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POSITIVELY HOMOGENEOUS QUASIDIFFERENTIABLE FUNCTIONS AND THEIR APPLICATIONS IN CO-OPERATIVE GAME THEORY

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PREFACE

In this paper, the author studies the properties of positively homogeneous functions, which represent a subclass of the set of quasidifferentiable functions. It is shown that these functions can be used to derive some new results in the theory of cooperative games.

This paper is a contribution to research on nondifferentiable optimization currently underway within the System and Decision Sciences Program.

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One interesting class of quasidifferentiable functions is that formed by the family of positively homogeneous functions. In this paper, the author studies the properties of these functions and uses them to derive some new results in the theory of cooperative games.

Key words: Homogeneous Quasidifferentiable Functions, Cooperative Games with Side Payments, Subdifferentials.

1. Introduction

We shall begin by recalling the definition of quasidifferentiability (for more information on the properties of quasidifferentiable functions see [5]). Let a finite-valued function $f: S \to E_1$ be defined on an open set $S \subset E_n$.

Definition 1 [5]. *A function* f *is said to be quasidifferentiable at a point* $x \in S$ *if it is differentiable at* x *in every direction* $g \in E_n$ *and there exist convex compact sets* $\underline{\partial} f(x) \subset E_n$ *and* $\overline{\partial} f(x) \subset E_n$ *such that*

$$\frac{\partial f(x)}{\partial g} = \max_{v \in \underline{\partial} f(x)} (v,g) + \min_{w \in \overline{\partial} f(x)} (w,g) \qquad \forall g \in E_n . \tag{1}$$

The pair of sets $Df(x) = [\underline{\partial} f(x), \bar{\partial} f(x)]$ is called a *quasi-differential* of the function f at the point x and the sets $\underline{\partial} f(x)$ and $\bar{\partial} f(x)$ are called a *subdifferential* and a *superdifferential*, respectively, of f at x .

In what follows we shall consider a positively homogeneous function f , i.e.,

$$f(\lambda x) = \lambda f(x) \qquad \forall\ \lambda \geqq 0 \ . \tag{2}$$

Let K be a convex cone in E_n with a compact base and a non-empty interior. We shall suppose that T is the base of this cone, where $\dim T < n$; let $\operatorname{ri} T$ denote the relative interior of the set T , and R_T the affine hull of T .

<u>Definition 2</u>. *A function* $f:T \to E_1$ *is said to be quasidifferentiable at a point* $x \in \operatorname{ri} T$ *if it is differentiable at this point in every direction* $g \in \tilde{R}_T = R_T - x$ *and convex compact sets* $\underline{\partial}_T f(x)$, $\bar{\partial}_T f(x) \subset \tilde{R}_T$ *exist such that*

$$\frac{\partial f(x)}{\partial g} = \max_{v \in \underline{\partial}_T f(x)} (v,g) + \min_{w \in \bar{\partial}_T f(x)} (w,g) \qquad \forall\ g \in \tilde{R}_T \ .$$

The following proposition is an immediate corollary of these definitions.

<u>Proposition 1</u>. *Let a function* $f:K \to E_1$ *be quasidifferentiable at a point* $x \in \operatorname{int} K$. *Then the function* $f|_{T(x,p)}$, *where* $T(x,p) = \{z \in K \mid (z-x,p) = 0\}$, $p \in E_n$, *is quasidifferentiable at* x , *and its quasidifferential is defined by the pair* [A,B] , *where*

$$A = Pr_p(\underline{\partial} f(x)) \quad , \quad B = Pr_p(\bar{\partial} f(x)) \; ,$$

and $Pr_p C$ *represents the orthogonal projection of a set* C *on the hyperplane*

$$H_p = \{z \in E_n \mid (z,p) = 0\} \; .$$

2. Quasidifferentiability of a positively homogeneous extension

Let us suppose that the function $\bar{f}: K \to E_1$ is the positively homogeneous extension to the cone K of a function f defined on the set $T(x,x)$, $x \in \operatorname{int} K$. Let f be quasidifferentiable at x .

Theorem 1. *The function* $\bar{f}$ *is quasidifferentiable at* x *and moreover*

$$D\bar{f}(x) = \left[\underline{\partial} f(x) + x \frac{f(x)}{\|x\|^2} \; , \quad \bar{\partial} f(x) \right] \; .$$

Proof. Since f is quasidifferentiable, the equality

$$\frac{\partial f(x)}{\partial h} = \max_{v \in \underline{\partial} f(x)} (v,h) + \min_{w \in \bar{\partial} f(x)} (w,h) \; , \tag{3}$$

holds for every direction

$$h \in H_x = \{v \in E_n \mid (v,x) = 0\}$$

and

$$\underline{\partial} f(x) \; , \; \bar{\partial} f(x) \subset H_x \; . \tag{4}$$

Let us consider an arbitrary direction $g \in E_n$ and suppose that

$$g \neq \lambda x \quad \text{for every} \quad \lambda \in E_1 . \tag{5}$$

Consider

$$F(x,g) = \lim_{\lambda \to +0} \left(\frac{\bar{\bar{f}}(x+\lambda g) - \bar{\bar{f}}(x)}{\lambda} \right) .$$

It is clear that

$$(x+\lambda g) \frac{\|x\|^2}{(x+\lambda g, x)} \in T(x,x) ,$$

where $\|x\|^2 = (x,x)$. Let

$$h = \frac{\|x\|^2}{(x+g,x)} (x+g) - x . \tag{6}$$

Then $h \in H_x$ and we have the following representation:

$$(x+\lambda g) \frac{\|x\|^2}{(x+\lambda g, x)} = x + \mu h ,$$

where

$$\lambda = \frac{\mu \|x\|^2}{\|x\|^2 + (g,x) - \mu (g,x)} .$$

(Note that $h \neq 0$ because $g \neq \lambda x$.) It is clear that $\lambda \to +0$ iff $\mu \to +0$, and thus we have

$$F(x,g) = \lim_{\lambda\to+0}\left(\frac{f(x+\mu h)\frac{(x+\lambda g,x)}{\|x\|^2} - f(x)}{\lambda}\right) =$$

$$= \lim_{\lambda\to+0}\left(\frac{f(x+\mu h) - f(x)}{\lambda} + f(x+\mu h)\frac{(g,x)}{\|x\|^2}\right) =$$

$$= \lim_{\mu\to+0}\left(\frac{f(x+\mu h) - f(x)}{\mu} \cdot \frac{\|x\|^2 + (g,x) - \mu(g,x)}{\|x\|^2}\right) +$$

$$+ f(x)\cdot\frac{(g,x)}{\|x\|^2} \quad .$$

Hence for every $g \neq \lambda x$ the derivative $\frac{\partial\bar{f}(x)}{\partial g}$ exists and

$$\frac{\partial\bar{f}(x)}{\partial g} = \frac{\partial f(x)}{\partial h} \cdot \frac{\|x\|^2 + (g,x)}{\|x\|^2} + \frac{f(x)}{\|x\|^2}(g,x) \quad , \tag{7}$$

where h is defined by (6).

From (3) we then get

$$\frac{\partial\bar{f}(x)}{\partial g} = \frac{\|x\|^2 + (g,x)}{\|x\|^2}\cdot\left[\max_{v\in\underline{\partial} f(x)}(v,h) + \min_{w\in\bar{\partial} f(x)}(w,h)\right] +$$

$$+ \frac{f(x)}{\|x\|^2}(g,x) \quad .$$

Since the function $\frac{\partial\bar{f}(x)}{\partial g}$ is positively homogeneous in g , it is enough to assume that g satisfies the condition $\|x\|^2 + (g,x) > 0$.

Then, taking (6) into account, we have

$$\frac{\partial\bar{f}(x)}{\partial g} = \max_{v\in\underline{\partial} f(x)}\left(v, g - x\frac{(g,x)}{\|x\|^2}\right) +$$

$$+ \min_{w \in \bar{\partial} f(x)} \left(w, g - x \frac{(g,x)}{\|x\|^2} \right) + \frac{f(x)}{\|x\|^2}(g,x) .$$

Since $\underline{\partial} f(x)$, $\bar{\partial} f(x) \subset H_x$,

$$\frac{\partial \bar{f}(x)}{\partial g} = \max_{v \in \underline{\partial} f(x)} (v,g) + \min_{w \in \bar{\partial} f(x)} (w,g) + \frac{f(x)}{\|x\|^2}(g,x) =$$

$$= \max_{v \in \underline{\partial} f(x) + \frac{f(x)}{\|x\|^2} x} (v,g) + \min_{w \in \bar{\partial} f(x)} (w,g) . \tag{8}$$

Now we have to check that this formula holds for $g \in H_x$ and $g=\lambda x$ for some $\lambda \neq 0$.

If $g \in H_x$, then $(g,x) = 0$ and

$$\max_{v \in \underline{\partial} f(x) + \frac{f(x)}{\|x\|^2} x} (v,g) = \max_{v \in \underline{\partial} f(x)} (v,g) .$$

Let us suppose that $g=\lambda x$ for some $\lambda \neq 0$. Then

$$\frac{\partial \bar{f}(x)}{\partial g} = \lim_{\mu \to +0} \left(\frac{\bar{f}(x+\mu\lambda x) - \bar{f}(x)}{\mu} \right) =$$

$$= \lim_{\mu \to +0} \left(\frac{(1+\mu\lambda) f(x) - f(x)}{\mu} \right) = \lambda f(x) .$$

But from (4) we have

$$\max_{v \in \underline{\partial} f(x) + \frac{f(x)}{\|x\|^2} x} (v,\lambda x) + \min_{w \in \bar{\partial} f(x)} (w,\lambda x) =$$

$$= \left(\frac{f(x)}{\|x\|^2} x, \lambda x \right) = \lambda f(x) ,$$

thus proving the theorem.

3. Game-theoretical applications of quasidifferentiable functions

Now let us consider the game-theoretical applications of quasidifferentiable functions. The study of so-called fuzzy or generalized games is currently attracting a great deal of interest. We will not go into the reasons for this here (but see J.-P. Aubin [1-3] on this topic): we shall simply recall the main definitions.

Let $I=1:n$ be a set of n players. We can then identify an arbitrary set $S \subset I$, called a *coalition*, with a characteristic vector e^S , where $e=\mathbb{I}=(1,\ldots,1) \in E_n$ and e^S is the projection of vector e on the subspace

$$\mathbb{R}^S = \{x \in E_n | x_i=0 \text{ for } i \notin S\} .$$

Thus the set of all coalitions is $\{0,1\}^n$.

The set of generalized (fuzzy) coalitions is, by definition, the convex hull $\text{co}\{0,1\}^n=[0,1]^n=I^n$. Hence a generalized coalition $\tau \in I^n$ associates with each player $i \in I$ a participation rate $\tau_i \in [0,1]$, which is a number between 0 and 1 .

Definition 3 [3]. *An n-person generalized cooperative game (with side payments) is defined by a positively homogeneous function* $v: [0,1]^n \to E_1$ *which assigns a payoff* $v(\tau) \in \mathbb{R}^1$ *to each generalized coalition* $\tau \in [0,1]^n$. *The function* v *is called the characteristic function of the game.*

Since v is positively homogeneous we can extend v to E_n^+ by setting

$$v(0) = 0$$

$$v(\tau) = \left(\sum_{i=1}^{n} \tau_i\right) v\left(\tau / \sum_{i=1}^{n} \tau_i\right)$$

for $\tau \in E_n^+$, $\tau \neq 0$.

We shall take the vector space E_n as the space of outcomes (or multi-utilities). Vector $x=(x_1,\ldots,x_n) \in E_n$ represents the utilities of the players; the utility of the generalized coalition τ is given by $(\tau,x)= \sum_{i=1}^{n} \tau_i x_i$. If $S \subset I$, then this utility is equal to $(e^S,x)= \sum_{i \in S} x_i$.

It is well-known (see, for example, [1,2]) that the directional derivative may be used to define the solutions of a game. In an extension of this idea, J.-P. Aubin has proposed that the Clarke subdifferential could be used to define a set of solutions to locally Lipschitzian games, i.e., games with a locally Lipschitzian characteristic function.

Definition 4 [3]. *We say that the Clarke subdifferential* $\partial_{Cl} v(\mathbb{I})$ *of* v *at* $\mathbb{I}$ *is the set of solutions* $S(v)$ *to a locally Lipschitzian game with characteristic function* v .

The following properties of the set $S(v)$ are worthy of note:

(a) $S(v)$ is non-empty, compact and convex

(b) $S(v)$ is Pareto-optimal, i.e., if $x \in S(v)$, then $\sum_{i=1}^{n} x_i = v(\mathbb{I})$

(c) $S(\lambda v) = \lambda S(v)$ for $\lambda \in E_1$

(d) $S(u+v) \subset S(u) + S(v)$

(e) If v is superadditive, then $S(v)$ coincides with the core of v

(f) If v is continuously differentiable at $\mathbb{I}$, then $S(v)=\nabla v(\mathbb{I})$, i.e., $S(v)$ contains only one element which coincides with the generalized Shapley value of the game v .

Definition 5. *A generalized game is said to be quasidifferentiable if its characteristic function is quasidifferentiable.*

Remark 1. Since quasidifferentiability is essential only on the diagonal of cube I^n then from Theorem 1 and the positive homogeneity of function v it is sufficient to assume that v is quasidifferentiable only at $\mathbb{I}$.

Let v be quasidifferentiable and its quasidifferential be $[\underline{\partial} v(\mathbb{I}), \bar{\partial} v(\mathbb{I})]$. From Proposition 1 we deduce that the function $v^1 = v|_{T(\mathbb{I},\mathbb{I})}$ is quasidifferentiable at $\mathbb{I}$ with a quasidifferential defined by the pair $[Pr_{\mathbb{I}}\, \underline{\partial} v(\mathbb{I}), Pr_{\mathbb{I}}\, \bar{\partial} v(\mathbb{I})]$. It is clear that the positively homogeneous extension $\bar{v}$ of the function v^1 on E_n^+ coincides with v ; the quasidifferential of this function at $\mathbb{I}$, which may be found using Theorem 1, is

$$[Pr_{\mathbb{I}}\, \underline{\partial} v(\mathbb{I}) + \frac{v(\mathbb{I})}{\|\mathbb{I}\|^2}\, \mathbb{I}, Pr_{\mathbb{I}}\, \bar{\partial} v(\mathbb{I})] .$$

It is also clear that this pair is in some sense "Pareto-optimal", since for $x \in Pr_{\mathbb{I}}\, \underline{\partial} v(\mathbb{I}) + \frac{v(\mathbb{I})}{\|\mathbb{I}\|^2}\, \mathbb{I}$ and $y \in Pr_{\mathbb{I}}\, \bar{\partial} v(\mathbb{I})$ we have

$$\sum_{i=1}^{n} (x_i+y_i) = (x+y, \mathbb{I}) = \left(\frac{v(\mathbb{I})}{\|\mathbb{I}\|^2}\, \mathbb{I}, \mathbb{I}\right) = v(\mathbb{I})$$

(because $Pr_{\mathbb{I}}\, \underline{\partial} v(\mathbb{I}), Pr_{\mathbb{I}}\, \bar{\partial} v(\mathbb{I}) \subset H_{\mathbb{I}}$) .

Let $D^{\pi} v(\mathbb{I})$ be a quasidifferential of v at $\mathbb{I}$ which is Pareto-optimal in the sense described above. We then have the following definition:

Definition 6. *The quasidifferential* $D^{\pi} v(\mathbb{I})$ *of the characteristic function* v *at the point* $\mathbb{I}$ *is called a quasisolution of the game.*

There are at least two reasons for using the term "quasi-solution". Firstly, it is known that quasidifferentials are not unique and are defined up to the equivalence relation. We should also note that a locally Lipschitzian function is not necessarily quasidifferentiable and vice versa. Moreover, it is obvious that a function which is both locally Lipschitzian and quasidifferentiable may have both a directional derivative and an upper Clarke derivative, which are essentially different quantities.

Quasisolutions also possess certain properties which go some way towards justifying their name.

1. If a characteristic function v is continuously differentiable at Π, then $D^{\pi}v(\Pi) = [\nabla v(\Pi), 0]$, where $\nabla v(\Pi)$ is the gradient of v at Π and a quasisolution can be identified with the generalized value of the game.
2. If v is concave (i.e., superadditive), then $D^{\pi}v(\Pi) = [0, \bar{\partial} v(\Pi)]$, where $\bar{\partial} v(\Pi)$ is the superdifferential of the concave function v and the quasisolution $D^{\pi}v(\Pi)$ can be identified with the core of the game.
3. Quasisolutions are linear on v.

<u>Remark 2</u>. In general, if one element of a quasidifferential is zero, then it is natural to regard the corresponding quasi-solution as a solution of the game.

Finally, using the properties of quasidifferentials we can find quasisolutions of the maximum and minimum games of a finite number of quasidifferentiable games, and thus we may speak about the calculus of quasisolutions.

Let us now consider the directional derivative

$$\frac{\partial v(\Pi)}{\partial g} = \lim_{\lambda \to +0} \frac{v(\Pi+\lambda g) - v(\Pi)}{\lambda} .$$

This value shows the marginal gain of coalition Π when a new coalition g joins the existing coalition Π . (We do not assume that $g \in E_n^+$, and hence this vector can have negative components. Such components may be interpreted as the "damage" caused to the corresponding players or alternatively as an indication that they should leave the whole set of players).

Since representation (1) holds for a quasidifferentiable game, it is interesting to consider the vectors x(g) and y(g) at which the corresponding maximum and minimum are attained. Since $\underline{\partial} v(\Pi)$ and $\bar{\partial} v(\Pi)$ are convex compact sets, the sets

$$\operatorname{Arg\,max} \{(x,g) \mid x \in \underline{\partial} v(\Pi)\}$$

and

$$\operatorname{Arg\,min} \{(y,g) \mid y \in \bar{\partial} v(\Pi)\}$$

consist of only one element for almost every $g \in S^{n-1}$.

Let G(v) denote the set of such g , and z(g)=x(g)+y(g) . Note that if the function v is both locally Lipschitzian and quasidifferentiable and also satisfies some additional property (which is too cumbersome to describe here--see Demyanov [4]), then the points z(g), $g \in G(v)$, describe all extreme points of the Clarke subdifferential of v at Π (the set of solutions proposed by J.-P. Aubin).

4. Solution of quasidifferentiable games

We shall now define the solution of a quasidifferentiable game, which we shall call an st-solution. We require the following additional definition:

Definition 7 [6]. *Let* K *be a compact convex set in* E_n . *The Steiner point of the set* K *is the point*

$$s(K) = \frac{1}{\sigma_n} \int_{S^{n-1}} \alpha p(K,\alpha) d\lambda \ , \tag{9}$$

where λ *is the Lebesque measure on the unit sphere* S^{n-1} *in* E_n , σ_n *is the volume of the unit ball in* E_n , α *is a variable vector on* S^{n-1} *and* $p(K,\cdot)$ *is the support function of* K .

Note that we always have $s(K) \in K$ and $s(-K)=-s(K)$. Let v be a quasidifferentiable characteristic function with quasidifferential

$$D^{\pi}v(\Pi) = [\underline{\partial}v(\Pi) \ , \ \bar{\partial}v(\Pi)] \ .$$

Definition 8. *The st-solution of a quasidifferentiable game with characteristic function* v *is the vector* st(v) *defined by the equality*

$$st(v) = s(\underline{\partial}v(\Pi)) + s(\bar{\partial}v(\Pi)) \ . \tag{10}$$

We first have to prove that this definition does not depend upon the pair defining a particular quasidifferential v (such a quasidifferential may not even be "Pareto-optimal"). This follows immediately from the linearity on K (with respect to vector addition of sets) of the function s defined by (9), and from the following obvious property of quasidifferentials: if $[A,B]$ is a quasidifferential of v at x, then the pair $[A_1,B_1]$ is also a quasidifferential of v at x if and only if

$$A - B_1 = A_1 - B . \tag{11}$$

Using the equality (11) and the linearity of s we get

$$s(A-B_1) = s(A_1-B) \Leftrightarrow$$

$$s(A)-s(B_1) = s(A_1)-s(B) \Leftrightarrow s(A)+s(B) = s(A_1)+s(B_1) .$$

The vector st(v) can be interpreted as the vector of average marginal utilities received by the players.

We shall now describe some properties of st-solutions.

<u>Proposition 2</u>. *If a generalized game is quasidifferentiable, then:*

1. *The mapping* $st:v \to st(v)$ *is linear in* v .
2. *The st-solution is Pareto-optimal, i.e.,*

$$\sum_{i=1}^{n} (st(v))_i = v(\mathbb{I}) .$$

3. *If* v *is continuously differentiable, then* $st(v)=\nabla v(\mathbb{I})$ *and the st-solution coincides with the generalized Shapley value of* v.
4. *If* v *is concave (superadditive), then* st(v) *is the Steiner point of the core of the game.*

The proof of this proposition follows immediately from Proposition 1, Theorem 1, and the definition of quasisolutions.

Now let us prove two more important properties of an st-solution: it satisfies the "dummy" axiom (Theorem 2) and is symmetric (Theorem 3).

Let a quasidifferentiable game have characteristic function v such that $v(x)=v(x^{I\setminus i})$ for every $x \in E_n^+$. Then for every $g \in E_n$ we have

$$\frac{\partial v(\mathbb{I})}{\partial g} = \lim_{\lambda \to +0}\left(\frac{v(\mathbb{I}+\lambda g) - v(\mathbb{I})}{\lambda}\right) = \tag{12}$$

$$= \lim_{\lambda \to +0}\left(\frac{v(\mathbb{I}^{I\setminus i}+\lambda g^{I\setminus i}) - v(\mathbb{I}^{I\setminus i})}{\lambda}\right) = \frac{\partial v(\mathbb{I}^{I\setminus i})}{\partial g^{I\setminus i}} .$$

It is clear that the function $\bar{v}=v|_{\mathbb{R}^{I\setminus i}}$ is quasidifferentiable at $\mathbb{I}^{I\setminus i}$ and its quasidifferential at this point is defined by the pair $[\mathrm{Pr}\,\underline{\partial}v(\mathbb{I}), \mathrm{Pr}\,\bar{\partial}v(\mathbb{I})]$, where Pr A is the projection of A on $\mathbb{R}^{I\setminus i}$. Hence, from (12), this pair is the quasidifferential of v at $\mathbb{I}$. Thus if $x \in \mathrm{Pr}(\underline{\partial}v(\mathbb{I}))$ and $y \in \mathrm{Pr}(\bar{\partial}v(\mathbb{I}))$, then $x_i=0$, $y_i=0$. From this we have $(st(v))_i=0$ and the following theorem holds.

Theorem 2. *If a quasidifferentiable game with characteristic function* v *is such that* $v(x)=v(x^{I\setminus i})$ *for every* $x \in [0,1]^n$, *then* $(st(v))_i=0$.

In other words, the function st(·) satisfies the so-called dummy axiom, which states that a (dummy) player who gives nothing to any coalition will also receive nothing.

"Nothing will come of nothing"

Shakespeare, *King Lear*

Suppose now that v is quasidifferentiable and π is a permutation of the set of players $I=1:n$. We shall define the game π^*v as follows: $\pi^*v(x)=v(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)})$.

Let $(\pi^{-1}x)_i = x_{\pi^{-1}(i)}$ and $(\pi x)_i = x_{\pi i}$.

<u>Theorem 3</u>. *The st-solution is symmetric, i.e.,* $st(\pi^*v)=\pi\ st(v)$.

<u>Proof</u>. If $[\underline{\partial}v(\mathbb{I}),\ \bar{\partial}v(\mathbb{I})]$ is a quasidifferential of v at $\mathbb{I}$, then

$$\frac{\partial\pi^*v(\mathbb{I})}{\partial g} = \lim_{\lambda\to+0}\left(\frac{\pi^*v(\mathbb{I}+\lambda g)-\pi^*v(\mathbb{I})}{\lambda}\right) =$$

$$= \lim_{\lambda\to+0}\left(\frac{v(\pi^{-1}\mathbb{I}+\pi^{-1}(\lambda g))-v(\pi^{-1}\mathbb{I})}{\lambda}\right) =$$

$$= \lim_{\lambda\to+0}\left(\frac{v(\mathbb{I}+\lambda(\pi^{-1}g))-v(\mathbb{I})}{\lambda}\right) = \frac{\partial v(\mathbb{I})}{\partial(\pi^{-1}g)} .$$

Hence

$$\frac{\partial\pi^*v(\mathbb{I})}{\partial g} = \max_{z\in\underline{\partial}v(\mathbb{I})}(z,\pi^{-1}g) + \min_{y\in\bar{\partial}v(\mathbb{I})}(y,\pi^{-1}g) =$$

$$= \max_{z\in\underline{\partial}v(\mathbb{I})}(\pi z,\pi(\pi^{-1}g)) + \min_{y\in\bar{\partial}v(\mathbb{I})}(\pi y,\pi(\pi^{-1}g)) =$$

$$= \max_{z\in\pi(\underline{\partial}v(\mathbb{I}))}(z,g) + \min_{y\in\pi(\bar{\partial}v(\mathbb{I}))}(y,g) .$$

Thus $[\pi(\underline{\partial}v(\mathbb{I})),\pi(\bar{\partial}v(\mathbb{I}))]$ is a quasidifferential of π^*v at $\mathbb{I}$. Since the Steiner point is invariant under orthogonal transformations of E_n , then

$$s(\pi\underline{\partial}v(\mathbb{I})) = \pi s(\underline{\partial}v(\mathbb{I})) \quad , \quad s(\pi\bar{\partial}v(\mathbb{I})) = \pi s(\bar{\partial}v(\mathbb{I}))$$

and hence

$$st(\pi * v) = \pi \cdot st(v) \ , \tag{13}$$

which is the proposition of the theorem.

It is clear that the above formula holds for every orthogonal transformation of E_n which leaves the vector $\mathbb{I}$ unchanged.

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