GEOMETRIC APPROACH TO ISERMAN DUALITY IN LINEAR VECTOR OPTIMIZATION

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ABSTRACT:

In recent years, there have been several reports on duality in vector optimization. However, there seem to be no unified approach to dualization. In the author's previous paper, a geometric consideration was given to duality in nonlinear vector optimization. In this paper, some relationship among duality, stability (normality) and condition of alternative will be reported on the basis of some geometric consideration. In addition, Isermann's duality in linear cases will be derived from the stated geometric approach.

1. Review of Duality, Stability and Condition of Alternatives in Scalar Optimization

Let X' be a subset of an n-dimensional Euclidean space R^n and let $f:X'\longrightarrow R$ and $g:X'\longrightarrow R^m$. Then for the following traditional scalar objective optimization problem

$$\text{(P):} \qquad \text{Minimize} \quad \{f(x) \mid x\varepsilon X'\subset R^n,\ g(x)\leqq 0\},$$

an associated dual problem is given by

$$\text{(D):} \qquad \text{Maximize} \quad \{\phi(u) \mid u\geqq 0,\ \phi(u)=\inf\{L(x,u) \mid x\varepsilon X'\}\}$$

Here the vector inequality $\leqq$ is the usual one which is componentwise.

Now set

$$X = X' \cap \{x\in R^n \mid g(x)\leqq 0\}.$$

$$X(z)=\{x\varepsilon X' \mid g(x)\leqq z\},$$

$$w(z)=\inf\{f(x) \mid x\varepsilon X',\ g(x)\leqq z\}$$

and

$$\text{epi } w = \{(z,y) \mid y\geqq w(z),\ X(z)\neq\phi\}.$$

Under some appropriate convexity condition, it is well known that the set epi w is convex.

Definition 1.1 The <u>duality</u> between the problems (P) and (D) implies to hold

$$\inf\ \{f(x) \mid x\varepsilon X'\subset R^n,\ g(x)\leqq 0\} = \max\ \{\phi(u) \mid u\geqq 0\}.$$

Definition 1.2 The problem (P) is said to be <u>stable</u> if the function w(z) is subdifferentiable at z=0.

Note 1.1 w(z) is subdifferentiable at z=0 if and only if there exists a nonvertical supporting hyperplane for epi W at (0,w(0)).

Theorem 1.1 The duality holds if and only if inf(P) is finite and (P) is stable[14].

Note 1.2 If our interests for duality is in the condition under which inf(P)=sup(D) holds, we can use the following <u>normality</u> condition[16]:

$$\mathrm{cl}\ Y_G = Y_{\mathrm{cl}G}$$

where

$$G = \{(z,y) \mid y \geq f(x),\ z \geq g(x),\ x \in X'\}$$

$$Y_G = \{y \in R^1 \mid (0,y) \in G,\ 0 \in R^m\}$$

$$Y_{\mathrm{cl}G} = \{y \in R^1 \mid (0,y) \in \mathrm{cl}G,\ 0 \in R^m\}.$$

Definition 1.3 <u>The condition of alternative</u> involving the pairs (f,X) and (ϕ, R^m_+) implies that for any $\alpha \in (-\infty,\infty)$ exactly one of the following (I_α), (II_α) holds:

(I_α) $\exists\ x \in X$ such that $f(x) < \alpha$

(II_α) $\exists\ u \in R^m_+$ such that $\phi(u) \geq \alpha$.

Theorem 1.2 The duality holds if and only if the condition of alternative involving the pairs (f,X) and (ϕ, R^m_+) holds[8].

2. Vector Optimization

Let X be a set of alternative in an n-dimensional Euclidean space R^n, and let $f=(f_1,\dots,f_p)$ be a vector-valued criterion function from R^n into R^p. For given two vectors y^1 and y^2 and a pointed cone K, the following notations for cone-order will be used:

$$y^1 \leqq_K y^2 \iff y^2 - y^1 \in K$$

$$y^1 \leq_K y^2 \iff y^2 - y^1 \in K\backslash\{0\}$$

$$y^1 <_K y^2 \iff y^2 - y^1 \in \mathrm{int}\ K$$

Furthermore, the K-minimal and the K-maximal solution set of Y are defined, respectively, by

$$\mathrm{Min}_K\ Y := \{\bar{y} \in Y \mid \text{no } y \in Y \text{ such that } y \leq_K \bar{y}\}$$

$$\mathrm{Max}_K\ Y := \{\bar{y} \in Y \mid \text{no } y \in Y \text{ such that } y \geq_K \bar{y}\}.$$

Throughout this paper, for any cone K in R^p we denote the positive dual cone of K by K^o, that is,

$$K^o := \{p \in R^p \mid \langle p,q \rangle \geq 0 \text{ for any } q \in K\}$$

where $\langle p,q \rangle$ denotes the usual inner products of p and q, i.e., $p^T q$. For a K-convex set Y, a K-minimal solution $\bar{y}$ is said to be <u>proper</u>, if there exists $\mu \in \text{int } K^o$ such that

$$\langle \mu, y \rangle \geq \langle \mu, \bar{y} \rangle \quad \text{for all } y \in Y.$$

Then, a general type of nonlinear vector optimization may be formulated as follows:

$$\text{(VP):} \qquad \text{D-minimize } f(x) \text{ subject to } x \in X,$$

where $f=(f_1,\ldots,f_p)$ and

$$X := \{x \in X' \mid g(x) \leq_Q 0, \quad X' \subset R^n\}.$$

For a while in this section, we impose the following assumptions:

(i) X' is a nonempty compact convex set.

(ii) D and Q are pointed closed convex cones with nonempty interior respectively of R^p and R^m.

(iii) f is continuous and D-convex.

(iv) g is continuous and Q-convex.

Under the assumptions, it can be readily shown that for every $z \in R^m$, both sets

$$X(z) := \{x \in X' \mid g(x) \leq_Q z\}$$

and

$$Y(z) := f[X(z)]$$

$$:= \{y \in R^p \mid y=f(x), \quad x \in X', \quad g(x) \leq_Q z\} \qquad (2.1)$$

are compact, X(z) is convex and Y(z) is D-convex. Let us consider the primal problem (VP) by embedding it in a family of preturbed problems with Y(z) given by (2.1):

(VP_z): D-minimize $Y(z)$.

Clearly the primal problem (VP) is identical to the problem (VP_z) with $z=0$. Now define the set Z as

$$Z := \{z \in R^m \mid X(z) \neq \emptyset\}.$$

It is known that the set Z is convex (see, for example, Luenberger[6]).

Associated with the problem (VP), the point-to-set map defined by

$$W(z) := \mathrm{Min}_D \, Y(z)$$

is called a **perturbation (or primal) map**. It is known that

(i) for each $z \in Z$, $W(z)$ is a D-convex set in R^p,

(ii) the map $W(z)$ is D-monotone, namely,

$$W(z^1) \subset W(z^2) + D$$

for any z^1, $z^2 \in Z$ such that $z^1 \leqq_Q z^2$, (iii) $W(\cdot)$ is a D-convex point-to-set map (Tanino-Sawaragi[17]).

A **vector-valued Lagrangian** function for the problem (VP) is defined on X' by

$$L(x,U) = f(x) + Ug(x).$$

Hereafter, we shall denote by $\mathcal{U}$ a family of all $p \times m$ matrices U such that $UQ \subset D$. Such matrices are said to be <u>positive</u> in some literatures (Ritter[13], Corley[2]). Note that for given $\mu \in D^o \setminus \{0\}$ and $\lambda \in Q^o$ there exist $U \in \mathcal{U}$ such that

$$U^T \mu = \lambda.$$

In fact, for some vector e of D with $\langle \mu, e \rangle = 1$,

$$U = (\lambda_1 e, \lambda_2 e, \ldots, \lambda_m e)$$

is a desired one.

The point-to-set map $\Phi: \mathcal{U} \longrightarrow \mathcal{P}(R^p)$ defined by

$$\Phi(U) = \mathrm{Min}_D \{L(x,U) \mid x \in X'\}$$

is called a **dual map**, where $\mathcal{P}(R^p)$ denotes the power set of R^p. Using this terminology, a dual problem associated with primal problem (VP) can be defined in parallel with ordinary mathematical programming as follows (Tanino-Sawaragi[17]):

$$(VD_T): \qquad \text{D-maximize} \bigcup_{U \in \mathcal{U}} \Phi(U).$$

It is known that (i) for each U, $\Phi(U)$ is a D-convex set in R^p, (ii) $\Phi(U)$ is a D-concave point-to-set map, i.e., for any $U^1, U^2 \in \mathcal{U}$ and any $\alpha \in [0,1]$

$$\Phi(\alpha U^1+(1-\alpha)U^2) \subset \alpha\Phi(U^1)+(1-\alpha)\Phi(U^2) + D.$$

Proposition 2.1 (Tanino-Sawaragi[17]) If $\bar{x}$ is a proper D-minimal solution to Problem (VP), and if the Slater constraint qualification holds, i.e., there exists $x \in X'$ such that $g(x) <_Q 0$, then there exists a p×m matrix $\bar{U} \in \mathcal{U}$, such that

$$f(\bar{x}) \in \mathrm{Min}_D \{f(x)+\bar{U}g(x) \mid x \in X'\}, \qquad \bar{U}g(\bar{x})=0.$$

Proposition 2.2 (Tanino-Sawaragi[17]) Under the same condition as Proposition 2.1,

$$\mathrm{Min}_D(VP) \subset \mathrm{Max}_D(VD_T).$$

In the following, we shall review several results regarding geometric duality of vector optimization different from that of Tanino-Sawaragi and show a geometric approach to Isermann duality[4] in linear cases.

3. Geometric Duality of Nonlinear Vector Optimization

For given two sets $A \subset R^n$ and $B \subset R^n$, define

$$A_1 := A + D$$

$$B_1 := B - D.$$

Throughout this chapter, we assume that A is closed.

Definition 3.1 The condition of alternative (CA1) for vector optimization implies that for any $\alpha \varepsilon A_1 \cup B_1$ exactly one of the following (I_α), (II_α) holds:

(I_α) $\exists$ $a \varepsilon A$ such that $a \leq_D \alpha$

(II_α) $\exists$ $b \varepsilon B$ such that $b \geq_D \alpha$.

Theorem 3.1 Suppose that $Min_D A \neq \phi$. Then the condition of alternative (CA1) for vector optimization holds if and only if

$$Min_D\ A \subset Max_D\ B.$$

A proof of this theorem, which was originally given by Luc[6], follows via the following lemma:

Lemma 3.1 Define the conditions D1, D2, A1 and A2 as follows:

D1: $\forall\ a \varepsilon A,\ \forall\ b \varepsilon B,\ \ a \not\leq_D b$

D2: $\forall\ a \varepsilon Min_D A,\ \exists\ b \varepsilon B,\ \ a \leq_D b$

A1: $\forall\ \alpha \varepsilon A_1 \cup B_1,\ \ II_\alpha \Longrightarrow \text{not } I_\alpha$

A2: $\forall\ \alpha \varepsilon A_1 \cup B_1,\ \ \text{not } I_\alpha \Longrightarrow II_\alpha$

Then, D1 is equivalent to A1, and D2 is equivalent to A2.

(proof): D1==>A1: From the condition II_α, there exists some $b \varepsilon B$ such that $b \geq_D \alpha$. Suppose to the contrary that the condition I_α holds, i.e., there exists some $a \varepsilon A$ such that $a \leq_D \alpha$. Then we have $a \leq_D b$, which is contradictive to D1.

A1==>D1: Putting $\alpha = b$, the condition II_b holds. Therefore, for any $b \varepsilon B$ we have not I_b due to A1, i.e., there exists no $a \varepsilon A$ such that $a \leq_D b$, which is identical to D1.

D2==>A2: The negation of I_α for any $\alpha \varepsilon A_1 \cup B_1$ implies that for any $\alpha \varepsilon A_1 \cup B_1$ there exists no $a \varepsilon A$ such that $a \leq_D \alpha$. It follows then from the definition of $A_1 \cup B_1$ that $\alpha \varepsilon Min_D A$ or $\alpha \varepsilon B_1$. $\alpha \varepsilon Min_D A$ with D2 yields that there exists some $b \varepsilon B$ such that $\alpha \leq_D b$, which is also obtained in case of

$\alpha\varepsilon B_1$ from the definition of B_1.

A2==>D2: For any $a\varepsilon \mathrm{Min}_D A$, I_a (i.e., α=a) does not hold. It follows then from the condition of A2 that there exists some $b\varepsilon B$ for any $a\varepsilon \mathrm{Min}_D A$ such that $a\leq b$.

Remark 3.1 The condition D1 is well known as the weak duality. It is easy to see that we have the strong duality from D1 and D2.

Definition 3.2 The condition of alternative (CA2) for vector optimization implies that for any $\alpha\varepsilon R^p$ exactly one of (I_α), (II_α) holds.

The following lemma is substantial for understanding a geometric relationship between the condition of alternative (CA2) and the duality of vector optimization:

Lemma 3.2 Denoting the weak D-minimum solution set of A_1 by $w\text{-}\mathrm{Min}_D A_1$ and setting $W(A_1)=w\text{-}\mathrm{Min}_D A_1 \setminus \mathrm{Min}_D A_1$, then under the condition of alternative (CA2), we have the following:

(i) $\mathrm{int}\ A_1 \cap \mathrm{int}\ B_1 = \emptyset$

(ii) $A_1 \cup B_1 = R^n$

(iii) $W(A_1) \cap B_1 = \emptyset$

(proof) If (i) is false, then there exists a point $\alpha\varepsilon R^n$ such that both I_α and II_α hold. Furthermore, if (ii) is false, then there exists a point $\alpha\varepsilon R^n$ such that neither I_α nor II_α of the condition of alternative (CA2) hold. Finally, if (iii) is false, there exists $\bar{b}\varepsilon W(A_1)\cap B_1$. Then, by setting $\alpha=\bar{b}$, both I_α and II_α hold.

Theorem 3.2 Suppose that $\mathrm{Min}_D A \neq \phi$. Then, if the condition of alternative (CA2) for vector optimization holds, then

$$\mathrm{Min}_D\ A = \mathrm{Max}_D\ B.$$

(Proof) $\mathrm{Min}_D\ A \subset \mathrm{Max}_D\ B$ follows in the same way as in the proof of Theorem 3.1. Next, we shall show $\mathrm{Max}_D B \subset \mathrm{Min}_D A$. Suppose that $\bar{b} \in \mathrm{Max}_D B$. From Lemma 3.2, we have $\partial A_1 = \partial B_1$. Then according to Lemma 4.2 of Nakayama[9],

$$\text{w-Min}_D\ A_1 = \text{w-Max}_D\ B_1$$

Therefore, it follows from (iii) of Lemma 3.2 that

$$\bar{b} \in \text{w-Min}_D A_1 \setminus W(A_1) = \mathrm{Min}_D\ A$$

This completes the proof.

Definition 3.3 The function f from R^n to R^r is said to be <u>subdifferentiable</u> at $\bar{x}$ if there exists a matrix U such that

$$f(x) \not\leq f(\bar{x}) + U(x-\bar{x}) \quad \text{for any } x \in R^r.$$

Definition 3.4 The problem (VP) is said to be <u>stable</u> if $W(z)$ is subdifferentiable at 0.

Theorem 3.3[15] Let $\mathrm{Min}_D(\mathrm{VP}) \neq \phi$. Then the problem (VP) is stable if and only if there exists solutions $\bar{x}$ to the primal problem and $\bar{U}$ to the dual problem such that

$$f(\bar{x}) \in \Phi(\bar{U}).$$

Geometric duality in multiobjective optimzation have been given by Jahn[5] and Nakayama[9-11]. There some devices for dualization were made in such a manner that the condition of alternative (A2) for vector optimization holds (Note Theorem 3.2 and Lemma 3.2). We shall review them briefly. As in the previous section, the convexity assumption on f and g will be also imposed here, but X' is not necessarily compact.

Define

$$G := \{(z,y)\in R^m \times R^p \mid y \geq_D f(x),\ z \geqq_Q g(x),\ x\in X'\},$$

$$Y_G := \{y\in R^p \mid (0,y)\in G,\ 0\in R^m,\ y\in R^p\}.$$

We restate the primal problem as

$$\text{(VP):} \qquad \text{D-minimize} \quad \{f(x) \mid x\in X\},$$

where

$$X := \{x\in X' \mid g(x) \leqq_Q 0,\ X'\in R^n\}.$$

Associated with this primal problem, the dual problem formulated by Nakayama[9] is as follows:

$$(\text{VD}_N): \qquad \text{D-maximize} \quad Y_{S(U)} \quad \text{where}$$

$$Y_{S(U)} := \{y\in R^p \mid f(x)+Ug(x) \not\leq_D y, \quad \text{for all } x\in X'\}.$$

On the other hand, the one given by Jahn[5] is

$$(\text{VD}_J): \qquad \text{D-maximize} \quad \bigcup_{\substack{\mu\,\in\,\text{int } D^o \\ \lambda\,\in\, Q^o}} Y_{H^-(\lambda,\mu)}$$

where

$$Y_{H^-(\lambda,\mu)} := \{y\in R^p \mid \langle\mu,f(x)\rangle + \langle\lambda,g(x)\rangle \geq \langle\mu,y\rangle \quad \text{for all } x\in X'\}.$$

Proposition 3.1 (weak duality)

(i) For any $y\in \bigcup_{U\in\mathcal{U}} Y_{S(U)}$ and for any $x\in X$,

$$y \not\leq_D f(x).$$

(ii) For any $y \in \bigcup_{\substack{\mu\in\,\text{int } D^o \\ \lambda\in D^o}} Y_{H^-(\lambda,\mu)}$ and for any $x\in X$

$$y \not\leq_D f(x).$$

Proposition 3.2 (Nakayama[9]) Suppose that G is closed, and that there is at least a properly efficient solution to the primal problem. Then, under the condition of Slater's constraint qualification,

$$(Y_G)^c \subset \bigcup_{\substack{\mu\in\,\text{int } D^o \\ \lambda\in\, Q^o}} Y_{H^-(\lambda,\mu)} \subset \bigcup_{U\in\mathcal{U}} Y_{S(U)} \subset \text{cl } (Y_G)^c.$$

Lemma 3.3 (Nakayama[9]) The following holds:

$$\mathrm{Min}_D\ (VP) = \mathrm{Min}_D\ Y_G.$$

Proposition 3.3 (strong duality)[9,5] Assume that G is closed, that there exists at least a D-minimal solution to the primal problem, and that these solutions are all proper. Then, under the condition of Slater's constraint qualification, the following holds:

(i) $\mathrm{Min}_D\ (VP) = \mathrm{Max}_D\ (VD_N)$

(ii) $\mathrm{Min}_D\ (VP) = \mathrm{Max}_D\ (VD_J)$.

In some cases, one might not so much as expect that the G is closed. In this situation, we can invoke to some apropriate normality condition in order to derive the duality. In more detail, see for examle, Jahn[5], Borwein-Nieuwenhuis[1], and Sawaragi-Nakayama-Tanino[15]. In linear cases, fortunately, it is readily seen that the set G is closed. In addition, we have G = epi W, if there exists no $x \in M$ such that $(C-UA)x \leq_D 0$ as will be seen later. Therefore, we can derive Isermann's duality[4] in linear cases via the stated geometric duality. We shall discuss this in the following section.

4. Geometric Approach to Isermann's Duality in Linear Cases

Let D, Q and M be pointed convex polyhedral cones in R^p, R^m and R^n, respectively. This means, in particular, that $\mathrm{int}\ D^o \neq \emptyset$. Isermann[4] has given an attractive dualization in linear cases. In the following, we shall consider it in an extended form.

(VP_I): D-minimize $\{Cx:\ x \in X\}$ where $X := \{x \in M:\ Ax \geq_Q b\}$.

(VD_I): D-maximize $\{Ub:\ U \in \mathcal{U}_0\}$

where $\mathcal{U}_0 := \{U \in R^{p \times m} |$ there exists $\mu \in \mathrm{int}\ D^o$ such that

$$U^T\mu\varepsilon Q^o \text{ and } A^T U^T \mu \leqq_{M^o} C^T\mu\}.$$

Then Isermann's duality is given by

Theorem 4.1

(i) $Ub \not\leq_D Cx$ for all $(U,x)\varepsilon\, \mathcal{U}_0 x X$.

(ii) Suppose that $\bar{U}\varepsilon\, \mathcal{U}_0$ and $\bar{x}\varepsilon X$ satisfy

$$\bar{U}b = C\bar{x}.$$

Then $\bar{U}$ is a D-maximal solution to the dual problem (VD_I) and $\bar{x}$ is a D-minimal solution to the primal problem (VP_I).

(iii) $\text{Min}_D\ (VP_I) = \text{Max}_D\ (VD_I)$.

Proposition 4.1 Let $f(x)=Cx$, $g(x)=Ax-b$ and $X'=M$, where C and A are $r\times m$ and $m\times n$ matrices, respectively and M is a pointed closed convex cone in R^n. Then every supporting hyperplane, $H(\lambda,\mu:\gamma)$ $(\gamma=\langle\mu,\bar{y}\rangle +\langle\lambda,\bar{z}\rangle$, for epi W at an arbitrary point $(\bar{z},\bar{y})$ such that $\bar{y}\varepsilon W(\bar{z})$ passes through the point $(z,y)=(b,0)$ independently of $(\bar{z},\bar{y})$. In addition, we have $\mu\varepsilon \text{int } D^o$, $\lambda\varepsilon Q^o$ and

$$C^T\mu - A^T\lambda \geqq_{M^o} 0. \qquad (4.1)$$

Conversely, if $\mu\varepsilon D^o$ and $\lambda\varepsilon Q^o$ satisfy the relation (4.1), then the hyperplane with the normal (λ,μ) passing through the point $(z,y)=(b,0)$ supports epi W.

(Proof): It has been shown in [9] that if the hyperplane $H(\lambda,\mu:\gamma)$ supports epi W, then $\mu\in D^o$ and $\lambda\in Q^o$. Further, since every efficient solution for linear cases is proper (See, for example, Sawaragi, Nakayama and Tanino[10]), we have $\mu\in \text{int } D^o$. Now, note that since $\hat{y}\in W(\hat{z})$, there exists $\hat{x}\in R^p$ such that

$$C\hat{x} = \hat{y}$$

$$b - A\hat{x} \leqq_Q \hat{z}.$$

Therefore, it follows from the supporting property of the hyperplane $H(\lambda,\mu:\gamma)$ that for any $(z,y)\in \text{epi } W$

$$\langle\mu,y\rangle + \langle\lambda,z\rangle \geq \langle\mu,\hat{y}\rangle + \langle\lambda,\hat{z}\rangle$$
$$\geq \langle\mu,C\hat{x}\rangle + \langle\lambda, b-A\hat{x}\rangle, \qquad (4.2)$$

where the last half part of (4. 2) follows from the fact that $\lambda \in Q^{o}$ and $\hat{z} - (b- A\hat{x}) \in Q$. Since $(b-Ax, Cx) \in \text{epi } W$ for any $x \in M$, the relation (4.2) yields that for any $x \in M$

$$\langle\mu, Cx\rangle + \langle\lambda, b-Ax\rangle \geq \langle\mu, C\hat{x}\rangle + \langle\lambda, b-A\hat{x}\rangle.$$

Consequently, for any $x \in M$

$$\langle C^{T}\mu-A^{T}\lambda, x-\hat{x}\rangle \geq 0$$

and hence for any $x-\hat{x} \in M$

$$\langle C^{T}\mu-A^{T}\lambda, x-\hat{x}\rangle \geq 0$$

Thereofore,

$$C^{T}\mu - A^{T}\lambda \geq_{M^{o}} 0. \qquad (4.3)$$

Seeing that the point $(b,0)$, which corresponds to $x=0$, belongs to epi W, it follows from (4.2) and (4.3) that

$$\langle\mu, y\rangle + \langle\lambda, z\rangle = \langle\lambda, b\rangle.$$

This means that the supporting hyperplane $H(\lambda,\mu:\gamma)$ passes through the point $(z,y)=(b,0)$ independently of the given supporting point $(\hat{z},\hat{y})$.

Conversely, suppose that $\mu \in D^{o}$ and $\lambda \in Q^{o}$ satisfy the relation (4.1). Recall that for every $(z,y) \in \text{epi } W$ there exists $x \in M$, which may depend on (z,y), such that

$$y \in Cx + D \quad \text{and} \quad z - (b - Ax) \in Q.$$

It follows, therefore, that for any $\mu \in D^{o}$ and $\lambda \in D^{o}$

$$\langle\mu, y-Cx\rangle \geq 0 \quad \text{and} \quad \langle\lambda, z-b+Ax\rangle \geq 0. \qquad (4.4)$$

Hence, by using the relation (4.1), we have from (4.4)

$$\langle\mu,y\rangle + \langle\lambda,z\rangle \geq \langle\lambda,b\rangle \qquad (4.5)$$

for every $(z,y) \in \text{epi } W$. The realtion (4.5) shows that the hyperplane $H(\lambda,\mu:\gamma)$ passing through the point $(b,0)$ and satisfying $C^T\mu \geq_{M^o} A^T\lambda$ supports epi W. This completes the proof.

The following lemma is an extension of the well known Stiemke's theorem and provides a key to clarify a relationship between Isermann's formulation and our geometric approach.

Lemma 4.1 There exists some $\mu \in \text{int } D^o$ such that

$$(C-UA)^T\mu \geq_{M^o} 0 \tag{4.6}$$

if and only if there exists no $x \in M$ such that

$$(C-UA)x \leq_D 0. \tag{4.7}$$

Proof: Suppose first that there exists some $\mu \in \text{int } D^o$ such that (4.6) holds. If some $x \in M$ satisfy (4.7), or equivalently,

$$(C-UA)x \in (-D)\backslash\{0\}$$

then since $\mu \in \text{int } D^o$

$$\langle \mu, (C-UA)x \rangle < 0$$

which contradicts (4.6). Therefore, there is no $x \in M$ such that (4.7) holds.

Conversely, suppose that there exists no $x \in M$ such that (4.7) holds. This means

$$(C-UA)M \cap (-D) = \{0\},$$

from which we have

$$((C-UA)M)^o + (-D)^o = R^n.$$

Hence for an arbitrary $\mu_0 \in \text{int } D^o$ there exists $\mu_1 \in ((C-UA)M)^o$ and $\mu_2 \in (-D)^o$ such that

$$\mu_0 = \mu_1 + \mu_2 \tag{4.8}$$

and thus

$$\mu_1 = -\mu_2 + \mu_0.$$

Since $-\mu_2 \in D^o$ and $\mu_0 \in \text{int } D^o$, it follows from (4.8) that we have $\mu_1 \in ((C-UA)M)^o \cap \text{int } D^o$. Consequently, recalling that $((C-UA)M)^o = \{\mu \mid (C-UA)^T\mu \geq_{M^o} 0\}$, the existence of $\mu \in \text{int } D^o$ satisfying (4.6) is established. This completes the proof.

Proposition 4.2 For linear cases with $b \neq 0$,

$$\bigcup_{U \in \mathcal{U}_0} \{Ub\} = \bigcup_{U \in \mathcal{U}_0} \Phi(U) = \bigcup_{\substack{\lambda \in Q^o \\ \mu \in \text{int } D^o}} Y_{H(\lambda,\mu)}$$

Proof: According to Proposition 4.1 with $f(x)=Cx$ and $g(x)=Ax-b$, for $\mu \in \text{int } D^o$ and $\lambda \in Q^o$ such that $C^T\mu \geq_{M^o} A^T\lambda$, we have

$$\langle \mu, f(x) \rangle + \langle \lambda, g(x) \rangle \geq \langle \lambda, b \rangle \quad \text{for all } x \in M.$$

Therefore, for $U \in R^{p\times m}$ such that $U^T\mu=\lambda$

$$\langle \mu, f(x)+Ug(x) \rangle \geq \langle \mu, Ub \rangle \quad \text{for all } x \in M,$$

which implies by virtue of the well known scalarization property and $\mu \in \text{int } D^o$ that

$$f(x) + Ug(x) \not\leq_D Ub \quad \text{for all } x \in X', \tag{4.9}$$

Hence for $U \in \mathcal{U}_0$

$$Ub \in \Phi(U),$$

which leads to $\bigcup_{U \in \mathcal{U}_0} \{Ub\} \subset \bigcup_{U \in \mathcal{U}_0} \Phi(U)$.

Next in order to show $\bigcup_{U \in \mathcal{U}_0} \Phi(U) \subset \bigcup_{\substack{\lambda \in Q^o \\ \mu \in \text{int } D^o}} Y_{H(\lambda,\mu)}$, suppose that $\bar{y} \in \Phi(U)$ for some $U \in \mathcal{U}_0$. Suppose further that $U^T\mu=\lambda$ and $C^T\mu \geq_{M^o} A^T\lambda$ for some $\mu \in \text{int } D^o$ and some $\lambda \in Q^o$. Then since from Lemma 4.1 we have $(C-UA)x \not\leq_D 0$ for all $x \in M$, we can guarantee the existence of an efficient solution $\bar{x} \in M$

for the vector valued Lagrangian $L(x,U)=Cx+U(b-Ax)$ such that $\bar{y}=C\bar{x}+U(b-A\bar{x})$. Moreover, since $L(.,U)$ is a convex vector-valued function over M for each U, due to the efficiency of $\bar{x}$ for $L(x,U)$ there exists $\bar{\mu}\in \text{int } D^{o}$ such that

$$\langle\bar{\mu},\ C\bar{x}+U(b-A\bar{x})\rangle \leq \langle\bar{\mu},\ Cx+U(b-Ax)\rangle \quad \text{for all } x\in M. \tag{4.10}$$

Hence, letting $\bar{\lambda}=U^{T}\bar{\mu}$

$$\langle\bar{\mu},\ \bar{y}\rangle \leq \langle\bar{\mu},\ y\rangle + \langle\bar{\lambda},\ z\rangle \quad \text{for all } (z,y)\in \text{epi } W. \tag{4.11}$$

which implies that $\bar{y}\in Y_{H(\lambda,\mu)}$. This establishes the desired inclusion.

Finally, we shall show $\bigcup_{\substack{\mu\in \text{int } D^{o} \\ \lambda\in D^{o}}} Y_{H(\lambda,\mu)} \subset \bigcup_{U\in\mathcal{U}_0} \{Ub\}$. Suppose now that $\bar{y}\in Y_{H(\lambda,\mu)}$ for some $\mu\in \text{int } D^{o}$ and $\lambda\in Q^{o}$. Since $(b,0)$ is a supporting point of $H(\lambda,\mu)$ for epi W according to Proposition 4.1, we have

$$\langle\mu,f(x)\rangle + \langle\lambda,g(x)\rangle \geq \langle\lambda,b\rangle \quad \text{for all } x\in X' \tag{4.12}$$

and

$$\langle\mu,\bar{y}\rangle = \langle\lambda,b\rangle \tag{4.13}$$

Since $b\neq 0$, recall that the relation (4.13) shows that two equations $U^{T}\mu=\lambda$ and $Ub=\bar{y}$ have a common solution $U\in R^{p\times m}$ (Penrose[12]). In other words, we have $\bar{y} = Ub$ for some $U\in R^{p\times m}$ such that $U^{T}\mu=\lambda$, which leads to $\bar{y}\in \bigcup_{U\in\mathcal{U}_0} \{Ub\}$. This establishes the desired inclusion.

Now we can obtain the Isermann duality for linear vector cases via Propositions 3.2-3.3 and 4.2 :

Theorem 4.1

For $b\neq 0$,

$$\text{Min}_D\ (P_I) = \text{Max}_D\ (D_I).$$

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