Working Paper

INCREASING RETURNS TO SCALE IN HETEROGENEOUS POPULATIONS

Robin Cowan

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International Institute for Applied Systems Analysis
A-2361 Laxenburg, Austria

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This research was conducted in conjunction with a summer research seminar on heterogeneity dynamics, under the direction of James W. Vaupel and Anatoli I. Yashin, in the Population Program at IIASA led by Nathan Keyfitz.

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Foreword

A group of eleven Ph.D. candidates from seven countries--Robin Cowan, Andrew Foster, Nedka Gateva, William Hodges, Arno Kitts, Eva Lelievre, Fernando Rajulton, Lucky Tedrow, Marc Tremblay, John Wilmoth, and Zeng Yi--worked together at IIASA from June 17 through September 6, 1985, in a seminar on population heterogeneity. The seminar was led by the two of us with the help of Nathan Keyfitz, leader of the Population Program, and Bradley Gambill, Dianne Goodwin, and Alan Bernstein, researchers in the Population Program, as well as the occasional participation of guest scholars at IIASA, including Michael Stoto, Sergei Scherbov, Joel Cohen, Frans Willekens, Vladimir Crechuha, and Geert Ridder. Susanne Stock, our secretary, and Margaret Traber managed the seminar superbly.

Each of the eleven students in the seminar succeeded in writing a report on the research they had done. With only one exception, the students evaluated the seminar as "very productive"; the exception thought it was "productive". The two of us agree: the quality of the research produced exceeded our expectations and made the summer a thoroughly enjoyable experience. We were particularly pleased by the interest and sparkle displayed in our daily, hour-long colloquium, and by the spirit of cooperation all the participants, both students and more senior researchers, displayed in generously sharing ideas and otherwise helping each other.

Robin Cowan succeeded in producing two papers over the course of the summer, the present paper on the long run dynamics of some demographic processes being one of them. It addresses an important question, building on some research started at IIASA by W. Brian Arthur. New ideas and fresh approaches are needed in demography: this paper is refreshingly innovative and may well represent the start of a significant direction for productive research.

Anatoli I. Yashin
James W. Vaupel

Abstract

This is very much a working paper. It presents some preliminary results having to do with the long run dynamics of certain types of demographic processes. A population is heterogeneous with regard to its preferences for two alternatives A and B. If the choice of alternatives displays increasing returns, i.e. the more one of the alternatives is chosen the more attractive it becomes, the long run properties of the system are, in general, not predictable. It may, however, have fixed points to one of which the system will converge. The stability of the fixed points depends very much on the correlation of the distribution of original preferences. As this is work in progress, suggested directions for future research are presented.

Acknowledgments

The author wishes to acknowledge many useful conversations with Brian Arthur, also the helpful comments and suggestions of James Vaupel and Anatoli Yashin. This research was funded in part by a grant from the Andrew Mellon Foundation through the Population Studies Committee at Stanford University.

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Introduction

Many times in a lifetime an individual is faced with choices between alternatives--where to go to university, whom to marry, whether or not to have another child and so on. Many of these decisions, when viewed by themselves, have no effect on people other than those directly involved. Where I decide to go to university may certainly have an effect on the financial status of my parents if they are footing the bill, but it is very unlikely to have any noticeable effect on the course of history. Neither, from an individual point of view, is my decision likely to to have any effect on the decisions that other people make. These statements seem unobjectionable when we are thinking of some particular individual. But suppose we view the world as a system, made up of many such individuals. Decisions can now take on another aspect . This aspect has to do with the capacity of a system of this sort to feed back on itself. What I mean by this is that cumulated individual decisions, which by themselves appear to be independent of each other and of what comes after them, may have an effect on future decisions. This is obviously true in many cases: The cumulative fertility and migration decisions of individuals this generation are certainly going to have an effect on the decisions of the next generation regarding social security, to take a timely example. There are, however, types of decisions where the engendered effect is not nearly so clear. I have in mind situations in which decisions with regard to particular choices have an effect on the decisions of future agents with regard to the same choice. For example, when I come to choose my school, many people have chosen schools before me. Any particular individual's decision will have no effect on mine. It may be, however, that the cumulative effect of the decisions of all those who have come before me is that one school is large and the other is small. It is clear that if I have preferences with regard to the size of the school to which I go, this cumulative effect will influence my decision making. This type of phenomenon is referred to in economics jargon as "returns to scale". To talk in terms of production, processes in which the activity becomes more or less efficient as it is done more are said to have increasing or decreasing returns to scale respectively. If the n-th unit of output needs less input than does the $n-1$-th unit of output, then the production process has increasing returns to scale. The production of children might be an example of one such process: A second child is almost certainly less expensive than a first--one can re-use such things as clothing, furniture, babysitting contacts and so on, and one has also accumulated an enormous amount of knowledge, so the amount of time devoted to learning things about child-rearing for the second child is considerably less than for the first. The cost of the second child is less than the cost of the first, so we say that, at least in this range, child-bearing has increasing returns to scale. My concern here, though, is with processes that are more social than something like child-rearing. In the case of child-rearing, the returns to scale are suffered by the parent who has the second child. They exist because that same parent has had the first child. If there are returns to scale in the case of the choice of university, they will not be suffered in the same way. That is to say that my choice of Queen's is not going to make Queen's look better or worse for me, but it may well do so for some one who is about to choose where he wants to go to school. This is what I mean be the process being social, as opposed to private. A demonstrative example of this kind of social returns to scale is in the choice of queues in a bank. As I enter the bank, I see several lines, each of which leads to a separate teller, and each of which I might join. Clearly the more people who, prior to my arrival, have joined the line to teller A, relative to the number of people who have joined any of the other lines, the less attractive line A will be to me. This is an example of decreasing returns to scale--the more something is done, the higher are the costs of doing it, or equivalently, the lower are the benefits. (In this case, the 'costs' must be seen in terms of what I lose by not choosing a different line.)

Demography

There are, I believe, many demographic phenomena which can be described at least partially in these terms. A clear example is the choice of language. In a bilingual country in which language use has not been well defined geographically, immigrants have to choose which of the languages they wish to learn. As well, parents may have to decide which language to speak at home for the purposes of teaching their children. This is certainly the case in the province of Quebec, (less so since 1977 and the passage of Bill 101), where either English or French is spoken, and also in other mainly rural regions in other provinces of Canada where the choice is between English and French (or often some other language if the area is populated largely by immigrants from the same country.) This phenomenon is common to many European countries, and also, to a much lesser extent, to the choice between English and Spanish in the southwestern United States. That there are increasing returns to choice of language is clear: Choosing the more common language increases one's potential social circle. Particularly if the community is small, or the proportion of the population speaking the less common language is small, choosing the more common language increases greatly the chances that the full range of social services--schools, law courts, hospitals and so on, will be available. The more people pick French, the more it costs to speak English.

Another such phenomenon would be the incidence of divorce. To a person who is unhappy in his marriage, there are both costs and benefits to getting a divorce. It rids one of unpleasant living situation perhaps, but it also makes life more difficult in several ways: One may lose part of one's acquaintances; it may well become more difficult to meet people; if one has children, there may be a substantial increase in the costs in terms of time spent of looking after them; there was also (not very long ago) a social stigma attached to being divorced. Virtually all of these problems are to some degree alleviated as more and more people are in the state of being divorced. As a larger number of divorced people want to spend time engaged in social activities, the number of facilities catering to single people rather than to couples will increase. This makes it easier to be single. As well, as there are more and more single parents, the quantity of such things as daycare centres will increase. (Indeed, it becomes possible to demand them as part of contract negotiations.) And, as has been apparent over the past several decades, the stigma attached to divorce slowly disappears. I will not press this last claim for it may be disputed that this is not a causal relationship, that rather the (exogenously) loose morals of the times are the explanation for the disappearance of this stigma, not the fact that many people in the past were willing to face it. (I doubt that morals become loose exogenously, and that a very good explanation for the current social acceptability of divorce is that so many people have been willing to face the stigmas of the past. If this general kind of explanation is correct, then it will also work to explain, at least in part, the rising incidence of non-marital cohabitation. It seems reasonable to believe that there have always been people who would have preferred cohabitation to marriage, but that for most of them the disapprobation suffered would have out-weighed the benefits, and so they married. It may also be that for some who would have preferred it, it never entered the realm of possibility, simply because it "wasn't done". For whatever reason, though, there came along an increasing number of people who felt that the gains overwhelmed the disapprobation and so did not bother with marriage. As there came to be more "normal" people in the state of cohabitation, it became more acceptable, and more people are finding that the gains, net of disapprobation, are positive. If this kind of armchair sociology is correct, then we have another example of increasing returns to scale.)

The final example of increasing returns which I will detail has to do with the choice of destination of migrants. In this exposition, I am thinking primarily of the emigration which took place before 1920 from Europe, primarily Great Britain and Ireland, to the New World. When an agent has decided to migrate, he must choose a destination. The majority of emigration from the U.K. and Ireland had as its destination either the United States or Canada. It seems to me that in the decision between the two destinations, substantial increasing returns to scale might be expected to be operating. There are several factors involved: In a migration to a frontier, it seems to make good sense to go to the more populous place (assuming that frontier remains to be settled). Once on the homestead, a settler would certainly want a reasonably sized settlement nearby from which to obtain the supplies that he could not produce himself--nails, lumber, sugar and so on. Though this may be an important factor for a certain era, it is not applicable if the immigrant is not a homesteader but rather a labourer. In this case, however, there are several other factors contributing to increasing returns to scale. Information about potential destinations is very important to migrants, and it is clearly easier to get information about the place to which the larger proportion of migrants has gone. There will be more letters home, either to the future migrant himself, or to his friends, and local newspapers will be more interested in the affairs of the destination in which there are more expatriots. There may well be information available about both places, but if the reports are more or less equally glowing, then the destination for which there are more reports will be stochastically preferred. Think of the potential migrant as arranging his information about each place in two separate distributions--he has a certain number of observations at each level of "goodness of destination". Then if the means of the two distributions are approximately equal, more observations will decrease the variance, and so make the more observed the more preferred. There is also the large factor of desiring to join friends and relatives. In a survey conducted among immigrants to the United States in 1909, fully 94% of those asked stated that they were going to join family or friends (Tomaske). There is also the fact that previous migrants are able to remit passage money to future migrants. In a similar survey to the one mentioned above, 30% of those asked responded that their passages had been financially assisted by some one who had previously migrated (Tomaske). These factors are all indications of increasing returns to scale.

It is clear that increasing returns to scale is a feature common to many demographic processes, and it is not at all difficult to find examples in other walks of life. It would be surprising indeed, though, if all of the returns increased in exactly the same manner: Changing the proportion of past migrants who have chosen Canada will have a certain effect on the desirability of Canada to future migrants. This effect is likely to be very different from the effect on the desirability of cohabitation of changing the proportion already cohabiting. It would be appropriate, then, to study the nature of processes subject to increasing returns in as general a way as possible.

One interesting aspect of such processes is their long run behaviour. Does the proportion of migrants who chose Canada as their destination settle down in the long run, or is it always moving? If all of the members of the population are identical, as is often assumed in demographic analysis, then the answer is clear--everyone does the same thing that the first person does. (If there are decreasing returns to scale, this is not necessarily true. In fact only in unusual cases will it be true.) If, however, the population is heterogeneous, some analysis is needed before such statements can be made.

In recent work, Arthur (1985a) has analysed a similar problem having to do with competing technologies. In his model the population is divided into two homogeneous sub-groups of equal size. These two groups are characterized by their respective gains from adopting one or other of two competing technologies. The total gain to an individual from adopting technology A is affected by the number of people who have adopted that technology before him. Here again is a system which exhibits returns to scale. Arthur analysed this system under several different returns regimes, viewing the proportion of adopters who have chosen technology A as a random walk. He found that under regimes of constant or decreasing returns to scale, the proportion of the people who have adopted technology A goes to 1/2 with probability 1 as time goes to infinity. With unbounded increasing returns, with probability 1 the proportion goes to zero or one, but to which it goes cannot be predicted. Under bounded increasing returns, the behaviour of the system depends very much on the nature of the bounds.

In related work, Arthur (1985b) has built a model of industrial location under increasing returns. In that model, a firm chooses its location based on the inherent attractiveness (to that firm) of the location, and on the returns the firm receives from there being other firms already located there. The firms of this model are much more heterogeneous than the adopters of his competing technologies model, in that firms may differ a great deal from each other with regard to how they view the inherent attractiveness of a particular location. Arthur obtained results very similar to those of his competing technologies model. An open question remains, however, whether the distribution of the prior tastes of the firms, or migrants, in particular the correlation of this distribution, can affect the outcome. It is this question that I wish to address in this paper, using a variation of Arthur's industrial location model. (I will talk now only in terms of migration, but the analysis, I think, is equally applicable, possibly with minor revisions, to the other phenomena discussed.) These prior tastes I will refer to as predilections, and they can be seen as the desire a migrant has for one or other of the destinations, ignoring any of the effects of returns to scale. We might ask each potential migrant, "Ignoring any possible considerations having to do with the number of people already there, how much do you want to go to Canada? How much do you want to go to the United States?". The answers to these questions will generate an ordered pair for each potential migrant, and these ordered pairs can be mapped onto the x-y plane (see Figure 1 in which each point represents a potential migrant).

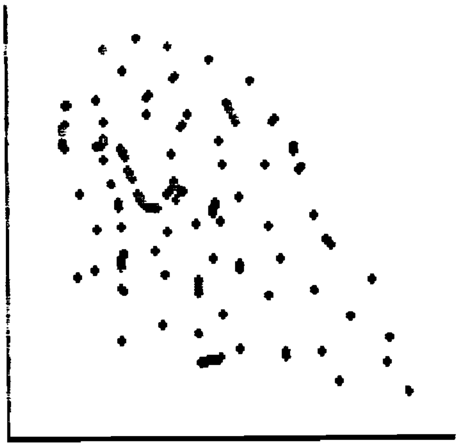

Figure 1. Map of preferences of potential migrants for two different destinations.

We can assume that these predilections form a probability distribution. Each migrant makes his choice of destination based on the utility he will receive from each of the two places. Every migrant's utilities will be determined by his predilections and by the level of returns to scale for each of the two places. So the migrant who has predilections (x, y), where x is his predilection for Canada, y for the United States, will see his utility from going to Canada as $x + H^c(N_c)$, where $H^c(.)$ is some function which represents the returns to scale resulting from Canada's population being N_c at the time of his choice. Likewise, his utility from going to the United States will be $y + H^u(N_u)$. When it comes time to decide where to go, the migrant simply chooses the destination which will give him the highest utility.

The Model

The model is very simple then. If we characterize each of the potential migrants by his coordinates on the $x-y$ plane, the migrant R_{xy} will go to Canada if and only if $x + H^c(N_c) \geq y + H^u(N_u)$. Otherwise, R_{xy} will go to the United States. From the migrants' point of view, at the time of his departure, his destination is perfectly determined--he knows N_c and N_u, so he simply works out $x + H^c(N_c)$ and $y + H^u(N_u)$, and goes to the appropriate place. From the point of view of an observer, however, the system is not so predictable. The problem is that the observer does not know which migrant is next. If we knew the order of the R_{xy}'s, the system would be completely deterministic. But the order is random--to the observer, the next migrant appears to be drawn randomly from the distribution of migrants. We can, however, make probabilistic statements about the next R_{xy}.

Since we are interested in the dynamics of the system and not the migrants themselves, the next R_{xy} is not really that important, or rather is so only incidentally. What we really care about is whether or not the next migrant goes to Canada. So what we want to know is whether for the next migrant $x + H^c(N_c) \geq y + H^u(N_u)$. And if we know N_c and N_u, we can make probabilistic statements about this.

At this point, it would be well to think about the nature of the increasing returns to scale functions H^u and H^c. What we actually care about is the inequality $x + H^c(N_c) \lesseqgtr y + H^u(N_u)$. This can be rearranged to $x \lesseqgtr y + [H^u(N_u) - H^c(N_c)]$. Let us let $G(N_c,N_u) = [H^u(N_u) - H^c(N_c)]$. $G(N_c,N_u)$ tells us how much more the United States has gained from returns to scale than Canada has at the levels of population current in the two countries. In other words, for a potential migrant whose predilections show no preference for either of the destinations, (i.e. $x = y$), $G(N_c,N_u)$ will tell us how much he prefers the United States to Canada at the population levels N_u and N_c, respectively. We would, I think, expect G to be a function of the ratios of the two populations. The reason for this is that in the eyes of a potential migrant, the difference between 1000 and 1100 is much bigger than is the difference between 100,000 and 100,100. This is true for two reasons: Of any one of his friends who have migrated, the probability that the friend has gone to Canada will be equal to the proportion of the total migrant population that has gone to Canada. (This is from the point of view of the observer who does not know who the next migrant will be, and so knows nothing about his friendship networks.) The second reason has to do with the services provided in each of the two places. Suppose that $N_u - N_c = g$, where g is a positive constant. We would expect that the level of services available in the United States would be higher than in Canada. It would also make sense, though, to believe that the difference in the levels of services available would be greater, the smaller is N_u. Suppose we think in terms of discrete levels of services, which have population thresholds associated with them. That is, when a population threshold is reached, the next level of services is added. This is one of the notions underlying hierarchy models in regional economics (Richardson, 1973). It is likely that the thresholds are closer together (in terms of absolute numbers) at lower levels of population. By the time the population has reached 100,000 we would not be likely to get the next level of services by adding 100 people. It is far more likely to be true that adding 100 people does make this difference when we begin with a population of 1000.

I assume now that the returns to scale are the same in the two countries. This means that $G(N_c, N_u)$ is symmetric about $\frac{1}{2}$. So if N is the total number of migrants ($N = N_c + N_u$) then $G(\alpha N, (1-\alpha)N) = -G((1-\alpha)N, \alpha N)$. Thus, $G(.5N, .5N) = 0$. Since G measures how much ahead one of the countries is in the returns to scale race, we need only say that G is positive if the country represented by the first argument has the larger population. The names of the countries do not matter. The convention I will adopt is that Canada will take the first argument, and so when the population of Canada is larger, G is positive. One more important simplification, I will assume that G is not a function of the total size of the migrant population, and so we can write $G(\alpha N, (1-\alpha)N) = H(\alpha)$, where α is the proportion of migrants who have gone to Canada. Since we are looking at systems which display increasing returns, as one of the destinations gets ahead in population, it gets ahead in attractiveness, so H is increasing in α. I assume that it is monotone.

I return now to the distribution of predilections. With regard to predilections alone, each migrant finds one of the two possible destinations preferable to the other. There will also be some migrants who are indifferent between the two. This set of potential migrants will be mapped onto a ray running through the origin at 45°. Along this ray, which I will refer to as the ray of indifference, $x = y$. Anyone below this ray prefers Canada, anyone above it prefers the United States. Now we take account of the increasing returns function. If $\alpha > .5$, there are more people in Canada. This means that for anyone whose predilections lay on the line of indifference, Canada is now preferred since $H(\alpha) > 0$ for $\alpha > \frac{1}{2}$. In fact, the whole distribution is shifted to the right, since for every potential migrant there is the same added bonus for going to Canada. Similarly, if $\alpha < .5$, there are more people in the United States, and so the US looks better than Canada to those who were previously indifferent. The US gains in everyone's eyes, so the distribution shifts up. This shifting of the distribution up or to the right, can be seen as equivalent to moving the ray of indifference down or up respectively, since what we are interested in is actually who prefers Canada, and who prefers the United States. The ray of indifference divides the distribution of migrants into these two groups, so if the distribution is shifted pointwise by a constant, it is equivalent to shifting the dividing line. Now to make life easy, we can simply scale $H(\alpha)$ so that it is equal to the shift of the ray of indifference (see Figure 2, which is the distribution of Figure 1 with contour lines drawn on it). Since the next migrant is drawn from the distribution, the probability that he prefers Canada is equal to the mass of the distribution which falls below the ray of indifference. So to make probabilistic statements about the destination of the next migrant, if we know the initial distribution of predilections, we need only know the location of the ray of indifference. This is a function of the proportion of migrants who have already gone to Canada. We have now come full circle, and the feedback nature of the system is apparent.

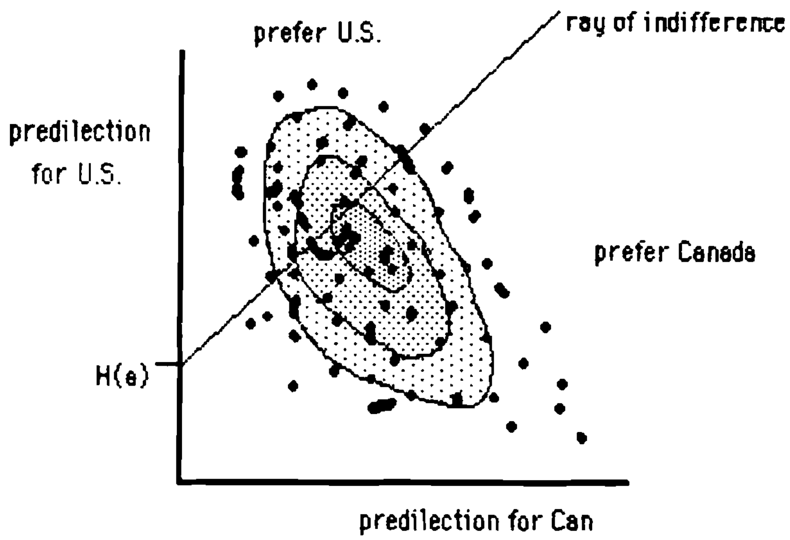

Figure 2. Map of preferences of potential migrants for two different destinations.

Results of the Model

Here it would be nice to begin an examination of a generic probability distribution in order to look at the dynamics of such a system. The interesting questions have to do with fixed points. From recent results of Arthur, Ermoliev and Kaniovski (1985), we know that with probability one, a model such as has been described will settle down to a stable fixed point. By a fixed point, I mean a value of α such that the probability that the next migrant goes to Canada is equal to α. What this means is that there is no systematic tendency for the proportion of the migrant population in Canada to change. If fixed points exist, and particularly if they are stable, then we may be able to make statements about the long run behaviour of the relative sizes of the populations of Canada and the United States.

Unfortunately, at this stage of research, a generic probability distribution is beyond reach, so I have retreated to a particular distribution. The distribution I use is Gumbel's bivariate exponential distribution (Johnson and Kotz, 1972, pp. 262). A bivariate exponential distribution is not wholly unreasonable if we appropriately choose the population about which to be concerned. Suppose the population are those people who have definitely decided to migrate. The decision process might be characterised as follows: Conditions in the home country become so bad that migration is decided upon--people believe that no matter how many people there are in the New World, conditions there are better than they are in the Old World. Then, having become potential migrants, people examine the populations in the two possible destinations to decide which would be the better place to go. This is clearly a model of push-migration, in that the momentum to migrate is generated in the source country. This is most appropriate in situations like the Irish potato famine, or for migrations due to religious persecution. The stochastic process of a pull-migration model has a different nature and so produces different results.[1] The exponential nature of the distribution implies that there are many people who have predilections which map them to points which are near the origin. In other words, many people do not have very strong predilections, though there are a few for whom they are very strong.

The joint distribution of predilections, then, is assumed to be

$$F_{xy}(x,y) = (1-e^{-x})(1-e^{-y})(1+\beta e^{-x-y}) \ .$$

And so

$$Pr(Y<y \mid X=x) = [1-\beta(2e^{-x}-1)][1-e^{-y}] + \beta(2e^{-x}-1)(1-e^{-2y}) \ ,$$

where β is 4 times the correlation coefficient, and is restricted to [-1,1]. (This unfortunately restricts the correlation of the distribution to $[\frac{-1}{4},\frac{1}{4}]$.) As noted before, the probability that the next migrant goes to Canada is the mass of the distri-

[1]For example if all migrants are pulled by relatives who have previously migrated, and the order in which previous migrants "pull" is not pre-determined, then the migration looks like a Polya process, and the proportion of migrants in Canada can settle down to any point in the interval [0,1]. If, however, some portion of the migrants are not pulled, but choose randomly between destinations (and then pull their relatives), the proportion of migrants in Canada will go to $\frac{1}{2}$. I am grateful to Brian Arthur for this point.

bution below the ray of indifference. This ray is the line $Y = x + H(\alpha)$. So what we want to know is $Pr(Y < x + H(\alpha))$. As an intermediate step, if we set $H(\alpha) = h$, we can write

$$Pr(Y < X + H(\alpha) \mid X = x) = [1 - \beta(2e^{-x} - 1)][1 - e^{-x-h}] + \beta(2e^{-x} - 1)(1 - e^{-2x-2h})$$

To find the unconditional probability, we must integrate over $x \in (0,\infty)$ to get

$$Pr(Y < x + h) = \int \{[1 - \beta(2e^{-x} - 1)][1 - e^{-x-h}] + \beta(2e^{-x} - 1)(1 - e^{-2x-2h})\} e^{-x} dx$$

where the final e^{-x} is from the marginal distribution of X.

After much smoke clears, and letting $Pr(Y < x + h)$ equal $Q(\alpha)$, we find that

$$Q(\alpha) = 1 - \frac{1}{2}e^{-h} + (\frac{\beta}{6})(e^{-h} - e^{-2h}) \ , \ \alpha \geq \frac{1}{2}$$

$$Q(\alpha) = \frac{1}{2}e^{h} - (\frac{\beta}{6})(e^{h} - e^{2h}) \ , \ \alpha \leq \frac{1}{2} \ .$$

$Q(\alpha)$ maps from the interval [0,1] onto [0,1] and is continuous, so by Brower's fixed point theorem, we know that at least one fixed point exists. That is, there is some $\overset{o}{\alpha}$ such that if the proportion of past migrants who have gone to Canada is equal to $\overset{o}{\alpha}$, then the probability that the next migrant goes to Canada is equal to $\overset{o}{\alpha}$.

We stated earlier that $H(\frac{1}{2}) = 0$. Substituting this into $Q(\alpha)$, where $\alpha = \frac{1}{2}$, we find that there is a fixed point at $\alpha = \frac{1}{2}$. Due to the symmetry of the problem, both in the distribution and in the returns to scale, this is not surprising. Because the problem is so symmetric, when we are looking for fixed points and their properties, we need only look on one side of the midpoint. That is to say that we can concentrate our attention on the interval $[\frac{1}{2},1]$.

One of the important questions to ask about fixed points is whether or not they are attractors. From recent results by Hill, Lane and Sudderth (1980), we know that a fixed point is an attractor if and only if it is a downcrossing of the diagonal, i.e. if the slope of the function $Q(\alpha)$ is less than 1 at the fixed point. (The set of attractors is the set of points to which the system will converge with non-zero probability.)

I will first examine the point $\alpha = \frac{1}{2}$ and then turn my attention to other possible fixed points. What we need to know in order to tell if a fixed point is an attractor or not, is the slope of the function $Q(\alpha)$ at that point. Differentiating $Q(\alpha)$ with respect to α we get

$$\frac{\partial Q}{\partial \alpha} = \frac{dH}{d\alpha}[\frac{1}{2}e^{-H(\alpha)} + \frac{\beta}{6}(-e^{-H(\alpha)} + 2e^{-2H(\alpha)})]$$

So at $\alpha = \frac{1}{2}$,

$$\frac{\partial Q}{\partial \alpha} = \frac{DH}{d\alpha}(\frac{1}{2} + \frac{\beta}{6}) .$$

When is this quantity greater than 1? Using the bounds on β, we can say that

$$\frac{1}{3}\frac{dH}{d\alpha} \leq \frac{\partial Q}{\partial \alpha} \leq \frac{2}{3}\frac{dH}{d\alpha} .$$

It is no surprise that the slope of $Q(\alpha)$ depends crucially on the returns function $H(\alpha)$. There are several conditions, though, under which we can draw conclusions about whether or not $\frac{1}{2}$ is an attractor.

1) If $\frac{dH}{d\alpha} > 3$ at $\alpha = \frac{1}{2}$ then $\frac{\partial Q}{\partial \alpha} > 1$ so $\frac{1}{2}$ is an upcrossing, and so is not an attractor. In other words, if increasing returns come into play very quickly and very powerfully, then with probability 1 the system will move away from the point $\frac{1}{2}$. This is as one would expect--if there are very large gains to going to the more populous place, then it seems very unlikely that the population will remain equally divided between the two places.

2) If $\frac{dH}{d\alpha} \leq \frac{3}{2}$ at $\alpha = \frac{1}{2}$, then $\frac{\partial Q}{\partial \alpha} < 1$, so $\frac{1}{2}$ is a downcrossing and so is an attractor. In other words, if the increasing returns are small near $\frac{1}{2}$, then the randomness of the order of migration will, with probability 1, keep the proportions at $\frac{1}{2}$ (given that that is where we start the system from).

3) If $\frac{3}{2} \leq \frac{dH}{d\alpha} \leq 3$, then whether or not $\frac{1}{2}$ is an attractor depends solely on β. To find that dependency, we simply take the derivative of $\frac{\partial Q}{\partial \alpha}$ with respect to β, at the value $\alpha = \frac{1}{2}$

$$\frac{\partial}{\partial \beta}(\frac{\partial Q}{\partial \alpha}) = \frac{dH}{d\alpha}\frac{1}{6} .$$

This value is non-negative, since we have assumed that $H(\alpha)$ is a monotonically increasing function. If we also assume that increasing returns to scale are operative at all points near $\frac{1}{2}$, then $\frac{\partial}{\partial\beta}(\frac{\partial Q}{\partial\alpha})$ is strictly positive. What this means is that the higher (positive) is the correlation of the distribution of predilections, the less likely (where the "sample space" is the set of functions $H(\alpha)$) it is that the system will settle down to $\frac{1}{2}$. This seems reasonable. If the correlation is high, (but symmetric about $\frac{1}{2}$) then there are many people whose predilections give close to equal value to Canada and the United States. They will readily change their preferences, and so send the proportion of potential migrants who prefer Canada away from $\frac{1}{2}$ if the population of one country gets slightly larger than the other. If, on the other hand, the distribution of predilections has a small (high negative) correlation, then there are relatively many people who have strong predilections and so will not easily change their preferences. For many of them, their predilections will outweigh the returns to scale when it is time to make their choice. (It is important to remember, in this regard, that if choice of destination were based on predilections alone, the proportion of migrants in Canada would go to $\frac{1}{2}$ with probability 1, due to the symmetry of the distribution.)

Something worth noting is that, as we can see from Figure 3, if $\frac{1}{2}$ is not an attractor, then there must be more fixed points in the range $(\frac{1}{2},1]$. If $\frac{1}{2}$ is not an attractor then it is un upcrossing, so $Q(\alpha)$ must re-cross the diagonal, possibly at 1, in order that it be defined on the whole range $[\frac{1}{2},1]$.

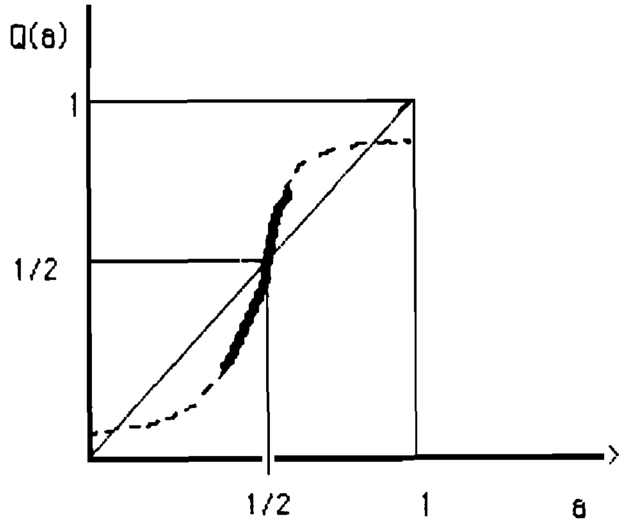

Figure 3. Probability that next migrant goes to Canada as a function of proportion of migrant population currently in Canada.

I turn attention now to other fixed points. One of the reasons for undertaking this enterprise was to examine the long run behaviour of this type of system. One very interesting event would be that in the long run all of the migrants went to the same destination. I ask now what would be necessary for this to take place. In other words, under what conditions is 1 (or 0) a an attracting fixed point? If 1 is a fixed point, then

$$1 = 1 - \frac{1}{2}e^{-H(1)} + \frac{\beta}{6}(e^{-H(1)} - e^{-2H(1)}) \quad ,$$

or equivalently,

$$0 = e^{-H(1)}(-\frac{1}{2} + \frac{\beta}{6} - e^{-H(1)}) \quad .$$

There are two possible solutions. The first is that $e^{-H(1)} = 0$. In this case, $H(1) = \infty$. Or more properly, $\lim_{\alpha \to 1} H(\alpha) = \infty$. This implies that the increasing returns to scale are unbounded. The second solution is that $e^{-H(1)} = \frac{\beta}{6} - \frac{1}{2}$. Or that $-H(1) = \ln(\frac{\beta}{6} - \frac{1}{2})$. If this solution is to be well defined, then it must be that $(\frac{\beta}{6} - \frac{1}{2}) \geq 0$. Or that $\beta \geq 3$. But there is a restriction (coming from the specification of the distribution) that $-1 \leq \beta \leq 1$. Thus the second solution is never well defined, so the only condition under which 1 is a fixed point is that the increasing returns to scale be unbounded, so that $H(1) = 0$. This stands to reason: Any outliers in the distribution indicate very strong predilections for one of the two destinations. If the returns are bounded, by finite m say, then there will be some potential migrants whose predilections for the United States are greater than m, since the distribution has infinite tails. These migrants in the tails make up a fixed proportion of the total migrant population, so the proportion who go to Canada can never get to 1--these outliers will always go to the United States.

If 1 is fixed, is it an attractor? To answer this question, we need to know whether $Q(\alpha)$ approaches 1 from above or below the ray $Q(\alpha) = \alpha$. Equivalently, we need to know whether the slope at 1 is less than or greater than 1.

$$\frac{dQ}{d\alpha}\Big|_{\alpha=1} = \lim_{\alpha \to 1} \frac{dH}{d\alpha}\left[\frac{1}{2}e^{-H(\alpha)} + \frac{\beta}{6}(-e^{-H(\alpha)} + 2e^{-2H(\alpha)})\right] .$$

Since $\lim \left[\frac{1}{2}e^{-H(\alpha)} + \frac{\beta}{6}(-e^{-H(\alpha)} + 2e^{-2H(\alpha)})\right] = 0$, then if $\frac{dH}{d\alpha} < \infty$, $\frac{dQ}{d\alpha}\Big|_{\alpha=1} = 0$, and 1 is stable. There are, however, functions $H(\alpha)$ such that $H(1) = \infty$ but $\frac{dQ}{d\alpha}\Big|_{\alpha=1} = 0$. In this case, it must be that

$$1 < \frac{dH}{d\alpha}\left[\frac{1}{2}e^{-H(\alpha)} + \frac{\beta}{6}(e^{-H(\alpha)} + 2e^{-2H(\alpha)})\right] \quad \text{at } \alpha = 1 .$$

This inequality places an upper bound on the speed at which $H(\alpha)$ increases as α approaches 1.

We can see from Figure 4 that if both $\frac{1}{2}$ and 1 are not attractors, (1 need not be a fixed point for this to be true) then there must be an attracting fixed point between them. That is to say that there exists an $\overset{o}{\alpha}$ such that $\frac{1}{2} < \overset{o}{\alpha} < 1$ and $\overset{o}{\alpha}$ is both fixed and attracting.

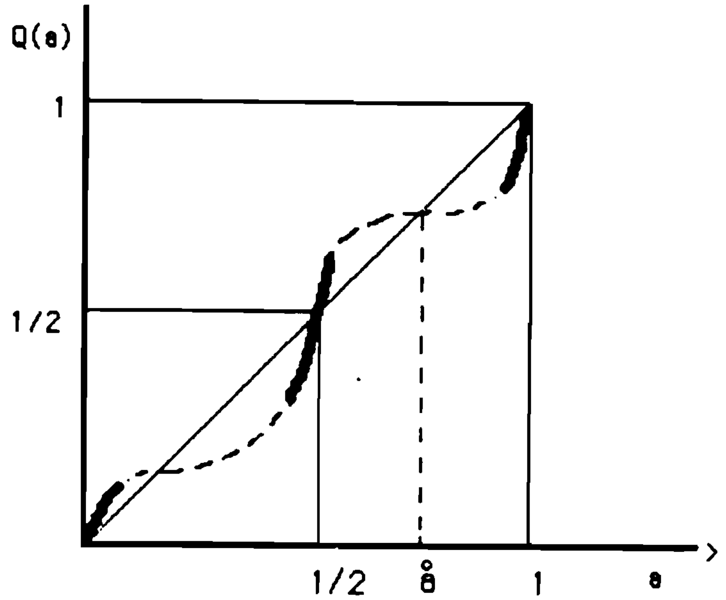

Figure 4. Probability that next migrant goes to Canada as function of proportion of migrant population currently in Canada.

At such a point, it must be that

$$\overset{o}{\alpha} = 1 - \frac{1}{2}e^{-H(\overset{o}{\alpha})} + \frac{\beta}{6}(e^{-H(\overset{o}{\alpha})} - e^{-2H(\overset{o}{\alpha})}) \ .$$

This is quadratic in $e^{-H(\overset{o}{\alpha})}$, so we can solve to get

$$H(\overset{o}{\alpha}) = -ln\left(\frac{1}{2} - \frac{3}{2\beta} \pm \left(\frac{3}{\beta}\right)\sqrt{\left(\frac{\beta}{6} - \frac{1}{2}\right)^2 + \frac{2\beta}{3}(1 - \overset{o}{\alpha})}\right) \ .$$

Unfortunately we cannot say very much about this without specifying $H(\alpha)$ more fully than has been done, and observing that the value $\beta = 0$ will need special treatment. It is worth noting, though, that the value $\overset{o}{\alpha}$ will be a function of β, and so of the correlation of the distribution.

Discussion

Part of the purpose of this paper has been to demonstrate that in a demographic process which exhibits increasing returns to scale, the long run outcome of the process is not always deterministic. We have examined some cases here in which one can make probabilistic statements about the outcome, but not much more. Part of the reason for the lack of specific statements has been the generality of the model. Were the increasing returns function to be specified more fully, more detail could be given about the outcome of the process. It is clear, though, that in a system of the type which has been modelled here, it is not generally true that we can predict the outcome. There will be particular cases in which we can, (if the only fixed point is $\frac{1}{2}$ for example) but in general the best we will be able to do is to specify the set of points to which there is a positive probability that the system will converge.

One interesting result that has appeared is the relationship between the convergence points of the system and the correlation of the distribution of predilections. Differences in predilections was the way in which our population was heterogeneous. The inference that might be drawn from this is that the form of heterogeneity is important to the long run dynamics of population systems. In this model, the differences that matter take a particular form--namely the correlation between the prior utilities gained from each of the two destinations. One might be tempted to suppose that the more similar the individuals of a population are, the more likely the population is to behave like a homogeneous one. In this model, however, what is crucial is how each individual views the two alternatives. The population may be very diverse with regard to the level af attractiveness of migration (i.e., the distance from the origin of the ordered pairs (x,y)), but if each individual is indifferent between the two destinations (i.e., $x = y$), then the population will appear to be homogeneous.

There are two next steps in this research. The first is to analyse the relationship between the correlation of the distribution and the location of arbitrary (non-extreme) fixed points. The second is to generalise the results to other less restricted probability distributions. I suspect that the same sort of relationship between the correlation of the distribution and fixed points will emerge.

The final remarks have to do with migration. In this model, migration has been treated as a pure birth process--People are born with regard to the New World when they arrive there. Where they are "born" depends, in the funny way described, on the populations of people already in each of the two countries. Unfortunately, with regard to migration, people also "die", that is to say, emigrate. In a fully blown model of migration, this should be taken into account. (It is also true that mortality has been ignored. Including mortality will not affect the results if we assume that there is no age bias to the choice of destination, and that both countries suffer the same mortality schedule.) I think that it can be argued that in the period that I have been referring to, emigration did not play that big a role. It is true that Canada lost population through emigration to the United States, but there is reason to believe that the U.S. was the original destination of many of these people, but because of differences in the regulations governing passenger traffic between Great Britain and the New World, they found it cheaper to travel through Canada. I have ignored emigration here because I have been trying to isolate the effects on a particular type of system of increasing returns to scale. Migration certainly exhibits increasing returns, and to the extent that they are an important factor, and to the extent that the population is heterogeneous in the appropriate way, it (and many other demographic processes) can be analysed using the type of model set out here.

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