ON OPTIMALITY CRITERIA

Yu. A. Rozanov

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Austria

International Institute for Applied Systems Analysis

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One of the future trends in optimal control theory, as well as in decision theory, is likely to concern more stable, robust, and reliable solutions in cases of uncertainty of considered systems. The first step in this direction may be the development of new and more flexible criteria of optimality. One of the possible criteria is suggested below with a hope of the future development of the corresponding optimization technique.

The typical situation is the following:

The system considered is characterized by some (abstract) parameter $\theta \in \Theta$ which is not known exactly. It is assumed that in a case of the system θ one has to maximize some benefit $f(\theta,u)$ choosing a proper "decision" $u \in U$. The decision $u = u(\theta)$ which is good enough for the system θ may be absolutely wrong for another system θ', $\theta' \neq \theta$, and the difficulty is to chose a reasonable decision $u \in U$ taking into account the possibility to make a blunder in our estimation of the real parameter $\theta \in \Theta$.

A number of observations lead us to suggest the following criterion of optimality: for some function $g(\theta,\cdot)$ we have to maximize its "expected" value

$$Eg(\cdot,f) = \int g\left[\theta, f(\theta,u)\right] P(d\theta) \to \underset{u \in U}{\text{maximum}} \quad . \qquad (1)$$

The main point is that we have to adjust the proper "utility function" g not only with respect to the probability distribution of the real benefit $f(\theta,u)$ but also with respect to the parameter $\theta \in \Theta$ itself. Here P means some preference measure (not necessarily probability distribution); in the most interesting cases it can be interpreted as the a priori distribution of $\theta \in \Theta$. Let us consider a few examples.

1. In the case where the function $g(\theta,\cdot)$ does not depend on θ, we deal with the criterion upon which most developed theories are based.

2. Suppose that we are satisfied with the decision u if for all $\theta \in \Theta$ the real benefit $f(\theta,u)$ is such that the pair $[\theta,f(\theta,u)]$ belongs to some "admissible set" $\Gamma \subseteq \Theta \times (-\infty,\infty)$. The admissible set Γ may, for example, consist of all pairs (θ,y) of the type

$$(\theta,y): \quad y \geq \max_{u \in U} f(\theta,u) - \varepsilon(\theta) \quad , \tag{2}$$

where $\varepsilon(\theta) \geq 0$ is some acceptable boundary. Of course, there can be no $u \in U$ such that

$$\left[\theta,f(\theta,u)\right] \subseteq \Gamma \text{ for all } \theta \in \Theta \quad .$$

In this case a preference function $P(u)$ can be defined as

$$P(u) = P\{\theta:[\theta,f(\theta,u)] \in \Gamma\} \quad , \tag{3}$$

and the optimality criterion might be as follows:

$$P(u) \to \underset{u \in U}{\text{maximum}} \tag{4}$$

Obviously the criterion (4) can be represented in the form (1) by using the corresponding utility function

$$g(\theta,y) = \begin{Bmatrix} 1 \text{ if } (\theta,y) \in \Gamma \\ 0 \text{ if } (\theta,y) \notin \Gamma \end{Bmatrix} \quad .$$

3. If we take the admissible set Γ of the form (2) with $\varepsilon(\theta) \equiv 0$, then the criterion (4) seems appropriate for risky decision making of the type "all" or "nothing." We choose the decision u^0 such that

$$\text{Probability of } \left\{ f(\theta,u^0) = \underset{u \in U}{\text{maximum}}\ f(\theta,u) \right\} = \text{maximum} \tag{5}$$

That is, we maximize the probability of having the maximum of the real benefit $f(\theta,u)$.

4. For another specific admissible set Γ, the criterion (4) gives us the well-known minmax principle developed in game theory for cautious decision making. Let

$$f_u = \liminf_{\theta \in \Theta_0} f(\theta,u)$$

be the lowest boundary of our benefit concerning some set $\Theta_0 \in \Theta$ under the decision u. Suppose that we are interested to receive at least the maximum of possible values f_u, $u \in U$

$$f_u \to \underset{u \in U}{\text{maximum}} \quad . \tag{6}$$

Obviously the decision $u \in U$ satisfies this minmax criterion if and only if

$$f(u,\theta) \geq M \text{ for almost all } \theta \in \Theta_0 \quad ,$$

where

$$M = \sup_{u \in U} f_u \quad .$$

Now it is easy to verify that by choosing the admissible set as

$$\Gamma = \theta_0 \times (M,\infty)$$

we can represent the minmax criterion (6) in the form (4) as well as in the form (1).

5. The general criterion (1) with its proper specifications (2)-(6) seems to be useful for a multicomponent (vector) optimization. In this case the discrete parameter θ is identified with the corresponding component considered; and the preference measure $P(\theta), \theta \in \Theta$, can be recognized as "Pareto coefficients" in the equation

$$Eg(\cdot, f) = \sum_{\theta \in \Theta} g[\theta, f(\theta, u)]P(\theta) \quad .$$