta > 0$ *such that*

$$f(x) \geq f(x_0) + \varepsilon \|x - x_0\| \quad \forall\, x \in \Omega \cap S_\delta(x_0) .$$

Proof. Take $\tilde{\varepsilon} > 0$ and set (see [1])

$$A_{\tilde{\varepsilon}}(x_0) = \left\{ g \in E_n \;\middle|\; \|g\|=1 \,,\; \left| \frac{\partial h(x_0)}{\partial g} \right| \leq \tilde{\varepsilon} \right\} .$$

The set $A_{\tilde{\varepsilon}}(x_0) \subset E_n$ is clearly compact, and if $\tilde{\varepsilon} = 0$ then $A_0(x_0) = \gamma_0(x_0) \cap S_1(0)$. It follows from Theorem 5 that there exists an $r(x_0) > 0$ such that

$$\min_{g \in A_0(x_0)} \frac{\partial f(x_0)}{\partial g} = r(x_0) > 0 ,$$

and therefore we can find $\bar{\varepsilon} > 0$ and $\bar{r}(x_0) > 0$ such that

$$\min_{g \in A_{\bar{\varepsilon}}(x_0)} \frac{\partial f(x_0)}{\partial g} = \bar{r}(x_0) > 0 .$$

Fix $\delta > 0$ and choose an arbitrary $x \in \Omega \cap S_\delta(x_0)$. If $\lambda = \|x - x_0\|$ and $g = \frac{1}{\lambda}(x - x_0)$ then

$$f(x) - f(x_0) = \lambda \left(\frac{\partial f(x_0)}{\partial g} + \frac{o(\lambda, g)}{\lambda} \right)$$
$$\frac{\partial h(x_0)}{\partial g} = -\left(\frac{o_1(\lambda, g)}{\lambda} \right) , \tag{17}$$

where

$$\frac{o(\lambda,g)}{\lambda} \xrightarrow[\lambda \to +0]{} 0 , \quad \frac{o_1(\lambda,g)}{\lambda} \xrightarrow[\lambda \to +0]{} 0$$

uniformly with respect to g , $\|g\|=1$. Set $\tilde{r} = \min\left\{ \bar{\varepsilon}, \frac{1}{2}\bar{r}(x_0) \right\}$. Then there exists a $\delta > 0$ such that

$$\max \left\{ \left| \frac{o(\lambda,g)}{\lambda} \right| , \left| \frac{o_1(\lambda,g)}{\lambda} \right| \right\} \le \tilde{r} \quad \forall \lambda \in (0,\delta] \ . \qquad (18)$$

Given such a δ , equations (17) are valid for any $x \in \Omega \cap S_\delta(x_0)$. This gives us

$$f(x) - f(x_0) \ge \|x - x_0\| \varepsilon ,$$

where $\varepsilon = \bar{r}(x_0) - \tilde{r}$.

Example 4. Consider the same function f and set Ω as in Example 3. If the inclusion

$$\bar{\partial} f(x_0) \subset \text{int cone } (\text{co}\{(-1,-1) , (1,-1)\})$$

is satisfied at $x_0 = (0,0) \in \Omega$, then x_0 is a strict local minimum point of function f on set Ω .

6. Reduction to the unconstrained case

Consider the function

$$F(x) = \max \{f(x) - f^* , h(x) , -h(x)\} ,$$

where $f^* = \inf_{x \in \Omega} f(x)$. Function F is quasidifferentiable on E_n . It is clear that if a point x_0 is a solution to problem (12) then x_0 is also a minimum point of F on E_n . We shall now write down a necessary condition for F to have a minimum on E_n at x_0 .

Since

$$\underline{\partial}F(x_0) = co\{A,B,C\} ,$$

where

$$A = \underline{\partial}f(x_0) - \overline{\partial}h(x_0) + \underline{\partial}h(x_0)$$

$$B = 2\underline{\partial}h(x_0) - \overline{\partial}f(x_0)$$

$$C = -2\overline{\partial}h(x_0) - \overline{\partial}f(x_0)$$

and

$$\overline{\partial}F(x_0) = \overline{\partial}f(x_0) + \overline{\partial}h(x_0) - \underline{\partial}h(x_0) ,$$

then the following result holds.

Proposition. *For a point* $x_0 \in \Omega$ *to be a minimum point of* f *on* Ω *it is necessary that*

$$-\overline{\partial}F(x_0) \subset \underline{\partial}F(x_0) . \tag{19}$$

Remark. In some cases condition (19) is a worse requirement for an extremum than condition (13). This can be illustrated by means of an example.

Example 5. Consider the same function h as in Example 3:

$$h(x) = \left| |x_1| + x_2 \right| , \quad x = (x^{(1)} , x^{(2)}) \in E_2 .$$

Let $x_0 = (0,0)$. It is not difficult to check that

$$\underline{\partial} h(x_0) - \bar{\partial} h(x_0) = \text{co}\ \{(1,1)\ ,\ (-1,1)\ ,\ (3,3)\ ,\ (-3,3)\}$$

$$2\underline{\partial} h(x_0) = \text{co}\ \{(0,0)\ ,\ (4,4)\ ,\ (-4,4)\}$$

and

$$\underline{\partial} h(x_0) - \bar{\partial} h(x_0) \subset 2\underline{\partial} h(x_0)\ . \tag{20}$$

However, inclusion (20) implies that any quasidifferentiable function f satisfies (19) (the necessary condition for a minimum on the set Ω) at the point $x_0=(0,0)$.

Theorem 7. *If functions* f *and* h *are quasidifferentiable, the convergence in* (15) *is uniform with respect to* $g \in E_n$, $\|g\|=1$, *and* $-\bar{\partial}F(x_0) \subset \text{int}\ \underline{\partial}F(x_0)$, *then* $x_0 \in \Omega$ *is a strict local minimum point of* f *on the set* Ω *described by* (2).

The proof is analogous to that of Theorem 11 in [4] .

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