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**MULTIOBJECTIVE DECISION MAKING –  
UTILITY THEORETIC APPROACH**

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## FOREWORD

One of the difficult problems in decision analysis relates to the situation, when the decision must be undertaken by a committee. There exist several formalizations of decision making process based on the utility function approach. This approach is however very difficult to apply in the group decision case, since the number of coefficients characterizing the utility function is very high and it is practically impossible to directly identify such utility function. Therefore, reduction of dimensionality of the parameter space is necessary.

In this paper a concept of convex dependence between two conflicting decision makers is presented. This concept was effectively used by the author to develop a decomposition principle of the group utility function as well as to formulate the conditions necessary to perform such a decomposition. The concept was successfully applied for a practical example.

The paper was presented by the author at IIASA and at the Symposium "Systems Analysis and Simulation". It is one of the results of a cooperation between the SDS and Japanese researchers in decision analysis.

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## MULTIOBJECTIVE DECISION MAKING

— UTILITY THEORETIC APPROACH —

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The purpose of this survey paper is to describe recent advances in multiattribute utility theory and group utility theory for multiobjective decision making. Firstly, single-attribute von Neumann-Morgenstern utility functions are briefly described. Secondly, for constructing multiattribute utility functions the concept of Keeney's utility independence among multiple attributes is described. Thirdly, the concept of convex dependence is introduced as a generalized concept of utility independence. Fourthly, for constructing group utility functions the concept of convex dependence among multiple decision makers is introduced. Algorithm of identifying multiattribute (and/or group) utility functions and some hypothetical examples for interpreting convex dependence are included.

(Multiobjective decision making; Utility theory; Multiattribute utility function; Group utility function; Utility independence; Convex dependence)

## 1. INTRODUCTION

Mathematical modeling of preferences has been widely studied in multiattribute decision analysis. Keeney and Raiffa (1976) have described in detail the standard approach. A preference representation function under risk can be constructed as a utility function, however, it is practically impossible to directly identify a multiattribute utility function. Therefore, it is necessary to develop conditions that reduce the dimensionality of the functions that are required to identify. These conditions restrict the form of a multiattribute utility function in a decomposition theorem.

Keeney and Kirkwood (1975) have extended the multiattribute utility theory for a decision maker to a group utility theory for multiple conflicting decision makers where a group utility function is constructed postulating the utility independence properties among the multiple decision makers.

In this paper after briefly describing a single-attribute utility function based on von Neumann and Morgenstern's (1944) expected utility hypothesis, additive, multilinear, and convex decompositions are described for multiattribute utility functions. These decompositions are based on additive and utility independence (Keeney and Raiffa, 1976) and convex dependence (Tamura and Nakamura, 1983) conditions, respectively. The concept of convex dependence is a generalized concept of utility independence where we consider the change of decision maker's attitude towards risk. This concept generates various decompositions which include Keeney's additive/multiplicative decompositions as special cases. For clarifying the interpretation of this concept an example of trading-off between environment and consumption is included



(Tamura and Nakamura, 1978).

For group decision making with multiple decision makers we describe the concept of convex dependence between two (conflicting) decision makers (Tamura and Yukimura, 1983). This concept can represent the change of attitude of each decision maker towards the group utility depending upon the utility level of the other decision maker. For clarifying the interpretation of this concept a hypothetical numerical example for siting a major airport is included.

As a possible directions for further research, value theoretic approach to riskless and/or risky preference representation is mentioned. In this approach the concept of strength-of-preference (Fishburn, 1970 and Dyer and Sarin, 1979) plays an important role.

## 2. UTILITY THEORY

### 2.1. Expected Utility Hypothesis of von Neumann-Morgenstern and Identification of a Single-Attribute Utility Function

Let

$$A = \{ a, b, \dots \}$$

be a set of alternative actions from which a decision maker must choose one action. Suppose the choice of  $a \in A$  will result in consequence  $x_1$  with probability  $p_1$ , and the choice of  $b \in A$  will result in consequence  $x_2$  with probability  $q_1$ , and so forth. Let

$$X = \{ x_1, x_2, \dots \}$$

be a set of all possible consequences. In this case

$$p_i \geq 0, \quad q_i \geq 0, \quad \dots \quad \text{for all } i$$

$$\sum_i p_i = \sum_i q_i = \dots = 1.$$

Let real function  $u$  be a utility function on  $X$ . Then the expected utilities of actions  $a, b, \dots$  are given, respectively, by

$$E_a = \sum_i p_i u(x_i), \quad E_b = \sum_i q_i u(x_i), \quad \dots \quad (1)$$

The assertion that the decision maker chooses an alternative action as if he maximizes his expected utility is called the expected utility hypothesis of von Neumann and Morgenstern (1944). In other words the decision maker chooses an action according to the normative rule

$$a \succ b \Leftrightarrow E_a > E_b, \quad a \sim b \Leftrightarrow E_a = E_b \quad (2)$$

where  $a \succ b$  denotes "a is preferred to b", and  $a \sim b$  denotes "a is indifferent to b". This rule is called the expected utility rule. Utility function which satisfies Eqns. (1) and (2) is uniquely obtained within the class of positive linear transformation.

Figure 1 shows a decision tree and lotteries which explain the above mentioned situation, where  $\ell_a, \ell_b, \dots$  denote lotteries which the decision maker comes across when he chooses the action  $a, b, \dots$ , respectively.

DEFINITION 1. A certainty equivalent of lottery  $\ell_a$  is an amount  $\hat{x}$  such that the decision maker is indifferent between the amount  $\hat{x}$  for certain and the lottery  $\ell_a$ .

From the expected utility hypothesis we obtain

$$u(\hat{x}) = E_a = \sum_i p_i u(x_i). \quad (3)$$

In a set  $X$  of all possible consequences, let  $x^0$  and  $x^*$  be the worst and the best consequences, respectively. Since utility function is unique within the class of positive linear transformations, let us normalize the utility function as

$$u(x^0) = 0, \quad u(x^*) = 1.$$

Let  $\langle x^*, p, x^0 \rangle$  be a lottery yielding consequences  $x^*$  and  $x^0$  with probabilities  $p$  and  $(1-p)$ , respectively. Especially when  $p=0.5$ , this lottery is called the fifty-fifty lottery and is denoted as  $\langle x^*, x^0 \rangle$ . Let  $x$  be a certainty equivalent of lottery  $\langle x^*, p, x^0 \rangle$ , that is

$$x \sim \langle x^*, p, x^0 \rangle$$

then

$$u(x) = pu(x^*) + (1-p)u(x^0) = p.$$

It is easy to identify a single-attribute utility function of a decision maker by asking the decision maker about the certainty equivalents of some fifty-fifty lotteries (Keeney and Raiffa, 1976). Let

$$x_{0.5} \sim \langle x^*, x^0 \rangle, \quad x_{0.25} \sim \langle x_{0.5}, x^0 \rangle, \quad x_{0.75} \sim \langle x^*, x_{0.5} \rangle$$

then

$$u(x_{0.5}) = 0.5u(x^*) + 0.5u(x^0) = 0.5$$

$$u(x_{0.25}) = 0.5u(x_{0.5}) + 0.5u(x^0) = 0.25$$

$$u(x_{0.75}) = 0.5u(x^*) + 0.5u(x_{0.5}) = 0.75.$$

If we plot the pairs  $(x^0, 0), (x_{0.25}, 0.25), (x_{0.5}, 0.5), (x_{0.75}, 0.75), (x^*, 1)$ , a diagram like Fig. 2 is obtained. By some curve fitting techniques, like least square method, a single-attribute utility function

$u(x)$  can be identified.

Attitude of a decision maker toward risk is described as follows:

DEFINITION 2. A decision maker is risk averse if he prefers the expected consequence  $\bar{x}$  ( $= \sum p_i x_i$ ) of any lotteries to that lottery. In this case

$$u(\bar{x}) > \sum_i p_i u(x_i). \quad (4)$$

If a decision maker is risk averse, his utility function is concave as shown in Fig. 2. Converse is also true. A decision maker is risk neutral (prone) if and only if his utility function is linear (convex).

## 2.2. Utility Decompositions Based on Additive and Utility Independence

The following results are the essential summary of Keeney and Raiffa (1976).

Let a specific consequence  $x \in X$  be characterized by two attributes (performance indices)  $Y$  and  $Z$ . For example, price and performance of cars, natural environment and economy of a nation, and so forth. In this case a specific consequence  $x \in X$  is represented by an ordered pair

$$x = (y, z), \quad y \in Y, \quad z \in Z.$$

A set of all possible consequences  $X$  can be written as a rectangular subset of a two-dimensional Euclidean space as  $X = Y \times Z$ . This consequence space is called two-attribute space. Although  $Y$  and  $Z$  could represent vector-attribute, both of these are regarded as single-attribute spaces here. Two-attribute utility function is defined on  $X = Y \times Z$  as  $u: Y \times Z \rightarrow \text{Re}$ .

DEFINITION 3. Attribute Y is utility independent of attribute Z, denoted  $Y(UI)Z$ , if conditional preferences for lotteries on Y given  $z \in Z$  do not depend on the conditional level  $z$ .

Let us assume that  $y^0$  and  $z^0$  are the worst level of the attributes Y and Z, respectively, and  $y^*$  and  $z^*$  are the best level of Y and Z, respectively.

DEFINITION 4. Given an arbitrary  $z \in Z$ , a normalized conditional utility function  $u_1(y|z)$  on Y is defined by

$$u_1(y|z) = [u(y,z) - u(y^0,z)]/[u(y^*,z) - u(y^0,z)] \quad (5a)$$

where it is assumed that  $u(y^*,z) > u(y^0,z)$ .

Similarly  $u_2(z|y)$  on Z is also defined by

$$u_2(z|y) = [u(y,z) - u(y,z^0)]/[u(y,z^*) - u(y,z^0)] \quad (5b)$$

where it is assumed that  $u(y,z^*) > u(y,z^0)$ .

From DEFINITION 4 we obtain

$$u_1(y^0|z) = u_2(z^0|y) = 0, \quad u_1(y^*|z) = u_2(z^*|y) = 1.$$

From DEFINITIONS 3 and 4 the following equations hold, if  $Y(UI)Z$ .

$$u_1(y|z) = u_1(y|z^0), \quad \text{for all } z \in Z.$$

In other words utility independence implies that the normalized conditional utility functions do not depend on the different conditional levels.

THEOREM 1.  $Y(UI)Z$ , if and only if

$$u(y,z) = au_1(y|z^0) + bu_2(z|y^0) + (1-a)u_1(y|z^0)u_2(z|y^*) - bu_1(y|z^0)u_2(z|y^0) \quad (6)$$

where  $u(y,z)$  is normalized as  $u(y^0, z^0) = 0$  and  $u(y^*, z^*) = 1$ , and  $a = u(y^*, z^0)$  and  $b = u(y^0, z^*)$ .

THEOREM 2. Attributes Y and Z are mutually utility independent, denoted Y(MUI)Z, if and only if

$$u(y,z) = au_1(y|z^0) + bu_2(z|y^0) + (1-a-b)u_1(y|z^0)u_2(z|y^0) \quad (7)$$

where  $u(y,z)$  is normalized, and a and b are defined as before.

THEOREMS 1 and 2 give decomposition theorems under the utility independence assumptions. It is clear from THEOREM 2 that if the attributes Y and Z are mutually utility independent, only one normalized conditional utility function needs to be assessed for each attribute. Since each normalized conditional utility function is a single-attribute utility function, it can be identified by asking the decision maker the certainty equivalents of some 50-50 lotteries as described in the previous section.

DEFINITION 5. Attributes Y and Z are additive independent, if, for arbitrarily chosen  $y' \in Y$  and  $z' \in Z$ ,

$$\langle (y,z), (y',z') \rangle \sim \langle (y,z'), (y',z) \rangle, \text{ for all } y \in Y, z \in Z.$$

THEOREM 3. Attributes Y and Z are additive independent, if and only if

$$u(y,z) = au_1(y|z^0) + bu_2(z|y^0) \quad (8)$$

where

$$a + b = 1.$$

Decomposition theorem 3 is a special case of the decomposition theorem 2 where  $a+b = 1$  in Eqn. (7). Therefore, the additive independence is a special case of mutual utility independence.

### 2.3. Utility Decompositions Based on Convex Dependence

The following results are due to Tamura and Nakamura (1983). This section deals with the case where

$$u_1(y|z) \neq u_1(y|z^0), \quad \text{for some } z \in Z$$

$$u_2(z|y) \neq u_2(z|y^0), \quad \text{for some } y \in Y$$

that is, utility independence does not hold between the attributes Y and Z.

DEFINITION 6. Attribute Y is n-th order convex dependent on attribute Z, denoted  $Y(CD_n)Z$ , if there exist distinct  $z_0, z_1, \dots, z_n \in Z$  and real functions  $\lambda_0, \lambda_1, \dots, \lambda_n$  on Z such that the normalized conditional utility function  $u_1(y|z)$  can be written as

$$u_1(y|z) = \sum_{i=0}^n \lambda_i(z) u_1(y|z_i), \quad \sum_{i=0}^n \lambda_i(z) = 1 \quad (9)$$

for all  $y \in Y$  and  $z \in Z$ , where

$$\lambda_i(z) = \begin{cases} \delta_{ij} & z = z_j, \quad j = 0, 1, \dots, n \\ \text{Real number,} & z \neq z_j, \quad j = 0, 1, \dots, n \end{cases}$$

$$i = 1, 2, \dots, n$$

$\delta_{ij}$  denotes Kronecker delta and  $n$  is the smallest nonnegative integer for which Eqn. (9) holds.

This definition says that if  $Y(CD_n)Z$ , any normalized conditional utility function on  $Y$  can be described as a convex combination of  $(n+1)$  normalized conditional utility functions with different conditional levels where the coefficients  $\lambda_i(z)$  are not necessarily nonnegative.

Geometric illustration of DEFINITION 6 is shown in Fig. 3. Suppose three arbitrary normalized conditional utility functions  $u_1(y|z_0)$ ,  $u_1(y|z_1)$  and  $u_1(y|z)$  are assessed on  $Y$  as shown in Fig. 3(a). If  $Y(CD_0)Z$ , all the normalized conditional utility functions are identical as shown in Fig. 3(b). If  $Y(CD_1)Z$ , an arbitrary normalized conditional utility function  $u_1(y|z)$  can be obtained as a convex combination of  $u_1(y|x_0)$  and  $u_1(y|z_1)$  as shown in Fig. 3(c).

For  $n = 0, 1, \dots$ , if  $Y(CD_n)Z$ , then  $Z$  is at most  $(n+1)$ th order convex dependent on  $Y$ . If  $Y(UI)Z$ , then  $Y(CD_0)Z$ , and  $Z(UI)Y$  or  $Z(CD_1)Y$ . In general if  $Y(CD_n)Z$ , then  $Z$  satisfies one of the three properties,  $Z(CD_{n-1})Y$ ,  $Z(CD_n)Y$  or  $Z(CD_{n+1})Y$ .

THEOREM 4. For  $n = 1, 2, \dots$ ,  $Y(CD_n)Z$ , if and only if

$$u(y, z) = au_1(y|z^0) + bu_2(z|y^0) + u_1(y|z^0)f(y^*, z) + \sum_{i=1}^{n^*} \sum_{j=1}^n c_{ij} G(y, z_i) G(y_j, z) \quad (10)$$

where



$$u(y^0, z^0) = 0, \quad u(y^*, z^*) = 1, \quad a = u(y^*, z^0), \quad b = u(y^0, z^*)$$

$$f(y, z) = u_1(y|z)[(1-a)u_2(z|y^*)+a-bu_2(z|y^0)] - au_1(y|z^0) \quad (11)$$

$$G(y, z) = a[(1-a)u_2(z|y^*)+a-bu_2(z|y^0)][u_1(y|z) - u_1(y|z^0)]$$

$c_{ij}$  is a constant, and summation  $i = 1$  to  $n^*$  means  $i = 1, 2, \dots, n-1, n^*$ .

THEOREM 5. For  $n = 1, 2, \dots$ ,  $Y(CD_n)Z$  and  $Z(CD_n)Y$ , if and only if

$$u(y, z) = au_1(y|z^0) + bu_2(z|y^0) + \sum_{i=1}^{n^*} \sum_{j=1}^{n^*} d_{ij} f(y, z_i) f(y_j, z) \\ + \sum_{i=1}^{n^*} \sum_{j=1}^{n^*} d'_{ij} G(y, z_i) H(y_j, z) \quad (12)$$

where  $u(y, z)$  is normalized,  $a$  and  $b$  are defined as before, Eqn. (11) holds,

$$H(y, z) = b[(1-b)u_1(y|z^*)+b-au_1(y|z^0)][u_2(z|y) - u_2(z|y^0)] \quad (13)$$

and  $d_{ij}$  and  $d'_{ij}$  are constants.

Exact expressions for  $c_{ij}$ ,  $d_{ij}$  and  $d'_{ij}$  can be found in Tamura and Nakamura (1983).

We have obtained two main convex decomposition theorems which can represent a wide range of utility functions. Moreover, when the utility on the arbitrary point  $(y_n, z_n)$  has a particular value, that is,  $d'_{ij} = 0$ , for all  $i, j$  in Eqn. (12), we can obtain one more decomposition of utility functions which does not depend on the point  $(y_n, z_n)$ . This decomposition still satisfies  $Y(CD_n)Z$  and  $Z(CD_n)Y$ , so we call this new

property reduced n-th order convex dependence and denote it by  $Y(RCD_n)Z$ .

We note that when  $d'_{ij} = 0$ , for all  $i, j$  and  $n = 1$ , Eqn. (12) reduces to Fishburn's (1974) bilateral decomposition

$$u(y, z) = au_1(y|z^0) + bu_2(z|y^0) + f(y, z^*)f(y^*, z)/f(y^*, z^*). \quad (14)$$

When  $n = 1$  and  $d'_{ij} \neq 0$ , that is  $Y(CD_1)Z$  and  $Z(CD_1)Y$ , Eqn. (12) reduces to

$$u(y, z) = au_1(y|z^0) + bu_2(z|y^0) + f(y, z^*)f(y^*, z)/f(y^*, z^*) \\ + d'G(y, z^*)H(y^*, z). \quad (15)$$

Equation (15) is Bell's (1979) decomposition under interpolation independence.

On two scalar attributes the difference between the conditional utility functions necessary to construct the previous decomposition models and the convex decomposition models is shown in Fig. 4. By assessing utilities on the heavy shaded lines and points, we can completely specify the utility function in the cases indicated in Fig. 4. As seen from Fig. 4, the advantage of the convex decomposition is that only single-attribute conditional utility functions need to be assessed even for high-order convex dependent case. Therefore, it is relatively easy to identify the utility functions.

Fishburn and Farquhar (1982) have established an axiomatic approach for selecting a basis of normalized conditional utility function.

#### 2.4. Interpretation of Convex Dependence

For describing the interpretation of convex dependence between two different attributes, we discuss the utility for environment and consumption (Tamura and Nakamura, 1978). In this problem there exists conflict between these two attributes, because the more we consume the more we pollute. It will be shown that the two attributes, environment and consumption, do not satisfy the utility independence property. Therefore, we may want to take into account the convex dependence property.

Let  $Y$  and  $Z$  be the attributes of environment and consumption, respectively, and  $e \in Y$  and  $c \in Z$  be the attribute levels. We restrict these attribute levels in  $e^0 \leq e \leq e^*$  and  $c^0 \leq c \leq c^*$  where  $e = e^0$  means environment is polluted and is in the worst level,  $e = e^*$  means no pollution exists and environment is clean,  $c = c^0$  means consumption is in the lowest level, and  $c = c^*$  in the highest level.

Now we consider how a normalized conditional utility function  $u_1(e|c)$  for environment changes depending upon the consumption level  $c$ . It is evident that the preference for environment changes according to the consumption level  $c$ . Comparing  $u_1(e|c^*)$  with  $u_1(e|c^0)$  these normalized conditional utility function is drawn schematically in Fig. 5 a). When  $c = c^*$ , even if the environment is deteriorated from the clean level ( $e = e^*$ ), the decrease of utility is not so rapid in compensation for high consumption, but as the environment level approaches to the worst level, the utility for the environment decreases rapidly. When  $c = c^0$ , the decrease of utility for environment is even for any environmental level, because the consumption level is suppressed to a low level.

Next, we consider how a normalized conditional utility function  $u_2(c|e)$  for consumption changes depending upon the environmental level. Comparing  $u_2(c|e^*)$  with  $u_2(c|e^0)$  these normalized conditional utility function is drawn schematically in Fig. 5 b). When  $e = e^*$ , utility for consumption increases according to the law of diminishing marginal utility. On the other hand, when  $e = e^0$ , they feel that the high consumption is a matter of course, therefore, the rate of increase of utility for the unit increase of the consumption level is very small when the consumption level is low. But the utility for consumption increases according to the law of diminishing marginal utility after the consumption level moves to higher level.

Accordingly, the trade-off between environment and consumption does not satisfy the utility independence property, and hence taking into account the convex dependence property we could construct an appropriate utility function.

### 3. ALGORITHM OF IDENTIFYING MULTIATTRIBUTE UTILITY FUNCTIONS

For identifying a multiattribute utility function under the convex dependence condition (including 0-th order), we need to find the order of convex dependence. This order can be assessed as in the following steps: Define

$$\underline{\lambda}_n \triangleq [ \lambda_0, \lambda_1, \dots, \lambda_{n-1}, \lambda^* ]^T \quad (16a)$$

$$\underline{u}_n \triangleq [ u_1(y_1|z_n), u_1(y_2|z_n), \dots, u_1(y_n|z_n), 1 ] \quad (16b)$$

$$U_n \triangleq \begin{bmatrix} u_1(y_1|z_0) & u_1(y_1|z_1) & \dots & u_1(y_1|z_{n-1}) & u_1(y_1|z^*) \\ u_1(y_2|z_0) & u_1(y_2|z_1) & \dots & u_1(y_2|z_{n-1}) & u_1(y_2|z^*) \\ \vdots & \vdots & & \vdots & \vdots \\ u_1(y_n|z_0) & u_1(y_n|z_1) & \dots & u_1(y_n|z_{n-1}) & u_1(y_n|z^*) \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}. \quad (16c)$$

Step 0. Normalized conditional utility functions  $u_1(y|z^0)$ ,  $u_1(y|z^*)$ ,  $u_2(z|y^0)$  and  $u_2(z|y^*)$  are assessed, and we draw the graph of them. If we can regard that

$$u_1(y|z^0) = u_1(y|z^*), \quad u_2(z|y^0) = u_2(z|y^*)$$

then we decide that  $Y(CD_0)Z$  and  $Z(CD_0)Y$ , that is,  $Y(MUI)Z$ . If not go to Step 1.

Step 1.  $n = 1$

Step 2. Normalized conditional utility functions  $u_1(y|z_0), u_1(y|z_1), \dots, u_1(y|z_n)$  and  $u_1(y|z^*)$  are assessed, and then  $\underline{u}_n$  and  $U_n$  are obtained.

Step 3. Linear equation

$$U_n \lambda_n = \underline{u}_n \quad (17)$$

is solved with respect to  $\lambda_n$ .

Step 4. We draw the graph of

$$f(y) = \sum_{i=0}^{n^*} \lambda_i u_1(y|z_i). \quad (18)$$

Step 5. The graph of  $f(y)$  is compared with the graph of  $u_1(y|z_n)$ . If we can regard that both curves are coincident within the allowable error, we decide that  $Y(CD_n)Z$ . If not,  $n+1 \rightarrow n$  and then go back to Step 2.

These steps can be easily realized by using a graphic terminal of a large computer.

Parameters  $a$  and  $b$  which appeared in THEOREMS 1 to 5 can be estimated as follows: We ask the decision maker the indifference probability  $p$  such that

$$(y^*, z^0) \sim \langle (y^*, z^*), p, (y^0, z^0) \rangle. \quad (19)$$

Then, we obtain

$$a = u(y^*, z^0) = pu(y^*, z^*) + (1 - p)u(y^0, z^0) = p. \quad (20)$$

Similarly, we ask the decision maker the indifference probability  $q$  such that

$$(y^0, z^*) \sim \langle (y^*, z^*), q, (y^0, z^0) \rangle. \quad (21)$$

Then, we obtain

$$b = q. \quad (22)$$

After obtaining the information for the order of convex dependence, normalized conditional utility functions and the scaling parameters  $a$  and  $b$ , we can construct a multiattribute utility function by using a decomposition form described in THEOREMS 1 to 5. For two attribute cases we could draw indifference curves for the multiattribute utility function in two attribute space  $Y \times Z$ .

#### 4. GROUP UTILITY THEORY

In the previous methods of social choice or group decision making the preference attitude of each decision maker (individual member of the group) has been described without taking into account the utility level (level of satisfaction) of the other decision makers, and such preference structures have been aggregated by some rule for group decision making. Keeney and Kirkwood's (1975) approach is also in this category. As the result we have often come across contradicting social decisions as seen in so-called voting paradox. As shown in Arrow's (1963) impossibility theorem there are no procedures for obtaining a group ordering of the various alternatives from the individual's ordinal rankings of the alternatives that is consistent with five reasonable criteria.

In real situation the preference attitude of each decision maker heavily depends on the outcomes or utility levels obtained by the other decision makers. For example a decision maker is satisfied with his low outcome and he feels that the group utility is relatively high even if his own utility level is low, when the other decision maker's utility level or outcome is lower than or equal to his outcome. On the contrary, the same decision maker is not satisfied with his high outcome when the other decision maker's outcome is higher than his outcome. Hence, the utility independence assumption among the multiple decision makers is not appropriate.

Essentially, in real social choice the individual's preference which is based only on his benefit, should not be reflected to the society. Instead, the individual's preference which is based on social

ethics or moral, should be reflected. Systematic methodologies for such societal decision have been missing and have been desired in many fields; economics, politics, behavioral science, operations research, and so forth.

In this section a group utility theory is described based on the concept of convex dependence. The group decision making by two (conflicting) decision makers is considered, where we discuss a systematic way of describing each decision maker's preference which depends on the utility level of the other decision maker. In other words, change of attitude of each decision maker towards the group utility is described depending upon the utility level of the other decision maker. Group utility function is then constructed by aggregating such preference of each decision maker. The following development is due to Tamura and Yukimura (1983).

DEFINITION 7. Let  $U_1 \times U_2$  denote the utility function space, and let  $u_1(x_1) \in U_1$ ,  $u_2(x_2) \in U_2$  denote the utility function of decision maker 1 (DM1) and DM2 on the multiattribute consequence spaces  $X_1$  and  $X_2$ , respectively, where  $x_i \in X_i$  ( $i = 1, 2$ ) denotes a specific consequence for DMi. A group utility function  $W(x_1, x_2)$  is assumed to be described as  $w[u_1(x_1), u_2(x_2)]$ .

We shall simplify the notation as follows:

$$w(u_1, u_2) = w[u_1(x_1), u_2(x_2)] \quad (23)$$

$$u_i^0 = u_i(x_i^0), \quad u_i^* = u_i(x_i^*), \quad i = 1, 2$$

where  $x_i^0$  and  $x_i^*$  denote the worst and the best consequences of DMi,



respectively, and hence  $u_1^0$  and  $u_1^*$  denote the utility level of DM1 for the worst and the best consequences, respectively. We will describe how to construct  $w(u_1, u_2)$  in the following.

DEFINITION 8. Given an arbitrary  $u_2 \in U_2$  a normalized conditional group utility function (NCGUF)  $w_1(u_1 | u_2)$  of DM1 on  $U_1$  is defined by

$$w_1(u_1 | u_2) = [w(u_1, u_2) - w(u_1^0, u_2)] / [w(u_1^*, u_2) - w(u_1^0, u_2)] \quad (24)$$

where it is assumed that  $w(u_1^*, u_2) > w(u_1^0, u_2)$ ,  $u_1^0 = 0$ ,  $u_1^* = 1$ . Then,  $w_1(u_1 | u_2)$  is normalized as

$$w_1(u_1^0 | u_2) = 0, \quad w_1(u_1^* | u_2) = 1, \quad \text{for all } u_2 \in U_2.$$

Similarly, NCGUF of DM2  $w_2(u_2 | u_1)$  can be defined by

$$w_2(u_2 | u_1) = [w(u_1, u_2) - w(u_1, u_2^0)] / [w(u_1, u_2^*) - w(u_1, u_2^0)] \quad (25)$$

where  $w(u_1, u_2^*) > w(u_1, u_2^0)$ ,  $u_2^0 = 0$ ,  $u_2^* = 1$ . It is also assumed that the group utility function  $w(u_1, u_2)$  is normalized so that

$$w(u_1^0, u_2^0) = 0, \quad w(u_1^*, u_2^*) = 1.$$

From mathematical point of view formulas of group utility functions are identical with those of multiattribute utility functions. In THEOREMS 1 to 5 if we replace the symbols as shown in Table 1, we could obtain the decomposition forms of group utility functions.

NCGUF (24) of DM1 represents his subjective preference structure for the group utility as a function of his own utility level under the condition that the utility level of DM2 is given. NCGUFs (24) and (25)

will play an important role for constructing a group utility function. Convex dependence between two decision makers is defined as follows:

DEFINITION 9. Utility of DM1 is said to be  $n$ -th order convex dependent on the utility of DM2, denoted  $U_1(CD_n)U_2$ , if NCGUF of DM1  $w_1(u_1|u_2)$  can be described as a convex combination of  $(n+1)$  NCGUFs of DM1  $w_1(u_1|u_2^i)$ ,  $i = 0, 1, \dots, n$  with different conditional levels.

For clarifying the interpretation of convex dependence between two (conflicting) decision makers, we deal with a decision making problem for siting a major airport, where we describe how to represent NCGUFs of each decision maker taking into account the situation of the other decision maker.

Let DM1 be the representative of the regional inhabitants and DM2 the representative of the developer of the airport. Existence of a benevolent dictator, who mediates DM1 and DM2 by assessing the scaling coefficients, is postulated. We can regard that DM1 wishes to construct the airport as far from the regional area as possible for reducing the environmental negative effect of the airport. On the other hand, DM2 wishes to construct the airport at the closer location to city for convenience and efficiency of the airport. Therefore, DM1 and DM2 are obviously conflicting.

We consider two mutually utility independent cases and one mutually first order convex dependent case and interpret each case.

Case A:  $U_1(MUI)U_2$

Suppose the certainty equivalents and the resulting NCGUFs of DM1 and DM2 are assessed as shown in Fig. 6. Convex NCGUFs in this figure

show that both DM1 and DM2 do not think that the group utility is high unless the utility levels of their own are very high, and that each DM's attitude towards the group utility does not depend on the utility level of the other DM. In other words, each DM is mutually utility independent and their attitude is selfish and stubborn.

In this case the group utility function is described as

$$w(u_1, u_2) = aw_1(u_1 | u_2^0) + bw_2(u_2 | u_1^0) + (1 - a - b)w_1(u_1 | u_2^0)w_2(u_2 | u_1^0). \quad (26)$$

Case B: Keeney-Kirkwood model

Suppose the certainty equivalents and the resulting NCGUFs of DM1 and DM2 are assessed as shown in Fig. 7. Linear NCGUFs in this figure show that NCGUF of each DM is equal to his own utility level. In this case each DM is again mutually utility independent and his attitude towards the group utility is stubborn but not as selfish as in Case A.

In this case  $w_1(u_1 | u_2^0) = u_1$ ,  $w_2(u_2 | u_1^0) = u_2$  in Eqn. (26), and the group utility function is described as

$$w(u_1, u_2) = au_1 + bu_2 + (1 - a - b)u_1u_2. \quad (27)$$

Case C:  $U_1(CD_1)U_2$  and  $U_2(CD_1)U_1$

Suppose the certainty equivalents and the resulting NCGUFs of DM1 and DM2 are assessed as shown in Fig. 8. Furthermore, suppose the first order convex dependence between DM1 and DM2 is assured. This means that preference attitude of DM1 (DM2) to the group utility varies

depending upon the utility level of DM2 (DM1). Even when the utility level of DM2 is at the worst level ( $u_2 = u_2^0$ ), the attitude of DM1 towards the group utility is selfish for low  $u_1$ , because the environmental impact of the airport to DM1 is straightforward. But for higher  $u_1$  the attitude of DM1 is changed to be more gentle.

On the other hand, when the utility level of DM2 is at the best level ( $u_2 = u_2^*$ ), the attitude of DM1 towards the group utility is always selfish. In other words, when the utility level of DM2 is high, DM1 does not feel that group utility is high unless his own utility level is very high. This can be interpreted that in the group decision DM1 is claiming equity by comparing his own utility level with that of DM2. When the utility level of DM1 is at the worst level ( $u_1 = u_1^0$ ), the attitude of DM2 is gentle and sympathetic, but when the utility level of DM1 is at the best level ( $u_1 = u_1^*$ ), the attitude of DM2 is changed to be slightly self-centered.

In this case the group utility function is described as Eqn. (15) where the symbols in Eqn. (15) are replaced according as Table 1.

As shown in Case C the concept of convex dependence makes it possible to describe in NCGUFs the change of attitude of each decision maker depending upon the utility level of the other decision maker. By using the group utility theory based on the concept of convex dependence, it is possible to offer clear information of various preference orderings for the alternatives depending upon the various attitude of each decision maker in the group and various cases for the values of scaling coefficients assessed by a benevolent dictator.

## 5. CONCLUDING REMARKS

Centering around the concept of convex dependence multiattribute utility theory and group utility theory are briefly surveyed for multi-objective decision making. A major advantage of the convex decomposition is that they include many previous decompositions such as additive, multiplicative, bilateral and interpolation decompositions as special cases. Therefore, depending upon the complexity of trade-offs the convex decompositions could provide more flexible multiattribute and/or group utility functions for modeling preferences of a decision maker or multiple (conflicting) decision makers.

Since the convex decompositions need only single-attribute utility functions even for high-order convex dependent cases, it is relatively easy to identify the multiattribute utility functions. Graphic terminals of a large-scale computer could be effectively used for this purpose.

We didn't include value theoretic approach (Dyer and Sarin, 1979, and Sarin, 1983) in this paper, however, riskless and/or risky preference representation based on the value theoretic approach is an important topic for further research. Under this approach it might be possible to discriminate a decision maker's strength-of-preference and the attitude towards risk.

The approach described in this paper are based on the expected utility hypothesis of von Neumann and Morgenstern (1944). Many paradoxes (e.g. Allais and Hagen, 1979, and Kahneman and Tversky, 1979) have been observed which violate particular axioms. For overcoming this difficulty nonlinear utility analysis (Nakamura, 1984) is being investigated.

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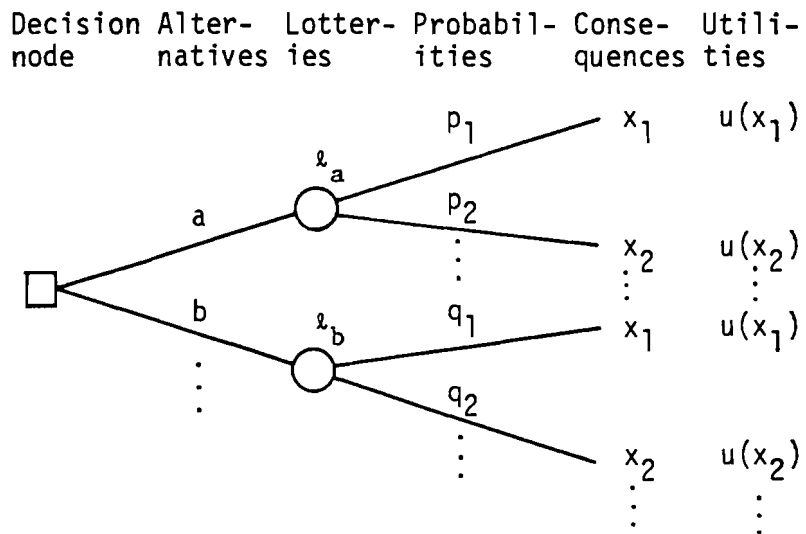


FIGURE 1 Decision tree and lotteries.

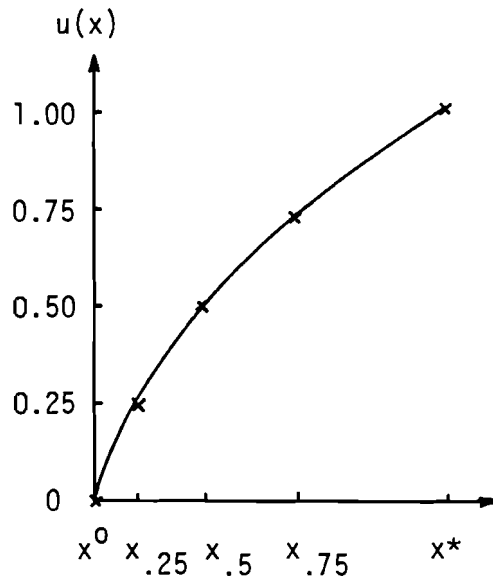


FIGURE 2 Single-attribute utility function.

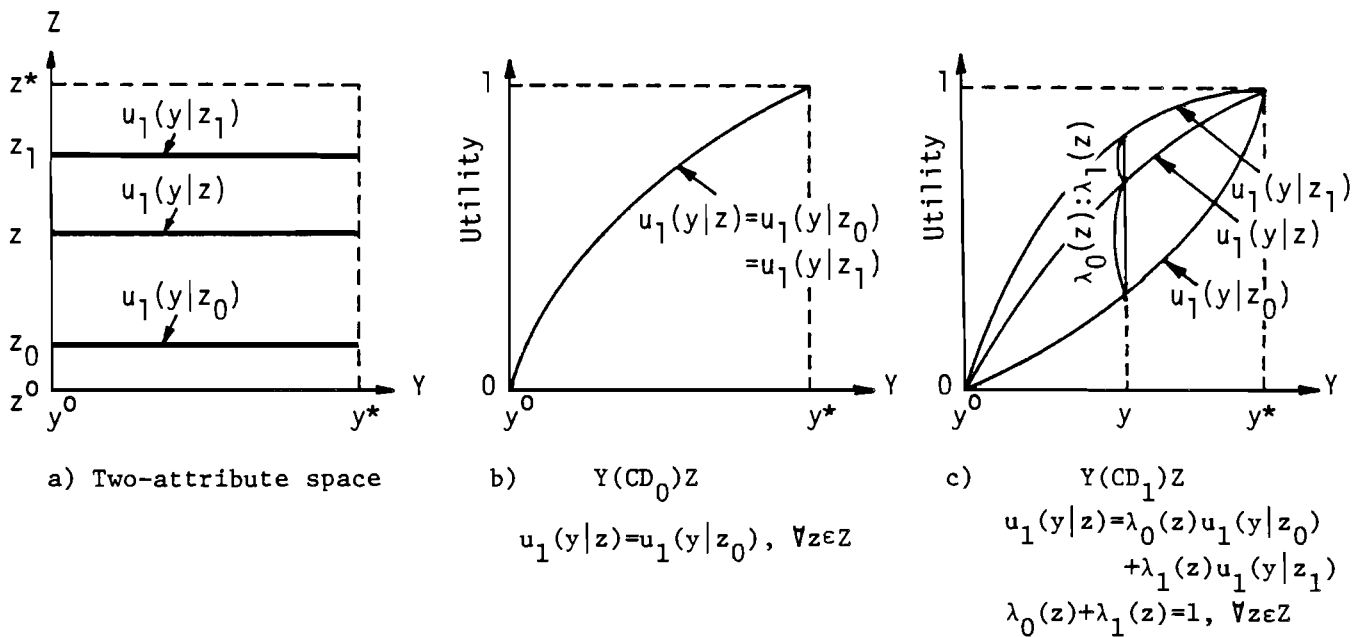
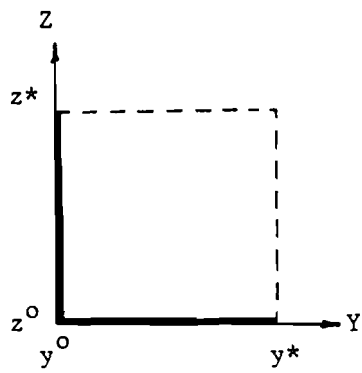
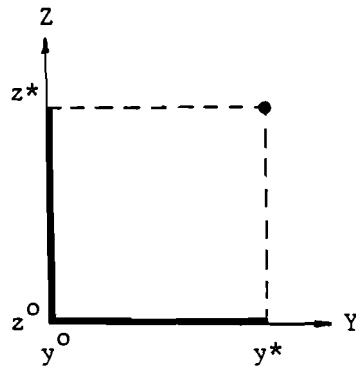


FIGURE 3 Relations among normalized conditional utility functions.

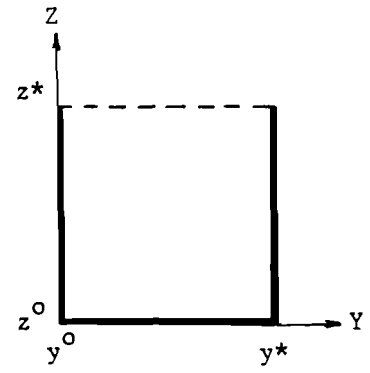




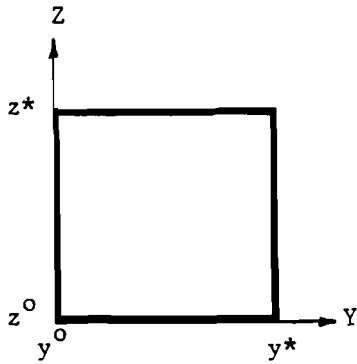
a) Additive model



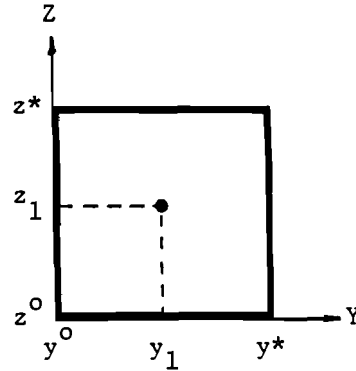
b) Multiplicative model  
 $Y(MUI)Z$  or  
 $Y(CD_0)Z, Z(CD_0)Y$



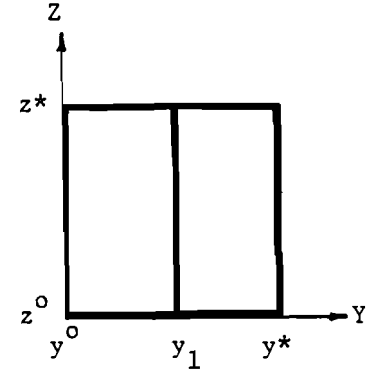
c)  $Y(UI)Z, Z(CD_1)Y$  or  
 $Y(CD_0)Z, Z(CD_1)Y$



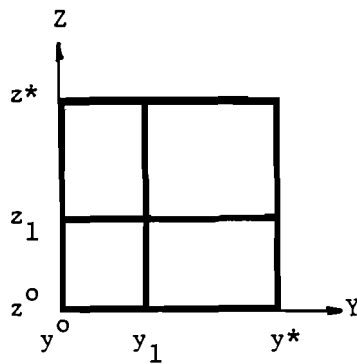
d) Fishburn's model  
 $Y(RCD_1)Z$



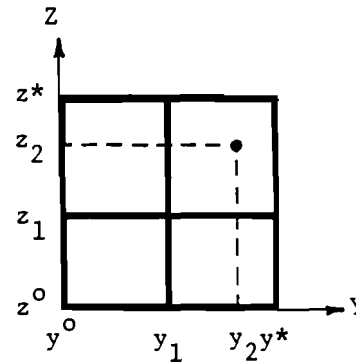
e) Bell's model  
 $Y(CD_1)Z, Z(CD_1)Y$



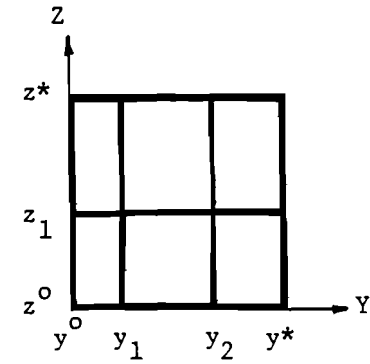
f)  $Y(CD_1)Z, Z(CD_2)Y$



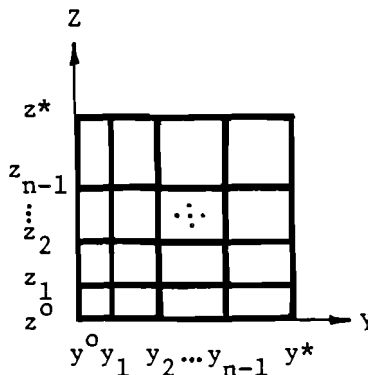
g)  $Y(RCD_2)Z$



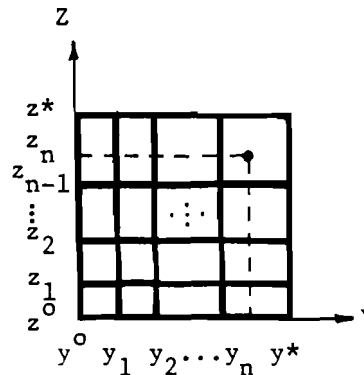
h)  $Y(CD_2)Z, Z(CD_2)Y$



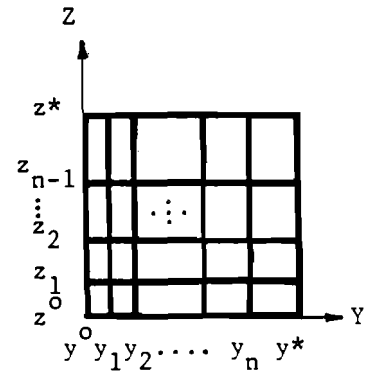
i)  $Y(CD_2)Z, Z(CD_3)Y$



j)  $Y(RCD_n)Z$



k)  $Y(CD_n)Z, Z(CD_n)Y$



l)  $Y(CD_n)Z, Z(CD_{n+1})Z$

FIGURE 4 Assigning utilities for heavy shaded consequences completely specifies the utility functions in the cases indicated. (Tamura and Nakamura, 1983)

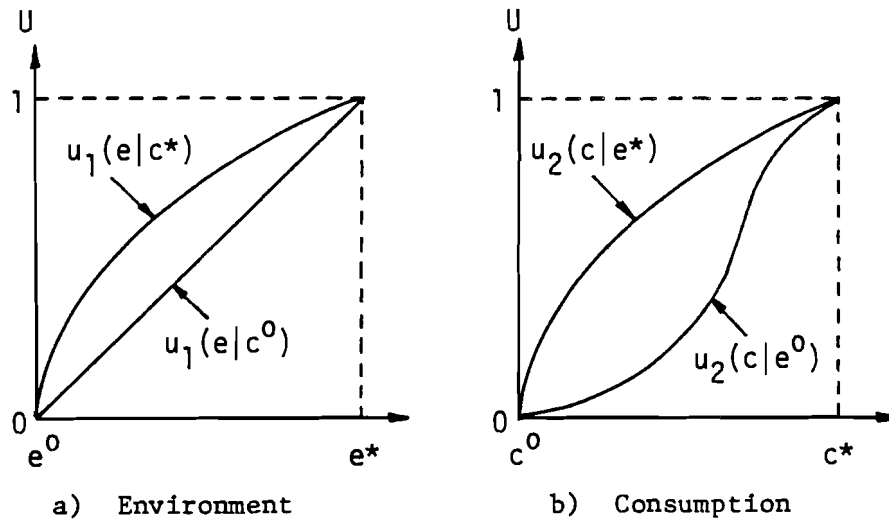


FIGURE 5 Normalized conditional utility functions for environment and consumption.

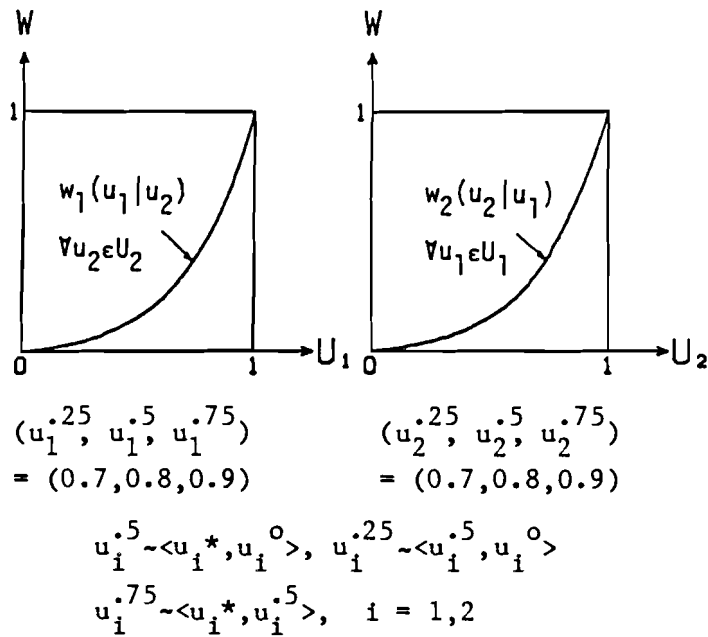


FIGURE 6 NCGUFs for Case A.

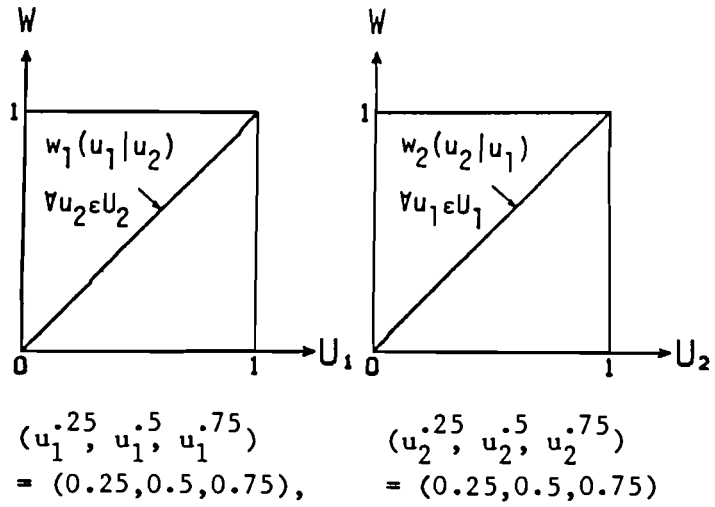


FIGURE 7 NCGUFs for Case B.

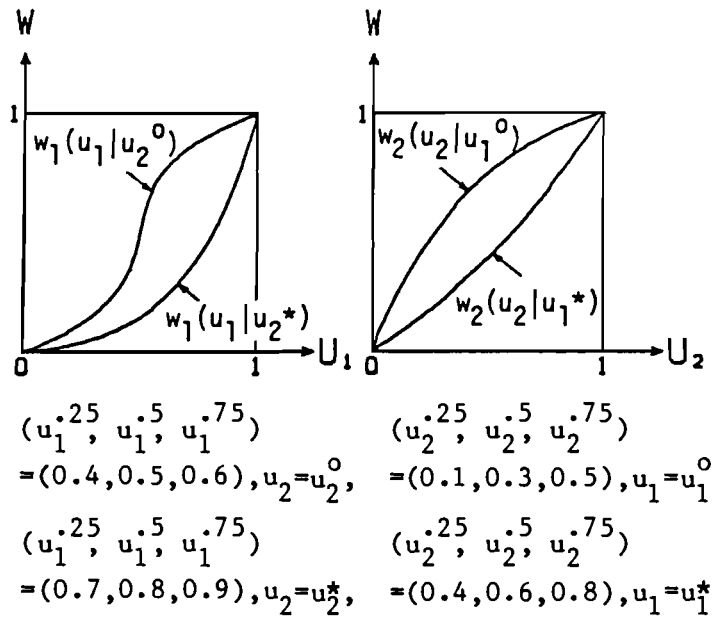


FIGURE 8 NCGUFs for Case C.

TABLE 1 Correspondence of symbols between multiattribute utility theory and group utility theory.

Multiattribute utility theory	Group utility theory
Two-attribute space: $Y \times Z$	Utility function space for two DMs: $U_1 \times U_2$
Attribute level: $y \in Y, z \in Z$	Utility level of each DM: $u_1 \in U_1, u_2 \in U_2$
Two-attribute utility function: $u(y, z)$	Group utility function for two DMs: $w(u_1, u_2)$
Normalized conditional utility functions: $u_1(y z), u_2(z y)$	NCGUFs: $w_1(u_1 u_2), w_2(u_2 u_1)$