Working Paper

ON THE SOLUTION SETS FOR UNCERTAIN SYSTEMS WITH PHASE CONSTRAINTS

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February 1986
WP-86-11

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PREFACE

One of the means of modelling a system with an uncertainty in the parameters or in the inputs is to consider a multistage inclusion or a differential inclusion. These types of models may serve to describe an uncertainty for which the only available data is a set-membership description of the admissible constraints on the unknown parameters.

A problem under discussion here deals with the specification of the "tube" of all solutions to a nonlinear multistage inclusion that arise from a given set and also satisfy an additional phase constraint. The description of this "solution tube" is important for solving problems of guaranteed estimation of the dynamics of uncertain systems as well as for the solution of other "viability" problems for systems described by equations involving multivalued maps.

On the Solution Sets for Uncertain Systems with Phase Constraints

A. B. Kurzhanskii

INTRODUCTION

This paper deals with multistage inclusions that describe a system with uncertainty in the model or in the inputs [1,2]. In particular this may be a difference scheme for a differential inclusion [3]. The solution to these inclusions is a multivalued function whose cross-section at a specific instant of time is the "admissible domain" for the inclusion.

The problem considered here is to specify a subset of solutions that consists of those "trajectories" which satisfy an additional phase constraint. These solutions are said to be "viable" with respect to the phase constraint [3]. The cross section of the set of all viable solutions is the attainability domain under the state constraint. The derivation of evolution equations for the latter domain is the objective of this paper.

The problem posed here is purely deterministic. However, the techniques applied to its solution involve some stochastic schemes. These schemes follow an analogy between some formulae of convex analysis [4,5] and those for calculating conditional mean values for specific types of stochastic systems [6,7] which was pointed out in [8,9].

A special application of the results of this paper could be the derivation of solving relations for nonlinear filtering under set-membership constraints on the "noise" and the description of the analogies between the theories of "guaranteed" and stochastic filtering.

1. Discrete-time Uncertain Systems

Consider a multistage process described by an n-dimensional recurrent inclusion

$$x(k+1) \in F(k, x(k)), \qquad k \geq k_0 \geq 0 \tag{1.1}$$

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$, $F(k, x(k))$ is a given multivalued map from $\mathbb{N} \times \mathbb{R}^n$ into $\operatorname{comp} \mathbb{R}^n$ ($\mathbb{N}$ is the set of natural numbers, $\operatorname{comp} \mathbb{R}^n$ is the set of all compact subsets of $\mathbb{R}^n$).

Suppose the initial state $x(k_0) = x^0$ of the system is confined to a preassigned set:

$$x^0 \in X^0 \ , \tag{1.2}$$

where X^0 is given in advance. A trajectory solution of system (1.1) that starts from point x^0 at instant k_0 will be denoted as $x(k \mid k_0, x^0)$. The set of all solutions for (1.1) that start from x^0 at instant k^0 will be denoted as $X(k \mid k_0, x^0)$ $(k \in \mathbb{N}\ ,\ k \geq k^0)$ with further notation

$$X(k \mid k^0, X^0) = \cup \{ X(k \mid k_0, x^0) \mid x^0 \in X^0 \} \ , \quad (k \in \mathbb{N}\ , k \geq k^0)$$

Let $Q(k)$ be a multivalued map from $\mathbb{N}$ into comp$\mathbb{R}^m$ and $G(k)$ be a single-valued map from $\mathbb{N}$ to the set of $m \times n$-matrices. The pair $G(k)$, $Q(k)$, introduces a state constraint

$$G(k)x(k) \in Q(k) \ , \quad k \geq k_0 + 1 \tag{1.3}$$

on the solutions of system (1.1).

The subset of $\mathbb{R}^n$ that consists of all the points of $\mathbb{R}^n$ through which at stage $s \in [k_0, \tau] = \{ k : k_0 \leq k \leq \tau \}$ there passes at least one of the trajectories $x(k \mid k_0, x^0)$, that satisfy constraint (1.3) for $k \in [k_0, \tau]$ will be denoted as $X(s \mid \tau, k_0, x^0)$.

The aim of this paper is first to study the sets $X(\tau \mid \tau, k_0, X^0) = X(\tau, k_0, X^0)$ and their evolution in "time" τ.

In other words, if a trajectory $x(k \mid k_0, x^0)$ of equation (1.1) that satisfies the constraint (1.3) for all $k \in [k_0, s]$ is named "viable until instant τ" ("relative to constraint (1.3)"), then our objective will be to describe the evolution of the set of all viable trajectories of (1.1). Here at each instant $k > k^0$ the constraint (1.3) may "cut off" a part of $X(k \mid k^0, x^0)$ reducing it thus to the set $X(k, k^0, x^0)$.

The sets $X(k, k^0, x^0)$ may also be interpreted as "attainability domains" for system (1.1) under the state space constraint (1.3). The objective is to describe evolution of these domains.

A further objective will be to describe the sets $X(s \mid \tau, k_0, x^0)$ and their evolution.

2. The Attainability Domains

From the definition of sets $X(s \mid \tau, k^0, x^0)$ it follows that the following properties are true.

Lemma 2.1. *Whatever are the instants* t, s, k, $(t \geq s \geq k \geq 0)$ *and the set* $\mathbb{F} \in \operatorname{comp} \mathbb{R}^n$, *the following relation is true*

$$X(t, k, \mathbb{F}) = X(t, s, X(s, k, \mathbb{F})). \tag{2.1}$$

Here $X(t, k, \mathbb{F}) = \bigcup \{ X(t, k, x) \mid x \in \mathbb{F} \}$.

Lemma 2.2. *Whatever are the instants* s, t, τ, k, l $(t \geq s \geq l;\ \tau \geq l \geq k;\ t \geq \tau)$ *and the set* $\mathbb{F} \in \operatorname{comp} \mathbb{R}^n$ *the following relation is true*

$$X(s \mid t, k, \mathbb{F}) = X(s \mid t, l, X(l \mid \tau, k, \mathbb{F})). \tag{2.2}$$

Relation (2.1) shows that sets $X(k, \tau, X)$ satisfy a *semigroup property* which allows to define a *generalized dynamic system* in the space $2^{\mathbb{R}^n}$ of all subsets of $\mathbb{R}^n$.

In general the sets $X(s \mid t, k, \mathbb{F})$ need not be either convex or connected. However, it is obvious that the following is true

Lemma 2.3. *Assume that the map* F *is linear in* x*:*

$$F^{(k)} x = A^{(k)} x + P$$

where $P \in \operatorname{conv} \mathbb{R}^n$. *Then for any set* $\mathbb{F} \in \operatorname{conv} \mathbb{R}^n$ *each of the sets* $X(s \mid t, k, \mathbb{F}) \in \operatorname{conv} \mathbb{R}^n$ $(t \geq s \geq k \geq 0)$.

Here $\operatorname{conv} \mathbb{R}^n$ stands for the set of all convex compact subsets of $\mathbb{R}^n$.

3. The One-Stage Problem

Consider the system

$$z \in F(x), \quad Gz \in Q, \quad x \in X,$$

where $z \in \mathbb{R}^n$, $X \in \operatorname{comp} \mathbb{R}^n$, $Q \in \operatorname{conv} \mathbb{R}^m$, $F(\kappa)$ is a multivalued map from $\mathbb{R}^n$ into $\operatorname{conv} \mathbb{R}^n$, G is a linear (single-valued) map from $\mathbb{R}^n$ into $\mathbb{R}^m$.

It is obvious that the sets $F(X) = \{ \bigcup F(x) \mid x \in X \}$ need not be convex.

Let Z, Z^* respectively denote the sets of al solutions for the following systems:

(a) $z \in F(X)$, $Gz \in Q$,

(b) $z^* \in \mathrm{co} F(X)$, $Gz^* \in Q$,

where co$F$ stands for the closed convex hull of $F(X)$.

The following statement is true

Lemma 3.1. *The sets Z, co$Z$, Z^* satisfy the following inclusions*

$$Z \subseteq \mathrm{co}\, Z \subseteq Z^* \tag{3.1}$$

Let $\rho(l \mid Z) = \sup \{ l'z \mid z \in Z \}$ denote the support function [4] of set Z. Also denote

$$\Phi(l, p, q) = (l - G'p, q) + \rho(-p \mid Q)$$

Then the function $\Phi(l, p, q)$ may be used to describe the sets co$Z$,Z^*.

Lemma 3.2. *The following relations are true*

$$\rho(l \mid Z) = \rho(l \mid \mathrm{co}\, Z) = \sup_q \inf_p \Phi(l, p, q) \quad , \quad q \in F(X),\ p \in \mathbb{R}^m \tag{3.2}$$

$$\rho(l \mid Z^*) = \inf_p \sup_q \Phi(l, p, q) \quad , \quad q \in F(X),\ p \in \mathbb{R}^m \tag{3.3}$$

It is not difficult to give an example of a nonlinear map $F(x)$ for which Z is nonconvex and the functions $\rho(l \mid \mathrm{co}\, Z)$, $\rho(l \mid Z^*)$ do not coincide, so that the inclusions $Z \subset \mathrm{co}\, Z$, co$Z \subset Z^*$ are strict.

Indeed, assume $X = \{0\}$, $x \in \mathbf{R}^2$

$$F(0) = \{x : 6x_1 + x_2 \le 3 \ , \ x_1 + 6x_2 \le 3 \ , \ x_1 \ge 0 \ , \ x_2 \ge 0\}$$

$$G = (0 \ , \ 1) \ , \ Q = (0 \ , \ 2) \ .$$

Then

$$Y = \{x : Gx \in Q\} = \{x : 0 \le x_2 \le 2\}$$

The set $F(0)$ is a nonconvex polyhedron 0 K D L in Figure 1 while set Y is a stripe. Here, obviously, set Z which is the intersection of $F(0)$ and Y, turns to be a nonconvex polyhedron 0 A B D L, while sets co Z , Z^* are convex polyhedrons 0 A B L and 0 A C L respectively (see Figures 2, 3). The corresponding points have the coordinates

A = (0 , 2), B = (1/2 , 2), C = (1 , 2), D = (3/7 , 3/7), K = (0 , 3), L = (3 , 0),
0 = (0 , 0).

Clearly $Z \subset \text{co}\ Z \subset Z^*$.

This example may also serve to illustrate the existence of a gap between (3.2) and (3.3).

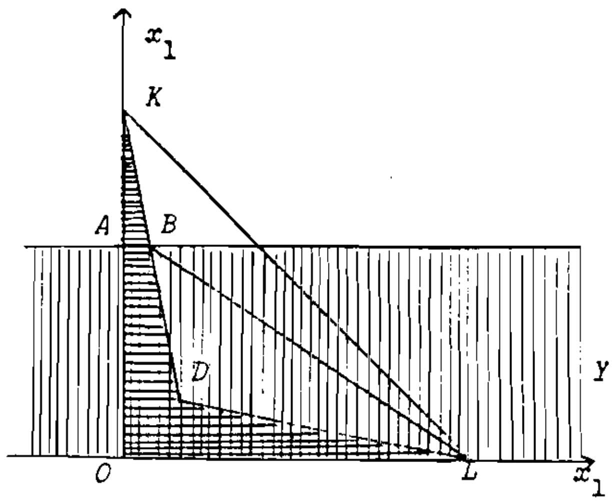

Figure 1

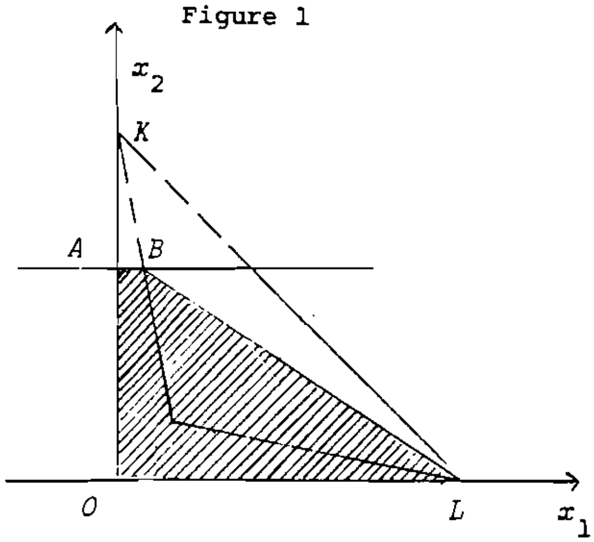

Figure 2

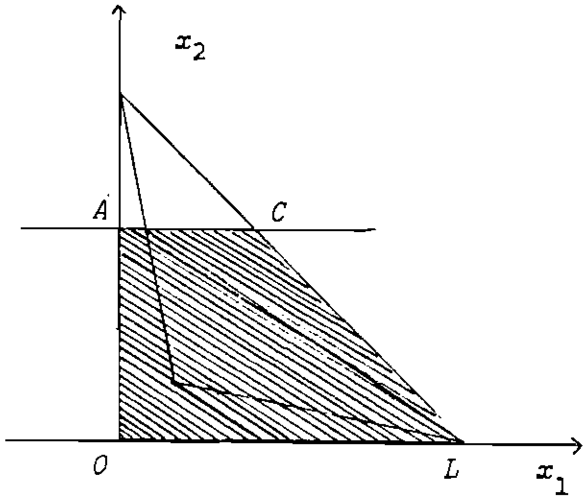

Figure 3

For a linear-convex map $F(x) = Ax + P$ ($P \in \text{conv}\,\mathbb{R}^n$) there is no distinction between Z, $\text{co}\,Z$, and Z^*:

Lemma 3.3 *Suppose $F(x) = Ax + P$ where $P \in \text{conv}\,\mathbb{R}^n$, A is a linear map from $\mathbb{R}^n$ in to $\mathbb{R}^n$. Then $Z = \text{co}\,Z = Z^*$.*

4. The One Stage Problem - An Alternative Approach.

The description of Z, $\text{co}\,Z$, Z^* may be given in an alternative form which, however, allows to present all of these sets as the intersections of some varieties of convex multivalued maps.

Indeed, whatever are the vectors l, p ($l \neq 0$) it is possible to present $p = Ml$ where M belongs to the space $\mathbb{M}^{m \times n}$ of real matrices of dimension $m \times n$. Then, obviously,

$$\rho(l \mid Z) = \sup_q \inf_M \Phi(l, Ml, q) = \rho(l \mid \text{co}\,Z) \quad , \quad q \in \boldsymbol{F}(X),\ M \in \mathbb{M}^{m \times n},$$

$$\rho(l \mid Z^*) = \inf_M \sup_q \Phi(l, Ml, q) \qquad q \in \boldsymbol{F}(X),\ M \in \mathbb{M}^{m \times n} \tag{4.1}$$

or

$$\rho(l \mid Z^*) = \inf \{ \Phi(l, Ml) \mid M \in \mathbb{M}^{m \times n} \}, \tag{4.2}$$

where

$$\Phi(l, Ml) = \{ \Phi(l, Ml, q) \mid q \in \text{co}\,\boldsymbol{F}(x) \} =$$

$$= \rho((E - G'M)l,\ \text{co}\,\boldsymbol{F}(X)) + \rho(-Ml \mid Q) \quad .$$

From (4.1) it follows that

$$Z \subseteq \bigcup_{q \in \boldsymbol{F}(X)} \bigcap_M R(M, q) \subseteq \bigcap_M \bigcup_{q \in \boldsymbol{F}(X)} R(M, q),\ M \in \mathbb{M}^{m \times n} \tag{4.3}$$

where

$$R(M, q) = (E_n - MG)q - MQ .$$

Similarly (4.2) yields

$$Z^* \subseteq \bigcap_M \bigcup_{q \in \text{co}\,\boldsymbol{F}(X)} \{ (E_n - MG)q - MQ \}. \tag{4.4}$$

Moreover a stronger assertion holds.

Theorem 4.1. *The following relations are true*

$$Z = \bigcup_{q \in F(X)} \bigcap_{M} R(M, q) \tag{4.5}$$

$$Z^{*} = \bigcap_{M} R(M, \mathrm{co} F(X)) \tag{4.6}$$

where $M \in \mathbb{M}^{m \times n}$.

Obviously for $F(x) = AX + P, (X, P \in \mathrm{co}\,\mathbb{R}^{n})$ we have $F(X) = \mathrm{co} F(X)$ and $Z = Z^{*} = \mathrm{co} Z$.

This *first* scheme of relations may serve to be a basis for constructing recurrent procedures. Another recurrent procedure could be derived from the following *second* scheme. Consider the system

$$z \in F(x) \tag{4.7}$$

$$Gx \in Q, \tag{4.8}$$

for which we are to determine the set Z of all vectors z consistent with inclusions (4.7), (4.8). Namely, we are to determine the restriction $F_Y(x)$ of $F(x)$ to set Y. Here we have

$$F_Y(x) = \begin{cases} F(x) \text{if} x \in Y \\ \phi \quad \text{if} x \in Y \end{cases}$$

where as before $Y = \{ x : Gx \in Q \}$.

Lemma 4.1 *Assume* $F(x) \in \mathrm{comp}\,\mathbb{R}^{n}$ *for any* x *and* $Q \in \mathrm{conv}\,\mathbb{R}^{m}$. *Then*

$$F_Y(x) = \bigcap_{L} (F(x) - LGx + LQ)$$

over all $n \times m$ *matrices* L, $(L \in \mathbb{M}^{n \times m})$.

Denote the null vectors and matrices as $\{0\}_m \in \mathbb{R}^m$, $\{0\}_{m,n} \in \mathbb{R}^{m \times n}$, the $(n \times n)$ unit matrix as E_n and the $(n \times m)$ matrix L_{mn} as

$$L_{mn} = \begin{vmatrix} E_m \\ \{0\}_{mn} \end{vmatrix}$$

Suppose $x \in Y$. Then $\{0\}_m \in Q - Gx$ and for any $(n \times m)$ -matrix L we have $\{0\}_n \in L\ (Q - Gx)$. Then it follows that for $x \in Y$.

$$F(x) \subseteq \bigcap_L (F(x) + L(Q - Gx)) \subseteq F(x)$$

On the other hand, suppose $x \,\overline{\in}\, Y$.

Let us demonstrate that in this case

$$\bigcap_L \{F(x) + L(Q - Gx)\} = \phi .$$

Denote $A = F(x)$, $B = Q - Gx$. For any $\lambda > 0$ we then have

$$\bigcap_L (A + LB) \subseteq (A + \lambda L_m B) \cap (A - \lambda L_m B)$$

Since $\{0\}_m \,\overline{\in}\, B$ we have $\{0\}_n \,\overline{\in}\, L_m B$. Therefore there exists a vector $l \in \mathbb{R}^n$, $l \neq 0$ and a number $\gamma > 0$ such that

$$(l, x) \geq \gamma > 0 \quad \text{for any} \quad x \in L_m B ,$$

Denote

$$L = \{x : (l, x) \geq \gamma\} .$$

Then $L \supseteq L_m B$ and

$$(A + \lambda L_m B) \cap (A - \lambda L_m B) \subseteq (A + \lambda L) \cap (A - \lambda L)$$

Set A being bounded there exists a $\lambda > 0$ such that

$$(A + \lambda L) \cap (A - \lambda L) = \phi .$$

Hence

$$\bigcap_L (A + LB) = \phi$$

and the Lemma is proved.

5. Statistical Uncertainty. The Elementary Problem.

Consider the system

$$z = q + \xi , \quad Gz = v + \eta , \tag{5.1}$$

where

$$q \in F(x) , \ v \in Q , \ x \in X$$

and ξ, η are independent gaussian random vectors with zero means ($E\xi = 0, E\eta = 0$) and with variances $E\xi\xi' = R$, $E\eta\eta' = N$, where $R > 0$, $N > 0$ ($R \in \mathbb{M}_n, N \in \mathbb{M}_m$).

Assuming at first that the pair $h = \{q, v\}$ is fixed, let us find the conditional mean $E(z \mid y = 0, h = h^*)$ under the condition that after one realization of the values ξ, η the relations

$$z = q + \xi, \quad y = -Gz + v + \eta = 0$$

are satisfied. After a standard calculation we have

$$\bar{z}_{y,h} = E(z \mid y = 0, \; h = h^*) = q + PG'N^{-1}(-Gq - Gv) + v \; ,$$

where $P^{-1} = R^{-1} + G'N^{-1}G$.

After applying a well-known matrix transformation [6]

$$P = R - RG'K^{-1}GR, \; K = N + GRG' \, ,$$

we have

$$E(z \mid y = 0, \; h = h^*) = (E - RG'K^{-1}G)q - RG'K^{-1}v \; .$$

The matrix of conditional variances is

$$E((z - \bar{z}_{y,h})'(z - \bar{z}_{y,h})) = P_y \; .$$

It does not depend upon h and is determined only by q, v and the element $\Lambda = RG'K^{-1}G$. Therefore it makes sense to consider the sets

$$W(\Lambda, q) = \bigcup \{ z_{y,h} \mid v \in Q \}$$

$$W(\Lambda) = \bigcup \{ z_{y,h} \mid q \in \boldsymbol{F}(X), \; v \in Q \}$$

and

$$W_*(\Lambda) = \bigcup \{ z_{y,h} \mid q \in \mathrm{co}\boldsymbol{F}(X), \; v \in Q \}$$

of conditional mean values. Each of the elements of these sets has one and the same variance P_y. The sets $W_*(\Lambda)$ and $W(\Lambda, q)$ are obviously convex while $W(\Lambda)$ may not be convex.

Lemma 5.1 *The following inclusions are true* ($Z \subset Z^*$)

$$Z \subset W(\Lambda), \; Z^* \subset W^*(\Lambda), \; W(\Lambda) \subset W^*(\Lambda) \; . \tag{5.2}$$

It can be seen that $W(\Lambda,q)$ has exactly the same structure as $R(M,q)$ of (4.3) (with only Λ substituted by M). Hence for the same reason as before we have

$$Z \subseteq \bigcup_{q \in F(X)} \bigcap_{D \in D} W(\Lambda,q) = \bigcap_{D \in D} W(\Lambda) \tag{5.3}$$

$$Z^* \subseteq \bigcup_{q \in \mathrm{co} F(X)} \bigcap_{D \in D} W(\Lambda,q) = \bigcap_{D \in D} W_*(\Lambda) \tag{5.4}$$

where the intersections are taken over the class $\boldsymbol{D}$ of all possible pairs $D = \{R,N\}$ of nonnegative matrices R,N of respective dimensions. However, a property similar to that of Lemma 4.1 happens to be true. Namely if by $\boldsymbol{D}(\alpha,\beta)$ we denote the class of pairs $\{R,N\}$ where $R = \alpha E_n$, $N = \beta E_m$, $\alpha > 0$, $\beta > 0$, then the element X will depend only upon two parameters α,β.

Theorem 5.1 *Suppose matrix G is of full rank m. Then the following equalities are true*

$$Z = \cap \{ W(\Lambda) | D \in \boldsymbol{D}(1,\beta), \beta > 0 \} \subseteq \mathrm{co} Z \subseteq \tag{5.5}$$

$$\subseteq \cap \{ W_*(\Lambda) | D \in \boldsymbol{D}(1,\beta), \beta > 0 \} = Z_* .$$

Here it suffices to take the intersections only over a one-parametric variety $D \in \boldsymbol{D}(1,\beta)$.

There are some specific differences between this scheme and the one of §4. These could be traced more explicitly when we pass to the calculation of support functions $\rho(l | Z)$, $\rho(l | Z^*)$ for Z, Z^*.

Lemma 5.2 *The following inequality is true*

$$\rho(l | Z^*) = f^{**}(l) \leq f(l) = \inf \{ \Phi(l,\Lambda) | D \in \boldsymbol{D}(1,\beta), \beta > 0 \} \tag{5.6}$$

where $f^{**}(l)$ is the second conjugate to $f(l)$ in the sense of Fenchel [4].

Moreover if we substitute $\boldsymbol{D}(1,\beta)$ in (5.6) for a broader class D then an exact equality will be attained, i.e.

$$\rho(l | Z^*) = f^{**}(l) = \inf \{ \Phi(l,\Lambda) | D \in \boldsymbol{D} \} \tag{5.7}$$

More precisely, we come to

Theorem 5.2 *Suppose matrix G is of full rank m. Then equality (5.7) will be true together with the following relation*

$$\rho(l | Z) = \rho(l | \mathrm{co} Z) = \sup \inf \{ \Phi(l,\Lambda,q) | d \in D | q \in F(X) \} \tag{5.8}$$

Problems (5.7), (5.8) are "stochastically dual" to (3.3, (3.2).

The results of the above may now be applied to our basic problem for multistage systems.

6. Solution to the Basic Problem

Returning to system (1.1)–(1.3) we will seek for the sequence of sets $X[s] = X(s, k_0, X^0)$ together with two other sequences of sets. These are

$$X^*[s] = X^*(s, k_0, X^0)$$

- the solution set of the system

$$x_{k+1} \in \mathrm{co}\, F(k, X^*[k]), \quad X^*[k_0] = X^0 \tag{6.1}$$

$$G(k+1) \in Q(k+1), \quad k \geq k_0 \tag{6.2}$$

and $X_*[s] = X_*(s, k_0, X^0)$ which is obtained due to the following relations:

$$X_*[s] = \mathrm{co}\, Z[s]$$

where $Z[k+1]$ is the solution set for the system

$$z(k+1) \in F(k, \mathrm{co}\, Z[k]), \quad Z[k^0] = X^0,$$

$$G(k+1)z(k+1) \in Q(k+1), \quad k \geq k_0 .$$

The sets $X_*[\tau]$, $X^*[\tau]$ are obviously convex. They satisfy the inclusions

$$X[\tau] \subseteq X_*[\tau] \subseteq X^*[\tau]$$

where each of the sets $X[\tau]$, $X_*[\tau]$, $X^*[\tau]$ lies within

$$Y(\tau) = \{ x : G(\tau)x \in Q(\tau) \}, \quad \tau \geq k_0 + 1 ,$$

The set $X^*[\tau]$ may therefore be obtained for example by either solving a sequence of problems (6.1), (6.2) (for every $k \in [k_0, s-1]$ with $X^*[k_0] = X^0$) (the first scheme of §4) or by finding all the solutions $\bar{x}[k] = \bar{x}(k, k_0, x^0)$ of the equation

$$x(k+1) \in (\mathrm{co}\, F)_{Y(k)}(k, x(k)), \quad x(k_0) \in X^0, \tag{6.3}$$

that could be prolongated until the instant $\tau + 1$ and finding the relation of this set to $X[\tau]$, $X_*[\tau]$, and $X^*[\tau]$.

Following the first scheme of §4 we may therefore consider the recurrent system

$$z(k+1) = (I_n - M(k+1)G(k+1))F^0(k, S(k)) + M(k+1)Q(k+1) \qquad (6.4)$$

$$S(k) = \{ \cap Z(k) | M(k) \}, \ k > k_0, \ S(k_0) = X^0, \qquad (6.5)$$

where $M(k+1) \in \mathbb{R}^{m \times n}$.

From Theorem 4.1 we may now deduce the result

Theorem 6.1 *The solving relations for $X[s]$, $X_*[s]$, $X^*[s]$ are as follows*

$$X[s] = S(s) \quad \text{for} \quad \boldsymbol{F}^0(k, S(k)) = \boldsymbol{F}(k, S(k)) \qquad (6.6)$$

$$X^*[s] = S(s) \quad \text{for} \quad \boldsymbol{F}^0(k, S(k)) = \text{co}\boldsymbol{F}(k, S(k)) \qquad (6.7)$$

$$X_*[s] = \text{co}S(s) \quad \text{for} \quad \boldsymbol{F}^0(k, S(k) = \boldsymbol{F}(k, \text{co}S(k)) . \qquad (6.8)$$

It is obvious that $X[\tau]$ is the exact solution while $X_*[\tau]$, $X^*[\tau]$ are convex majorants for $X[\tau]$. Clearly by interchanging and combining relations (6.7), (6.8) from stage to stage it is possible to construct a variety of other convex majorants for $X[\tau]$. However for the linear case they all coincide with $X[\tau]$.

Lemma 6.1 *Assume $\boldsymbol{F}^0(k, S(k)) = A(k)S(k) + P(k)$ with $\boldsymbol{P}(k)$, X^0 being closed and compact. Then $X[k] = X^*[k] = X_*[k]$ for any $k \geq k_0$.*

Consider the system

$$Z(k+1) = (I_n - M(k+1)G(k+1))\boldsymbol{F}^0(k, Z(k)) - M(k+1)Q(k+1), Z(k_0) = X^0, \qquad (6.9)$$

denoting its solution as

$$\begin{aligned} &Z(k; M_k(\cdot)) \quad \text{for} \quad F^0(k, Z) = \boldsymbol{F}(k, Z) \\ &Z_*(k, M_k(\cdot)) \quad \text{for} \quad F^0(k, Z) = \boldsymbol{F}(k, \text{co}Z) \\ &Z^*(k, M_k(\cdot)) \quad \text{for} \quad \boldsymbol{F}^0(k, Z) = \text{co}\boldsymbol{F}(k, Z) \end{aligned}$$

Then the previous suggestions yield the following conclusion

Theorem 6.2 *Whatever is the sequence $M_s(\cdot)$, the following solving inclusions are true*

$$X[s] \subseteq Z(s, M_s(\cdot)) \qquad (6.10)$$

$$X_*[s] \subseteq Z_*(s, M_s(\cdot))$$

$$X^*[s] \subseteq Z^*(s, M_s(\cdot)), \quad s > k_0,$$

with $Z(s, M_s(\cdot)) \subseteq Z_*(s, M_s(\cdot)) \subseteq Z^*(s, M_s(\cdot))$.

Hence we also have

$$X[s] \subseteq \cap \{ Z(s, M_s(\cdot)) | M_s(\cdot)) \} \quad (6.11)$$

$$X_*[s] \subseteq \cap \{ Z_*(s, M_s(\cdot)) | M_s(\cdot)) \} \quad (6.12)$$

$$X^*[s] \subseteq \cap \{ Z^*(s, M_s(\cdot)) | M_s(\cdot)) \} \quad (6.13)$$

over all $M_s(s)$.

However a question arises which is whether (6.11)−(6.13) could turn into exact equalities.

Lemma 6.2 *Assume the system (1.1), to be linear: $\mathbf{F}(k, x) = A(k)x + P(k)$ with sets $P(k)$, $Q(k)$ convex and compact. Then the inclusions (6.11)−(6.13) turn into the equality*

$$\mathbf{X}[s] = \mathbf{X}^*[s] = \cap \{ Z_s(\cdot, M_s(\cdot)) \} = \cap \{ Z_s^*(\cdot, M_s(\cdot)) \} \quad (6.14)$$

Hence in this case the intersections over $M(k)$ could be taken either in each stage as in Theorem 6.1 (see (6.6), (6.7)) or at the final stage as in (6.14).

Let us now follow the second scheme of §4, considering the equation

$$x(k+1) \in \tilde{\mathbf{F}}_{Y(k)}(k, x(k)), \quad x^0 = x(k_0), \quad x^0 \in X^0, \quad (6.15)$$

and denoting the set of its solutions that starts at $x^0 \in X^0$ as $X^0(k, k_0, x^0)$ with

$$\cup \{ x^0(k, k_0, x^0) | x^0 \in X^0 \} = X^0(k, k_0, X^0) = X^0[k].$$

According to Lemma 4.1 we substitute (6.15) by the equation

$$x(k+1) \in \bigcap_L (\tilde{F}^*(k, x(k)) - LG(k)x(k) + LY(k)) \quad , \quad x^0 \in X^0,$$

and the calculation of $X^0[k]$ should thence follow the procedure

$$\tilde{X}[k+1] = \bigcup_{x \in \tilde{X}(k)} \bigcap_L (\tilde{F}(k, x) - LG(k)x + LQ(k)), \quad X(k_0) = X^0. \quad (6.16)$$

Denote the whole solution "tube" for $k_0 \le k \le s$ as $\tilde{X}^s_{k_0}[\cdot]$. Then the following assertion will be true.

Theorem 6.3 *Assume $\tilde{X}^s_{k_0}[k]$ to be the cross-section of the tube $\tilde{X}^s_{k_0}[\cdot]$ at instant k. Then*

$$X[s] = \tilde{X}^{s+1}_{k_0}[s] \text{ if } \tilde{F}(k, x) = \mathbf{F}(k, x)$$

$$X^* = \tilde{X}_{k_0}^{s+1}[s] \text{ if } \tilde{F}(k,x) = \mathrm{co}F(k,x)$$

Here $\tilde{X}_{k_0}^{s}[s] \supset \tilde{X}_{k_0}^{s+1}[s]$ and the set $\tilde{X}_{k_0}^{s}[s]$ may not lie totally within $Y(s)$.

The solution of equation (6.16) is equivalent to finding all the solutions for the inclusion

$$x(k+1) \in \bigcap_L (\tilde{F}(k,x) - LG(k)x + LQ(k)), \quad x(k_0) \in X^0 \tag{6.17}$$

Equation (6.17) may be substituted by a system of "simpler" inclusions

$$x(k+1) \in \tilde{F}(k,x(k)) - L(k)G(k)x(k) + L(k)Q(k), \quad x(k_0) \in X^0 \tag{6.18}$$

for each of which the solution set for $k_0 \leq k \leq s$ will be denoted as

$$\tilde{X}_{k_0}^{s}(\cdot,k_0,X^0,L(\cdot)) = \tilde{X}_{k_0}^{s}[\cdot,L(\cdot)]$$

Theorem 6.4 *The set $X_{k_0}^{s}[\cdot,L(\cdot)]$ of viable solutions to the inclusion*

$$x_{k+1} \in \tilde{F}(k,x(k)) \qquad x(k_0) \in X^0$$

$$G(k)x(k) \in Q(k), \qquad k_0 \leq k \leq s$$

is the restriction of set

$$X_{k_0}^{s+1}[\cdot] = \bigcap_L \tilde{X}_{k_0}^{s+1}[\cdot,L]$$

defined for stages $[k_0, \ldots, s+1]$ to the stages $[k_0,\ldots,s]$. The intersection is taken here over all constant matrices L.

However a question arises, whether this scheme allows also to calculate $X_{k_0}^{s}[s]$. Obviously

$$X_{k_0}^{s} \subseteq \bigcap_{L[\cdot]} \tilde{X}_{k_0}^{s+1}[s,L[\cdot]] \tag{6.19}$$

over all sequences $L[\cdot] = \{L(k_0), L(k_0+1),\ldots,L(s+1)\}$. Moreover the following proportion is true.

Theorem 6.5 *Assume $\tilde{F}(k,x)$ to be linear-convex: $\tilde{F}(k,x) = A(k)x + P(k)$, with $P(k)$, $Q(k)$ convex and compact. Then (6.19) turns to be an equality.*

7. Solution to the Basic Problem. "Stochastic" Approximations.

The calculation of $X[s]$, $X_*[s]$, $X^*[s]$ may be also performed on the basis of the results of §5. Namely system (6.6), (6.7) should now be substituted by the following

$$Z(k+1) = (I_n - F(k+1)G(k+1))F^0(k, H(k)) - F(k+1)Q(k+1) \tag{7.1}$$

$$H(k+1) = \{ \cap Z(k+1) | D(k+1) \in \boldsymbol{D}(1,\beta) \} \tag{7.2}$$

$$F(k+1) = R(k)G'(k+1)K^{-1}(k+1),\ F(k_0) = X^0 \tag{7.3}$$

$$K(k+1) = N(k+1) + G(k+1)R(k)G'(k+1)$$

$$D(k+1) = \{ R(k), N(k+1) \}$$

Theorem 7.1 *Assume that in Theorem* 6.1 $S(k)$ *is substituted by* $H(k)$ *and* $M(k)$ *by* $F(k)$. *Then the result of this theorem remains true.*

If set $Q(k)$ of (1.3) is of specific type

$$Q(k) = y(k) - \tilde{Q}(k)$$

where $y(k)$ and $\tilde{Q}(k)$ are given, then (1.3) is transformed into

$$y(k) \in G(k)x(k) + \tilde{Q}(k) \tag{7.4}$$

which could be treated as an equation of observations for the uncertain system (7.1). Sets $X[s]$, $X_*[s]$, $X^*[s]$ therefore give us the guaranteed estimates of the unknown state of system (1.1) on the basis of an observation of vector $y(k)$, $k \in [k_0, s]$ due to equation (7.4). The result of Theorem 7.1 then means that the solution of this problem may be obtained via equations (7.1)–(7.3), according to formulae (6.8)–(6.10) with $M(k)$, $S(k)$ substituted respectively by $F(k)$, $H(k)$. The deterministic problem of nonlinear "guaranteed" filtering is hence approximated by relations obtained through a "stochastic filtering" approximation scheme.

8. The Set $X(s \mid t, k, \mathbb{F})$.

Assume that the sequence $y[k, t]$ is fixed. Let us discuss the means of constructing sets $X(x \mid t, k, \mathbb{F})$, with $s \in [k, t]$. From the respective definition one may deduce the assertion

Lemma 8.1 The following equality is true

$$X(s \mid t, k, \mathbb{F}) = X(s \mid s, t, X(t, k, \mathbb{F})) \tag{8.1}$$

Here the symbol $X(s \mid s, t, \mathbb{F})$, taken for $s \le t$, stands for the set of states $x(s)$ that serve as starting points for all the solutions $x(k, s, x(s))$ that satisfy the relations

$$x(k+1) \in F(k, x(k)), \; x(t) \in \mathbb{F}$$

$$x(k) \in Y(k), \quad s \le k \le t$$

Corollary 8.1 Formula (8.1) may be substituted for

$$X(s \mid t, k, \mathbb{F}) = X(s, k, \mathbb{F}) \cap X(s \mid t, k, \mathbb{R}) \tag{8.2}$$

where $\mathbb{R}$ is any subset of $\mathbb{R}^n$ that includes $X(t, k, \mathbb{F})$.

Thus the set $X(s \mid t, k, \mathbb{F})$ is described through the solutions of two problems the first of which is to define $X(s, k, \mathbb{F})$ (along the techniques of the above) and the second is to define $X(s \mid s, t, \mathbb{R})$. The solution of the second problem will be further specified for $\mathbb{F} \in \text{comp}\mathbb{R}^n$ and for a closed convex Y.

The underlying elementary operation is to describe X^* - the set of all the vectors $x \in \mathbb{R}^n$ that satisfy the system

$$z \in F(x), \quad z \in Y$$

$$(X^* = \{x : F(x) \cap Y \ne \phi\})$$

In view of Lemma 4.1 we come to

Lemma 8.2 Th set X^* may be described as

$$X^* = \cup \{ \cap \{Ex - MF(x) + MY \mid M \in \mathbb{M}^{n \times n}\} \mid x \in \mathbb{R}^n \} \quad .$$

From here it follows:

Theorem 8.1

The set $X(s \mid s, t, \mathbb{R})$ may be described as the solution of the recurrent system (in backward "time")

$$X[k] = Y(k) \cap X^*[k] \tag{8.3}$$

where

$$X^*[k] = \cup \{ \cap \{Ex - MF(x) + MX[k+1] \mid M \in \mathbb{M}^{n \times n} \} \mid x \in \mathbb{R}^n \},$$

$$s \le k \le t, \; X[t] = Y[t].$$

Finally we will specify the solution for the linear case

$$x(k+1) \in A(k)\, x(k) + P(k)\, Y(k) = \{x : G(k)\, x \in Q(k)\} .$$

Assume

$$X = \{x : z \in Ax - P \,,\, x \in Y \,,\, z \in Z\} \ ,\ Y = \{x : Gx \in Q\} \tag{8.4}$$

where $A \in \mathbb{M}^{n \times n}$, $G \in \mathbb{M}^{m \times n}$, P , Q , Z are convex and compact.

Lemma 8.3 The set X may be defined as

$$\rho(l \mid X) = \inf\{\rho(\lambda \mid P) + \rho(\lambda \mid Z) + \rho(p \mid Q)\}$$

over all the vectors $\lambda \in \mathbf{R}^n$, $p \in \mathbf{R}^m$ that satisfy the equality $l = A'\lambda + G'p$.

The latter relation yields:

Lemma 8.4 The set X may be defined as

$$X \subseteq L'(Z + P) + M'Q = H(L \,,\, M) \tag{8.5}$$

whatever are the matrices $L \in \mathbb{M}^{n \times n}$ and $M \in \mathbb{M}^{m \times n}$ that satisfy the equality $L'A + M'G = E_n$. Moreover the following equalities are true

$$X = \{\cap H(L \,,\, M) \mid L \,,\, M\} \tag{8.6}$$

$$\rho(l \mid X) = \inf\{\rho(l \mid H(L \,,\, M)) \mid L, M\}$$

over all $L \in \mathbb{M}^{n \times n}$, $M \in \mathbb{M}^{m \times n}$.

Corollary 8.2 Suppose $|A| \neq 0$. Then conditions (8.5), (8.6) may be substituted for

$$X \subseteq (E_n - M'G)\, A^{-1}(Z + P) + M'Q = H(M) \,,$$

$$X = \cap\{H(M) \mid M\} \,,\ \rho(l \mid X) = \inf\{\rho(l \mid H(M)) \mid M\}$$

where

$$M \in \mathbb{M}^{m \times n} .$$

The latter relations may be used for recurrent procedures. These are either

$$X[k] = \cap\{H_k(L \,,\, M) \mid L\, A(k) + MG(k) = E_n\} \,, \tag{8.7}$$

$$H_k(L\ ,\ M) = L'(X[k+1] + P(k)) + M'Q(k)\ ,\ X[t] = Y[t]\ ,$$

$$s \le k \le t$$

with

$$X(s \mid s\ ,\ t\ ,\ Y[t]) = X[s] \qquad (8.9)$$

or

$$X[k] \subseteq H_k(L(k)\ ,\ M(k))\ ,\ X[t] = Y[t] \qquad (8.10)$$

$$s \le k \le t$$

with

$$X(s \mid s\ ,\ t\ ,\ Y(t)) = \bigcap \{X[s] \mid L_s(\cdot)\ ,\ M_s(\cdot)\} \qquad (8.11)$$

where

$$L_s(\cdot) = (L(s)\ , \ldots ,\ L(t))\ ;\ M_s(\cdot) = (M(s)\ , \ldots ,\ M(t))$$

Theorem 8.2 The set $X(s \mid s\ ,\ t\ ,\ Y)$ may be derived due to either equations (8.7) - (8.9) or (8.10), (8.11).

Remark As mentioned in the sequel to § 7, set $Y(t)$ may be generated due to a measurement equation

$$G(k)\,x \in y(k) - \tilde{Q}(k) = Q(k)$$

where $\tilde{Q}(k)$ is the restriction on the "noise" in the observations. Then each of the sets $X(\tau\ ,\ k_0 X^0)$ gives a " guaranteed" estimate for the unknown state of the system (1.1) on the basis of the available measurement $y(\cdot) = (y(k_0)\ , \ldots ,\ y(k))$ obtained due to equation

$$Y(k) \in G(k)\,x(k) + Q^*(k)$$

$$k_0 \le k \le \tau$$

Thus sets $X(\tau\ ,\ k_0\ ,\ X^0)$ solve the "filtering" problem, whilst $X(s \mid \tau\ ,\ k_0\ ,\ X^0)$ gives the solution of either the interpolation ("refinement") problem (if $k_0 \le s \le \tau$) or the extrapolation problem (if $k_0 \le \tau \le s$).

In § 7 the approximation of $X(\tau\ ,\ k_0\ ,\ X^0)$ was given through stochastic filtering procedures. The same approach may be propagated to give an alternative approximation scheme for sets $X(s \mid s\ , k_0\ , X^0)$.

The schemes of this paper allow to treat nonlinear systems. However in the linear case they do not coincide with the procedures given in [2,10] for solving guaranteed estimation problems with set-membership instantaneous constraints.

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