ON THE DETERMINATION OF THE DEGREE OF A POLYNOMIAL

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FOREWORD

Within the framework of the Economic Structural Change Program, a cooperative research activity of IIASA and the University of Bonn, FRG, a project is carried out on "Statistical and Econometric Identification of Structural Change"; the project involves studies on the formal aspects of the analysis of structural changes. On the one hand, they include statistical methods to detect non-constancies, such as stability tests, detection criteria, etc., and on the other hand, methods which are suitable for models which incorporate non-constancy of the parameters, such as estimation techniques for time-varying parameters, adaptive methods, etc.

The present paper discusses a decision procedure for the determination of the degree of a polynomial which is based on stage-wise rejective hypotheses testing. It can be applied to the problem mentioned, but also to similar regressor or parameter selection situations, such as the determination of a trend surface, a distributed lag structure, or the order of an autoregressive process.

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Abstract:

Starting from a method suggested by T.W.Anderson (1971) stagewise rejective test procedures for determining the degree of a polynomial are proposed. Accounting for the special structure of the problem, Holm's (1979) individual significance levels can be improved. If the critical limits for the individual tests of the simultaneous test procedure are chosen in an appropriate dependence on the sample size, the test procedure provides a weakly consistent estimate of the correct order of polynomial. The corresponding theorem is proved for a general procedure for determining the correct subset of a finite number of model parameters.

Key-words: Degree of a polynomial, regressor subset selection, stagewise rejective tests, weak convergence.

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1. Introduction

In some situations of regression analysis, the regressor subset selection problem has the structure of deciding within a sequence of nested hypotheses. Typical situations of this type are the case where a polynomial, a trend surface, or a distributed lag structure of unknown order are to be estimated. A related situation arises when the order of an autoregressive process is to be estimated.

Corresponding statistical selection procedures should keep the order of the model so large as necessary and so small as possible: Given the true order to be r, a choice less than r leads to biased estimates of the model parameters whereas the choice of an order larger than r results in a loss of efficiency and could lead, e.g., to an erroneous interpretation of explanatory variables which in fact are irrelevant for the dependent variable. One requirement to be met is that, asymptotically for an increasing number of observations, the true order should be obtained.

In recent years it has become common practice to use 'model fitting criteria' for the selection of the appropriate model (Akaike, 1974; Amemiya, 1980; Mallows, 1973; Parzen, 1974; Schwarz, 1978). For the linear model situation Geweke & Meese (1981) have investigated different criteria for estimating the true order: They have established that only Schwarz's (1978) SBIC criterium provides a weakly consistent estimation procedure. For AR models, Hannan & Quinn (1979) suggest a strongly consistent order estimation procedure. Pötscher (1983) used simultaneous Lagrange multiplier statistics in order to test the parameters of ARMA models; he proved the strong consistency of his procedure for determining the correct order if the significance levels for the individual tests tend to zero in an appropriate way. It should be noted that the use of model fitting criteria, e.g., Akaike's AIC, is equivalent to simultaneousely looking on the likelihood ratio statistics when testing all possible pairs of models, the critical limits depending on the difference of the number of model parameters and the number of observations.

In the following the multiple test approach for the individual regression coefficients is used for determining the order of a polynomial regression. The formulation of the hypotheses is in accordance with the procedure introduced by Anderson (1971). The multiple test procedure controls the probability of erroneously including a term of higher than the true order of the polynomial. It is an improved version of a stagewise rejective test (Marcus et al., 1976; Holm, 1979) taking into account the nested structure of the hypotheses to be tested simultaneously. In the asymptotic case of an increasing number of observations the procedure can easily be adapted to serve as a weakly consistent estimation procedure for the order of the polynomial. This property of weak convergence to the true model, moreover, is - under fairly general assumptions on the model with a finite number of parameters - valid for any parameter selection problem and does not depend on the nested structure discussed in the paper..

2. A Stagewise Rejective Test Procedure

Let

$$f(x, \theta) = \theta_0 + \theta_1 x + \dots + \theta_q x^q \qquad (1)$$

be a regression function in form of a polynomial of degree q. If such a polynomial is to be used as a descriptive device for a data set, it often should summarize the overall characteristics of the data. For this purpose the polynomial should be of fairly low degree. The degree of the polynomial with a satisfactory fit is rarely known to the investigator in advance. It general the investigator might be able to give the possible lowest degree m and the highest degree q; he then is left with the multiple decision problem of deciding whether the degree is m,m+1,...,q.

Anderson (1971) formalizes the problem as a decision problem between q-m+1 mutually exclusive parameter sets

$$
\begin{aligned}
H_q &: \theta_q \neq 0 \\
H_{q-1} &: \theta_q=0,\ \theta_{q-1}\neq 0 \\
&\vdots \\
H_{m+1} &: \theta_q=\ldots=\theta_{m+2}=0,\ \theta_{m+1}\neq 0 \\
H_m &: \theta_q=\ldots=\theta_{m+1}=0\ .
\end{aligned}
\tag{2}
$$

An alternative formulation is a decision problem between the hypotheses

$$
\begin{aligned}
H^*_q &: \theta_q=0 \\
H^*_{q-1} &: \theta_q=\theta_{q-1}=0 \\
&\vdots \\
H^*_{m+1} &: \theta_q=\ldots=\theta_{m+1}=0\ .
\end{aligned}
\tag{3}
$$

where, for i=m+1,...,q,

$$H^*_i = \bigcup_{j=m}^{i-1} H_j\ . \tag{4}$$

Anderson supposes that the investigator wants to control directly the probability of errors of saying that coefficients are not zero when they are zero or correspondingly of choosing a higher degree than suitable, and that, given these probabilities, he wants to minimize the probabilities of saying coefficient are zero when they are not, or correspondingly of choosing a lower degree than suitable.

To the set of q-m null hypotheses $H^*_q,\ldots,H^*_{m+1}$ a stagewise rejective test procedure (Holm, 1979) can be applied. Such a procedure keeps a multiple level of significance ; this means that, whichever of the null hypotheses $H^*_q,\ldots,H^*_{m+1}$ are true, the probability of an erroneous rejection of a true null hypothesis is always bounded by α (see, e.g., Sonnemann, 1982).

Let $Y_{m+1},\ldots,Y_q$ be the respective statistics for testing the null hypotheses

$$\tilde{H}_{om+1}: \theta_{m+1}=0$$

$$\tilde{H}_{om+2}: \theta_{m+2}=0$$
$$\vdots \qquad (5)$$
$$\tilde{H}_{oq} : \theta_q=0$$

which refer to the q-m real-valued scalar parameters $\theta_{m+1},\ldots,\theta_q$. To cope with the two-sided test situation, the set of null hypotheses (5) is replaced by the set of q-m pairs of one-sided null hypotheses of the form

$$\tilde{H}^{\le}_{oi}: \theta_i\le 0 \ , \ \tilde{H}^{\ge}_{oi}: \theta_i\ge 0, \ i=m+1,\ldots,q, \qquad (6)$$

where

$$\tilde{H}_{oi} = \tilde{H}^{\le}_{oi} \cap \tilde{H}^{\ge}_{oi} \ . \qquad (7)$$

The i-th pair of (6) is tested by means of the test statistics

$$p^{\le}_i = P(T_i\ge y_i)$$
$$p^{\ge}_i = P(T_i\le y_i) \quad . \qquad (8)$$

Here, the random variable T_i has the distribution of the test statistic Y_i, given $\theta_i=\ldots=\theta_q=0$. The quantity y_i is the observed value of the test statistic Y_i. Usually, $p^{\le}_i$ and $p^{\ge}_i$ are denoted as the observed error probabilities. It is assumed that, independently of the true values of the parameters θ_j, $j<i$, and independently of the values of any nuisance parameters, the following inequalities hold for all i and $0\le\alpha<1$:

$$\left.\begin{array}{l} P\{p^{\le}_i \le \frac{\alpha}{2}\} \le \frac{\alpha}{2} \\ \\ P\{p^{\ge}_i \le \frac{\alpha}{2}\} \le \frac{\alpha}{2} \end{array}\right\} \quad \text{if} \quad \theta_i=\ldots=\theta_q=0 \quad . \qquad (9)$$

Then the two-sided test statistic is defined by

$$p_i = \min\{p^{\le}_i, p^{\ge}_i\} \qquad (10)$$

and obeys

$$P\{p_i \le \frac{\alpha}{2}\} \le \alpha, \quad \text{if } \theta_i=\ldots=\theta_q=0 \qquad (11)$$

The condition $\alpha<1$ assures that never both hypotheses $\tilde{H}^{\le}_{oi}$ and $\tilde{H}^{\ge}_{oi}$ can be rejected at the same time (Holm, 1979).

The stagewise rejective test procedure is based on the set (3) of null hypotheses which fulfills

$$H_i^* = \bigcap_{j=i}^{q} \tilde{H}_{oj} \ . \tag{12}$$

Then a level α-test for any such null hypothesis H_i^* is given by the critical region

$$\min_{j \in \{i,...,q\}} p_j \le \frac{\alpha}{2(q-i+1)} \ . \tag{13}$$

This follows since under H_i^*

$$P\{\text{ reject } H_i^*\} = P\{\bigcup_{j=i}^{q}\{p_j \le \frac{\alpha}{2(q-i+1)}\}\} \le$$

$$\le \sum_{j=i}^{q} P\{p_j \le \frac{\alpha}{2(q-i+1)}\} \le (q-i+1) \cdot \frac{\alpha}{q-i+1} = \alpha$$

by use of the Bonferroni's inequality and equations (9) and (11).

The stagewise rejective procedure is defined as follows:

Procedure: Reject the hypothesis H_k^*, if

$$p_k = \min_{j \in \{m+1,...,q\}} p_j \le \frac{\alpha}{2(q-m)} = \alpha_{(m+1)} \ ; \tag{14}$$

if $p_k > \alpha_{(m+1)}$ accept H_{m+1}^* and stop testing. At the second stage reject H_ℓ^*, if

$$p_\ell = \min_{j \in \{k+1,...,q\}} p_j \le \frac{\alpha}{2(q-k)} = \alpha_{(k+1)} \ ; \tag{15}$$

if $p_\ell > \alpha_{(k+1)}$ accept H_{k+1}^* and stop further testing. At the third stage the procedure is performed as at stage 2, replacing k by ℓ; and so on.

Theorem 1: The above defined multiple test procedure for the set of q-m null hypotheses $H_q^*,...,H_{m+1}^*$ provides the multiple level of significance α.

Proof: Marcus et al.(1976) have introduced so-called closed testing procedures, which keep the multiple level of significance α. For these test procedures it is required, that the finite set of null hypotheses to be tested is closed under intersection.

Any null hypothesis then is rejected if not only this hypothesis but also all other null hypotheses restricting the parameters to a subset of its parameter space are rejected in a level α-test.

Obviously the set of null hypotheses given by the H_i^*, $i=m+1,\ldots,q$, is closed under intersection, since for any subset $J\subseteq\{m+1,\ldots,q\}$ of indices it holds that

$$\bigcap_{j\in J} H_j^* = H_r^* \ , \ r = \min_{j\in J} j \ . \tag{16}$$

The construction implies that if H_k^* is rejected at the first stage all null hypotheses H_j^* with $H_j^*\subset H_k^*$ (viz. $H_{m+1}^*,\ldots,H_{k-1}^*$) are rejected in a level α-test based on (10), too. The same argument applies at the further stages of the procedure. □

Anderson (1971) in addition discusses the case where the interest in the different degrees of the polynomial is not the same: He gives a few hints how to choose individual significance levels for testing the individual hypotheses H_{0i}. Basically his advice tends to make q fairly large and the individual significance levels small for large degrees i ('if high degrees are not needed, the probability is small that a high degree is decided on').

To cope with this situation in the stagewise rejective procedure, positive weights $w_{m+1},\ldots,w_q$ can be defined, expressing the relative importance of the parameters $\theta_{m+1},\ldots,\theta_q$ for the multiple decision problem (cf. Holm, 1979): if $w_i>w_j$, θ_i is of more importance for the decision problem than θ_j is.

Modified Procedure: This procedure is performed in analogy to the original one, replacing the p_i by

$$p_i' = p_i/w_i \ , \ i=m+1,\ldots,q \ , \tag{17}$$

and the level α-test for testing the null hypothesis H_i^* by

$$\min_{j\in\{i,i+1,\ldots,q\}} p_j' \le \frac{\alpha}{2\sum_{j=i}^{q} w_j} = \alpha'_{(i)} \ . \tag{18}$$

At the first stage reject H_k^*, if

$$p'_k = \min_{j \in \{m+1,...,q\}} p'_j \leq \alpha'_{(m+1)} \quad ;$$

if $p'_k > \alpha'_{(m+1)}$ accept H_{m+1}^* and stop. At the second stage reject H_ℓ^*, if

$$p'_\ell = \min_{j \in \{k+1,...,q\}} p'_j \leq \alpha'_{(k+1)} \quad ;$$

if $p'_\ell > \alpha(k+1)$ accept H_{k+1}^* and stop; and so on.

Lemma 1: The modified test procedure for the set of q-m null hypotheses $H_q^*,...,H_{m+1}^*$ also provides the multiple level of significance α.

The proof is equivalent to that of Theorem 1.

The advantage of the procedures is obvious. If q=5, m=0, and, say, H_3^* is rejected at the first stage, in case of equally weighting the remaining two parameters θ_4 and θ_5 are individually tested at the two-sided level $\alpha/2$ only. This possible use of larger individual significance levels as compared to the classical Bonferroni type procedure increases the probability of correctly including non-zero polynomial terms.

It should be reminded that for the proposed procedures of simultaneously testing the set of null hypotheses H_i^*, i=m+1,...,q, it is required only that - under H_i^* - a level α-test exists for the respective coefficient of degree i independently of the coefficients corresponding to degrees j<i. A test of the degree of a polynomial can either be based on the coefficients of orthogonal polynomials or on those of the simple powers of the regressor variable.

Particularly in cases of small degrees of the polynomial one might be interested in directed decisions, i.e., $\theta_i<0$ or $\theta_i>0$. In such cases one would require a probability of at least $1-\alpha$ that the joint conclusion does neither contain false rejections of true null hypotheses nor false directional decisions. Closed test procedures do in general not fulfill this requirement; there are counter-examples even for independent test statistics as shown by Popper Shaffer (1980). This author, however, gives necessary conditions for the distribution functions in the case of independent test statistics. Bauer et al. (1985) give a general procedure of the Bonferroni-Holm type which meets this requirement: this procedure which cannot be further improved for the general situation is only slightly superior to Holm's procedure applied to 2k one-sided hypotheses.

3. Weak Consistency of a General Multiple Test Procedure for Determining the Correct Subset of Model Parameters

In this Section a general procedure will be proposed for selecting the model parameters by multiple testing, the method being valid also under the special structure of hypotheses given in the previous Section. It will be shown that this method is weakly consistent for estimating the correct subset of non-zero parameters.

Let us assume that one has to decide, which of the finite number of q parameters $\theta_1,\dots,\theta_q$ are non-zero and therefore have to be included into the model. Without loss of generality the set $I_0 = \{r+1,\dots,q\}$ denotes the indices of the parameters $\theta_{r+1}=\dots=\theta_q=0$, whereas $I_1 = \{1,\dots,r\}$ denotes the indices of the non-zero parameters $\theta_1\neq 0,\dots,\theta_r\neq 0$.

Let $\hat{\theta}_{in}$, $i=1,\dots,q$, be estimates of θ_i and $\hat{\sigma}^2_{in}$ estimates of the variances σ^2_{in} (>0) of the $\hat{\theta}_{in}$ obtained from a sample of size n.

General Multiple Test Procedure: Estimate the index sets I_0 and I_1 by $\hat{I}_0$ and $\hat{I}_1$, respectively, with $\hat{I}_0\cap\hat{I}_1=0$ and $\hat{I}_0\cup\hat{I}_1=\{1,\dots,q\}$, so that

$$\hat{I}_0 = \{j\in\{1,\dots,q\}: |\hat{\theta}_{jn}|(\hat{\sigma}^2_{jn})^{-1/2}\leq c_j(n)\} ,$$
$$\hat{I}_1 = \{1,\dots,q\}-\hat{I}_0,$$

where $c_j(n)$, $j=1,\dots,q$, are (increasing) functions with $c_j(n)\to\infty$.

Theorem 2: Assume

(a) $E[(\hat{\theta}_{in} - \theta_i)^2]\ \sigma_{in}^{-2}$ is bounded

(b) $\dfrac{\hat{\sigma}_{in}}{\sigma_{in}} \overset{p}{\to} 1$

(c) $\sigma_{in} \cdot c_i(n) \to 0$.

Then $P(\hat{I}_0 \neq I_0) \to 0$.

Before proving this result it should be noted that from $c_i(n) \to \infty$ and condition (c) it follows

$$\sigma_{in} = \sqrt{E[(\hat{\theta}_{in} - E(\hat{\theta}_{in}))^2]} \to 0$$

This fact together with (a) implies convergence in the quadratic mean for the $\hat{\theta}_{in}$, $i=1,\ldots,q$:

$$E\,[(\hat{\theta}_{in} - \theta_i)^2] \to 0$$

Proof: In part A it is shown that the probability for $\hat{I}_0$ not including all the indices $r+1,\ldots,q$ of the zero parameters tends to zero. In part B it will be proved that the probability for $\hat{I}_0$ to contain at least one of the indices $1,\ldots,r$ tends to zero, too.

(A) Let, for $i=r+1,\ldots,q$, $c_{i0}(n)$ be a function, so that $c_{i0}(n) \to \infty$ and $c_{i0}(n)(c_i(n))^{-1} \to 0$. Given any $c_i(n)$ with $c_i(n) \to \infty$ such a $c_{i0}(n)$ can always be found.

For any particular $\hat{\theta}_{in}$, $i=r+1,\ldots,q$, Chebychev's inequality leads to

$$P\{|\hat{\theta}_{in}|\sigma_{in}^{-1} \geq c_{io}(n)\} \leq \frac{E[\hat{\theta}_{in}^2]\sigma_{in}^{-2}}{c_{io}^2(n)} \leq \frac{M}{c_{io}^2(n)} ,$$

due to assumption (a) such a finite $M>0$ can always be found. Clearly, this probability tends to zero because of $c_{i0}(n) \to \infty$. From the above inequality follows that

$$|\hat{\theta}_{in}|(\sigma_{in}c_i(n))^{-1} \overset{p}{\to} 0 ,$$

if $c_{i0}(n)(c_i(n))^{-1} \to 0$, as has been assumed. Hence also

$$|\hat{\theta}_{in}|(\sigma_{in}c_i(n))^{-1}\sigma_{in}(\hat{\sigma}_{in})^{-1} \overset{p}{\to} 0$$

since $\hat{\sigma}_{in}\sigma_{in}^{-1} \xrightarrow{P} 1$ as stated in (b). That means that for any $\varepsilon > 0$

$$P\{|\hat{\theta}_{in}|(\hat{\sigma}_{in}c_i(n))^{-1} > \varepsilon\} \to 0$$

and

$$P\{|\hat{\theta}_{in}|\hat{\sigma}_{in}^{-1} > \varepsilon c_i(n)\} \to 0$$

this statement also being valid for $\varepsilon = 1$. This completes the first part of the proof.

(B) Let $0 < \delta < 1$ be a fixed number. Then for $i = 1, \ldots, r$

$$P\{|\hat{\theta}_{in}|\hat{\sigma}_{in}^{-1} \le c_i(n)\} = P\{|\hat{\theta}_{in}|\sigma_{in}^{-1} \frac{\sigma_{in}}{\hat{\sigma}_{in}} \le c_k(n)\} \le$$

$$\le P\{|\hat{\theta}_{in}|\sigma_{in}^{-1} \frac{\sigma_{in}}{\hat{\sigma}_{in}} \le c_i(n),\ |\frac{\sigma_{in}}{\hat{\sigma}_{in}} - 1| < \delta\} + P\{|\frac{\sigma_{in}}{\hat{\sigma}_{in}} - 1| \ge \delta\} \le$$

$$\le P\{|\hat{\theta}_{in}|\sigma_{in}^{-1}(1-\delta) \le c_i(n)\} + P\{|\frac{\sigma_{in}}{\hat{\sigma}_{in}} - 1| \ge \delta\} .$$

The second summand tends to zero because of (b). The first summand can be transformed as follows:

$$P\{|\hat{\theta}_{in}|\sigma_{in}^{-1} \le c_i(n)(1-\delta)^{-1}\} =$$

$$= P\{(|\theta_i| - |\hat{\theta}_{in}|)\sigma_{in}^{-1} \ge |\theta_i|\sigma_{in}^{-1} - c_i(n)(1-\delta)^{-1}\} \le$$

$$\le P\{|\theta_i - \hat{\theta}_{in}|\sigma_{in}^{-1} \ge |\theta_i|\sigma_{in}^{-1} - c_i(n)(1-\delta)^{-1}\} \le$$

$$\le E[(\hat{\theta}_{in} - \theta_i)^2]\sigma_{in}^{-2} \cdot (|\theta_i|\sigma_{in}^{-1} - c_i(n)(1-\delta)^{-1})^{-2} =$$

$$= E[(\hat{\theta}_{in} - \theta_i)^2](|\theta_i| - \sigma_{in}c_i(n)(1-\delta)^{-1})^{-2} .$$

The application of Chebychev's inequality depends on

$|\theta_i|\sigma_{in}^{-1} - c_i(n)(1-\delta)^{-1}$

probability of fitting a polynomial of higher order than the true one is bounded by α, independently of n. To get weak consistency of the multiple test procedure for determining the correct order, the critical limits must in a particular way depend on the sample size. Testing in a linear model setting, this means that for increasing numbers of observations the significance levels for the individual tests of regression parameters should decrease: The corresponding critical limits must tend to infinity slower than the inverse of the standard deviation of the respective parameter estimators.

The result in Theorem 2 is not confined to the special structure of nested hypotheses. It is generally applicable to subset selection in statistical models with a finite number of parameters.

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being positiv. This will always be the case for sufficiently large values of n due to condition (c) and $|\theta_i| > 0$. The convergence in the quadratic mean of $\hat{\theta}_{in}$ assures that also the first summand tends to zero. This completes the proof. □

Remark: The assumption (a) is, together with (c), more stringent than the assumption of convergence in the quadratic mean for the $\hat{\theta}_{in}$. Assumption (c) and convergence in probability of $\hat{\theta}_{in}$ to θ_i would induce convergence in the quadratic mean but not suffice for the proof of the first part of the Theorem. If the estimators for the parameters are unbiased as in the 'linear model' situation, condition (a) is trivially fulfilled. If only asymptotic unbiasedness is assured some restriction on the bias is needed: The quadratic bias must tend to zero at least as fast as the variance.

In the linear model situation, i.e., $Y = X\theta+\varepsilon$ with $\varepsilon \sim N(0,\sigma^2 I_n)$, the general test procedure simply consists of simultaneous t-tests for the individual parameters. If it is assumed that $(X'X)n^{-1} \to Q$, Q being a positive definite matrix, conditions (a) and (b) hold when the usual variance estimate based on the complete model is applied. Condition (c) then requires that $c_i(n)n^{-1/2} \to 0$, e.g., $c_i(n)=c_i n^{\gamma_i}$ with $0<\gamma_i<1/2$ and $0<c_i<\infty$.

In the multiple test situation of Section 2 the smallest individual significance level to be used is $\alpha_{(m+1)}=\alpha/(q-m)$. If the simultaneous significance level α depending on n is chosen according to $(q-m).(2\pi)^{-1/2}\exp(-c^2 n^{2\gamma}/2).(cn^{\gamma})^{-1}$ with $0<c<\infty$ and $0<\gamma<1/2$, then the smallest $\alpha_{(i)}$, and hence all others, will fulfill the requirement of Theorem 2.

4. Concluding Remarks

Corresponding to the fixed simultaneous significance level α for the multiple test procedure discussed in Section 2, the

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