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OPTIMAL HARVESTING POLICY FOR THE LOGISTIC GROWTH MODEL

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INTRODUCTION

Logistic growth functions have been widely used to study optimal management of fisheries (e.g. Clark, 1976), forests (Kilkki and Vaisanen, 1969; Andersson and Lesse, 1984) and mammal populations (see for instance Spence, 1973). Although this simple model does not capture many of the important elements of biological dynamics, it does possess the critical element of saturation, the slowing of biomass accumulation as a "carrying capacity" is reached. Thus the theoretical results based on this very simple growth model are useful in understanding the results of more complex and more realistic optimal management problems. In this light, this paper makes four contributions.

First, we introduce constraints on the control variable (Heaps and Neher, 1979). Rarely if ever are harvest levels in realistic problems completely unconstrained, so this first complication of the traditional bionomic model provides an important added degree of realism.

Second, we study the situation where the boundary conditions of the state variable are specified. Generally the resource manager is not free to choose the initial resource conditions (indeed much of resource management is concerned with decisions when these conditions are judged to be somehow undesirable) so this complication of the model is important. Terminal conditions are sometimes specified by law or administrative direction. Furthermore, it is frequently computationally infeasible to solve realistic planning models for an infinite time horizon. Examination of the system behavior near a specified terminal value is consequently useful to understanding applied management problems.

Third, we show that the solution derived by applying Pontryagin's maximum principle is indeed globally optimally. Other treatments of this problem do not attend to this important detail of sufficiency.

Finally, we state and solve the dual optimization problem. Very often the duals of many complex management problems are far easier to solve than are the primals. In our case the dual problem also has a very clear interpretation for management: maximize the terminal inventory stock subject to the condition that harvest levels should never fal below a prescribed level.

The conclusion outlines a somewhat more realistic model where the biological system is characterized by an age-class model, and indicates the kind of issues which are interesting in that context.

1. OPTIMAL HARVESTING POLICY

System

$$\dot{x} = Kx(1-x) - ux \quad , \quad t \in [0,T] \tag{1}$$

where K, T are given positive numbers, x and u are the state and control variables correspondingly. The point above the character stands for the time derivative.

Boundary Conditions

$$x(0) = x_0 > 0 \quad , \quad x(T) = x_T > 0 \tag{2}$$

Constraints

$$0 \le u(t) \le \bar{u} \quad , \quad t \in [0,T] \ . \tag{3}$$

The pair $x(t), u(t)$, which satisfies the conditions (1)–(3), will be called admissible.

Objective functional

$$I = -\int_0^T u(t)x(t)e^{-\delta t}\mathrm{d}t \quad , \quad \delta \ge 0 \tag{4}$$

It is required to find out an admissible pair $x^0(t), u^0(t)$ such that the value $I(x^0(t), u^0(t))$ is minimal among the values I for all admissible pairs x, u.

Maximum Principle for the Above Problem

If the optimal pair $x^0(t), u^0(t)$ exists, then one can try to find it by applying the traditional technique of the maximum principle of Pontryagin [1962]. For the above problem it involves the introduction of the Hamiltonian function

$$H(t,\psi,x,u) = \psi(t)[Kx(1-x) - ux] + \lambda_0 uxe^{-\delta t} \quad , \quad \lambda_0 \geq 0 \tag{5}$$

where $\psi(t)$ is the adjoint function satisfying the equation:

$$\dot{\psi} = \frac{\partial H}{\partial x}|_{x^0,u^0} = -\psi[K - 2Kx^0(t) - u^0(t)] - \lambda_0 u^0(t)e^{-\delta t} \quad . \tag{6}$$

By the maximum principle the optimal pair $x^0(t), u^0(t)$ has a property that for any $t \in [0,T]$

$$M(t) \triangleq H(t,\psi(t),x^0(t)u^0(t)) = \max_{0 \leq u \leq \bar{u}} H(t,\psi(t),x^0(t),u) \quad , \tag{7}$$

where $(\psi(t), \lambda_0)$ is not zero vector and $M(t)$ is a continuous function of t and for $t \in (t_0,T)$ has to satisfy to the equation:

$$M(t) = H(T,\psi(T),x^0(T),u^0(T)) - \int_t^T \frac{\partial H}{\partial \tau}(\tau,\psi(\tau),x^0(\tau),u^0(\tau))d\tau$$

$$= H(t_0,\psi(t_0),x^0,(t_0)u^0(t_0)) + \int_{t_0}^t \frac{\partial H}{\partial \tau}(\tau,\psi(\tau),x^0(\tau),u^0(\tau))d\tau. \tag{8}$$

Equation (7) can be used to define firstly u^0 as a function of t, x, ψ.

$$u^0 = u(t,x,\psi) \quad ,$$

Since

$$H(t,\psi(t),x(t),u(t)) = -\psi Kx(1-x) + ux(-\psi + \lambda_0 e^{-\delta t}) \, ,$$

from (7) and positiveness of $x(t)$ it follows that

$$u^0(t,x,\psi) = \begin{cases} \bar{u} & , \text{ when } (-\psi(t) + \lambda_0 e^{-\delta t}) > 0 \\ u_s & , \text{ when } (-\psi(t) + \lambda_0 e^{-\delta t}) \equiv 0 \\ & \quad \text{(so-called singular control),} \\ 0 & , \text{ when } (-\psi(t) + \lambda_0 e^{-\delta t}) < 0 \end{cases} \tag{9}$$

Then one can eliminate the variable u from equations (1) and (6) and come to their solution as to the two-point boundary value (*tpbv*) problem. This solution, if successful, will give $x^0(t)$ and $u^0(t)$. There is no regular way to solve the tpbv problem if it is nonlinear (as ours). The following is an attempt to get this solution for all possible combinations of boundary conditions (2).

Singular part of the solution

If there is an interval for which $\psi(t) = \lambda_0 e^{-\delta t}$, then within it as follows from (6), and from (7),

$$x_s(t) = x^0(t) = \frac{K - \delta}{2K} = \text{const}, \tag{10}$$

$$u_s(t) = u^0(t) = \frac{K + \delta}{2} = \text{const} \quad (\leq \bar{u}) \tag{11}$$

At the same interval

$$M(t) = \lambda_0 e^{-\delta t} K \frac{1 - \frac{\delta^2}{K^2}}{4} \tag{12}$$

Since $\psi(t) = \lambda o\ e^{-\delta t}$, and $(\psi(t), \lambda o)$ is nonzero it implies that $\lambda o > 0$ and can be taken as $\lambda o = 1$ for example.

In general $x_0 \neq x_s(0)$ and $x_T \neq x_s(T)$. Therefore, the optimal pair $x^0(t), u^0(t)$ cannot consist only of $x_s(t)$ and $u_s(t)$. At least in the vicinity of $t = 0$ and $t = T$ the optimal pair should generally differ from $x_s(t), u_s(t)$.

The Structure of the Optimal Solution

Knowing the singular part of the optimal solution on the interval $[t_i, t_s]$, $0 < t_i \leq t_s < T$, one can show that for all possible boundary conditions (values x_0 and x_T) the optimal control $u^0(t)$ consist of the saturation portions near boundaries, spanned by the singular control in between.

To show this, one should check near boundaries the existence of a function $\psi(t)$ for which $u^0(t, \psi, x)$ generates the admissible trajectory $x(t)$. We will do this by studying the admissibility of trajectory for the right and left boundary conditions (2), separately.

The Right Boundary

The Case $x_T > x_s(T)$. To the boundary condition $x_T > x_s(T)$ one can "ascend" from $x_s(t)$ with control $u(t) \equiv 0$, $t > t_s$ Time of departure t_s from $x_s(t)$ can be chosen from the condition to "hit" x_T at time T. If this control is optimal here, then from (9) the corresponding $\psi(t)$ should be such that

$$\psi(t) > \lambda_0 e^{-\delta t}. \tag{13}$$

The Case $x_T < x_s(T)$. To the boundary condition $x_T < x_s(T)$ when $\bar{u} > \frac{K + \delta}{2}$, and $x_T > 1 - \frac{\bar{u}}{k}$ one can "descend" starting from the singular level at $t = t_s$ with the control $u(t) = \bar{u}$. If this is the optimal control, then from (9) the function $\psi(t)$, corresponding to it, should be such that

$$\psi(t) < \lambda_0 e^{-\delta t} \quad . \tag{14}$$

The Left Boundary

The Case $x_0 < x_s(0)$. From this boundary condition one can "ascend" with control $u(t) \equiv 0$ to the singular level. For this ascent to be the part of the optimal trajectory it is necessary that

$$\psi(t) > \lambda_0 e^{-\delta t} \quad ,0 \le t < t_i \quad , \tag{15}$$

The Case $x_0 > x_s(0)$. From this boundary condition one can "descend" with control $u(t) = \bar{u}$ (when $\frac{k+\delta}{2} < \bar{u}$) to the singular level. For this descent to be part of the optimal trajectory it is necessary (as follows from (9)), that

$$\psi(t) < \lambda_0 e^{-\delta t} \quad , \; 0 \le t < t_i \tag{16}$$

To prove that the inequalities (13)-(16) are fulfilled for the chosen controls we introduce the new variable ρ by the formula

$$\psi(t) = \lambda_0 \, e^{-\delta t} \, \rho(t) \tag{17}$$

The inequalities $\psi(t) > \lambda o \; e^{-\delta t}$ and $\psi(t) < \lambda_0^{e^{-\delta t}}$ will become equivalent to $\rho(t) > 1$ and $\rho(t) < 1$ correspondingly. On the singular part $\rho(t) \equiv 1$. From (6) we can get that

$$\dot{\rho} = -\rho(K - \delta - 2K\,x) + u(\rho - 1). \tag{18}$$

This equation is simpler than (6) and will more easily bring us to our goal.

Case $u^0 \equiv 0$ $\quad$ ($x_T > x_s(T)$ and $x_0 < x_s(0)$)

For $u \equiv 0$

$$\dot{\rho} = -\rho(K - \delta - 2Kx)$$
$$\dot{x} = Kx - Kx^2$$

From these two equations and the fact that in the "singular" interval $\rho \equiv 1$ one can find

$$\rho = \left\{\frac{1-x_s}{1-x}\right\}^{2(1-x_s)} \left\{\frac{x_s}{x}\right\}^{2x_s} , \; > 1 \text{ for } x \ne x_s \, . \tag{19}$$

It proves the optimality of $u^0(t) = 0$ at the boundaries for the cases $x_T > x_s(T)$ and $x_0 < x_s(0)$.

Case $u^0(t) = \bar{u}$ $\;$ ($x_T < x_s(T)$ and $x_s > x_s(0)$)

When $u(t) = \bar{u}$, then by substitution $x = \frac{K-u}{K} y$ to (18) and (1) we come to the equation

$$\frac{d\rho}{dy} = 2\rho \, \frac{y_s - y}{y(y-1)} + \frac{u}{K-u} \, \frac{(\rho-1)}{y(y-1)} \tag{20}$$

with $y > 1$, $y_s = \frac{K}{K-u} x_s$.

It follows from (20) and $\rho(y_s) = 1$ that

$$\frac{d\rho}{dy}\Big|_{ys} = 0 \text{ and } \frac{d^2\rho}{d_y^2}\Big| y_s < 0$$

From this and (20) one can conclude that p is decreasing (being positive, see p. 7-8) when $y < y_s$ and y is decreasing and when $y > y_s$ and y is increasing

That gives also the required proof for optimality of $u^0(t) = \bar{u}$ at the boundaries for the cases $x_T < x_s(T)$ and $x_0 > x_s(0)$. By these four possible boundary conditions the structure of optimal solution was proven valid.

Equation for the Boundary Portion

For the intervals with $u^0(t) = 0$ and $u^0(t) = \bar{u}$ the state equation has the form

$$\dot{x} = ax - Kx^2$$

with $a = K$ or $K - \bar{u}$ correspondingly. The solution for $t \geq t_0$ and $a \neq 0$ is

$$x(t) = \frac{a}{(ax^{-1}(t_0) - K)e^{-a(t-t_0)} + K} \tag{21}$$

and when $a = 0$

$x(t) = (x^{-1}(t_0) + K \cdot (t - t_0)^{-1}$.

For given boundary conditions x_0 and x_T this solution can be used to define the values of t_i and t_s ($t_i < t_s$)-moments of time for joining the boundary portions of optimal solution with the singular arc (10).

If for the given boundary conditions $t_s \leq t_i$ then the optimal solution has no portion with the singular arc and consists of only two conjuncted boundary portions.

2. THE GLOBAL OPTIMALITY OF THE SOLUTION GIVEN BY THE MAXIMUM PRINCIPLE FOR THE ABOVE PROBLEM

To prove this we will use the approach to global optimality developed by Krotov (1962, 1963). In this approach one can prove the global optimality of $x^0(t), u^0(t)$ from the discussed problem by constructing the function

$$R(t,x,u) = \frac{\partial\Psi(t,x)}{\partial t} + \frac{\partial\Psi(t,x)}{\partial x}(Kx - Kx^2 - ux) + uxe^{-\delta t} \tag{22}$$

where $\Psi(t,x)$ is the so-called Krotov's function, and by checking for this function the fulfillment of the following condition

$$R(t,x^0(t),u^0(t)) = \max_{u,x} R(t,x,u), \; 0 \leq t \leq T \tag{23}$$

subject to $0 \leq u \leq \bar{u}$.

Let $\Psi(t,x) = \psi(t)x$ and let $\psi(t)$ be the adjoint variable from the discussed problem.

With such Krotov's function the maximum of $R(t,x,u)$ with respect to u is reached along $u = u^0(t)$ since

$$\max_u R = \frac{\partial\Psi}{\partial t} + \max_u H(t,\psi,x,u) \tag{24}$$

where $H(t,\psi,x,u)$ is the Hamiltonian for our problem, which reaches its maximum with respect to u along $u = u^0(t)$.

Since $R(t,x,u)$ is the quadratic function of x, we will check the validity of (23) with respect to x by calculating $\frac{\partial R}{\partial x}$ and $\frac{\partial^2 R}{\partial x^2}$.

$$\frac{\partial R}{\partial x} = \frac{d\Psi_x}{dt} + \Psi_x[(K - 2x - u)] + ue^{-\delta t} =$$

$$= \frac{d\psi}{dt} + \psi(K - 2x - u) + ue^{-\delta t} = 0$$

and due to (6) with $\lambda_0 = 1$.

$$\frac{\partial^2 R}{\partial x^2} = -2K\psi .$$

If $\psi(t) \geq 0$ then (24) holds and the global optimality for $x^0(t), u^0(t)$ is proven. Let us check this.

On the singular arc $\psi(t) = \lambda_0 e^{-\delta t} > 0$. Then for the cases with boundary conditions $x_T > x_s(T)$ and $x_0 < x_s(0)$ it was shown that

$$\psi(t) > \lambda_0 e^{-\delta t} > 0 .$$

For the cases $x_0 > x_s(0)$ and $x_T < x_s(T)$ due to (17) the positiveness for $\psi(t)$ follows from $\rho(t) > 0$. This inequality is true because as follows from (7), (7a), (8) for $x(t) > x_s(0)$, $t < t_t$,

$$\rho(t) > \frac{\bar{u}\; x - e^{\delta(t-tt)} K \frac{1 - \frac{\delta^2}{K^2}}{4} - \delta \bar{u} e^{\delta \cdot t}\, x \int_t^{tt} e^{-\delta\tau} d\tau}{\bar{u}x - K \cdot x\,(1-x)}$$

$$= e^{\delta(t-t_t)} \frac{\bar{u}\; x - K \frac{1 - \frac{\delta^2}{K^2}}{4}}{\bar{u}x - Kx(1-x)} > e^{\delta(t-tt)} \frac{\bar{u}x_s - Kx_s(1-x_s)}{\bar{u}x_s - Kx_s(1-x_s)} > 0 .$$

and for $x(t) < x_s(T)$, $t > t_s$

$$\rho(t) > \frac{\bar{u}\; x - e^{\delta(t-tt)} K \frac{1 - \frac{\delta^2}{K^2}}{4} + \delta \bar{u} e^{\delta t}\, x \int_{ts}^{t} e^{\delta\tau} d\tau}{\bar{u}x - K \cdot x(1-x)}$$

$$= e^{\delta}(t-t_s) \frac{\bar{u}x - K \frac{\frac{1-\delta^2}{K^2}}{4}}{\bar{u}x - Kx(1-x)} > e^{\delta(t-ts)} \frac{ux_s - Kx_s(1-x_{s1})}{\bar{u}x - Kx(1-x)} > 0 .$$

This proves the global optimality for $x^0(t), u^0(t)$

3. THE DUAL PROBLEM

System and Boundary Conditions

See (1) and (2).

Constraints

$$0 \le u(t) \le \bar{u} \le , \quad t \in [0,T]$$

$$\int_0^T u(t)x(t)e^{-\delta t}dt \ge W > 0 \ . \tag{25}$$

Objective functional

$$I = -x(T) \tag{26}$$

The optimization problem (1), (2), (25), (26) can be tracted as a maximization of an inventory stock at the given moment T with total harvesting no less than the prescribed volume W. The straightforward elaboration led to the same conditions of maximum principle for this problem as in Section 1 with the following additions: $\psi(T) \ge 0$ and the sign of λ_0 is initially not specified now.

This means that our considerations concerning the structure of optimal control for the first problem are applicable here too and will produce the same conclusions as before: namely, for all possible initial conditions (values of x_0) near boundaries the optimal control $u^0(t)$ consists of the saturation portions ($u^0(t) = 0$ or $u(t) = \bar{u}$) spanned in between by the same singular control $u_s^0(t) = \frac{K+\delta}{2}$, $t_i \le t \le t_s$.

$$\text{If } x_0 > x_s(0) \ , \ \text{then } u^0(t) = 0 \ , \ t \le t_i \ ,$$

$$\text{if } x_0 < x_s(0) \ , \ \text{then } u^0(t) = \bar{u} \ , \ t \le t_i$$

After $t = t_i$ the optimal trajectory $x^0(t) = x_s(t) = \dfrac{1 - \frac{\delta}{K}}{2}$ till the time $t = t_s < T$ which is defined by the moment when

$$\int_0^{t_s} u^0(t)x^0(t)e^{-\delta t}dt = W \ .$$

Here for $t_s < t \le T$ $u^0(t) = 0$, and $x^0(T) > x_s(T)$.

If $t_s > T$ then t_s time of "departure" $\bar{t}_s$ from $x_s(t)$ is defined as

$$\int_0^{\bar{t}_s} u^0(t)x^0(t)e^{-\delta t}dt + \bar{u}\int_{\bar{t}_s}^{T} x^0(t)dt = W$$

and in this case for $\bar{t}_s < t \le T$ $u^0(t) = \bar{u}$ and $x^0(T) > x_s(T)$.

4. CONCLUSIONS

Bioeconomic models based on logistic biological dynamics are widely used in the fisheries, forestry and renewable resource literature. We present a rather complete solution to the resource management problem with logistic growth, showing the effect of control constraints and arbitrary boundary conditions as well as demonstrating the sufficiency of the maximum principle solution and solving the dual problem. These solutions have some utility in their own right for prescribing optimal management policies. Furthermore, they suggest how a more realistic system might behave.

How would one complicate this model to capture the next degree of realism? Let us consider the case of forest growth. In many parts of the world forests regenerate after either natural catastrophic disturbances (e.g. fire, windthrow or insect defoliation) or anthropogenic ones (timber harvesting, agricultural abandonment). The dynamics of the resulting even-aged forests can be characterized by the aging of individual stands and the regeneration of new stands through the harvest of old ones. Optimal control can be studied in this context.

Heaps (1984) and heaps and Neher (1979) examined the continuous time process. While some of the characteristics of the solutions have been derived, others have not. In particular, the temporal asymptotic behavior is not well understood: under what circumstances does the rate of harvest converge? What is the nature of the asymptotic age structure of the forest? While the logistic model provides some insight into the development of a renewable resource, it obscures the answer to some of these interesting questions.

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