WORKING PAPER

ON THE GUARANTEED STATE ESTIMATION PROBLEM FOR PARABOLIC SYSTEMS

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FOREWORD

This report deals with the problem of guaranteed estimation of the state of a distributed system on the basis of available measurements. The disturbances in the initial distribution, in the system inputs and in the available observations are assumed to be unknown in advance. No statistical information on these is given and it is only the restrictions on the possible realizations of these functions that are taken to be available. The inverse problem which arises here therefore reduces to the description of the "informational set" of all solutions that are consistent with the system equation, the available measurement and the constraints on the uncertainties. A minimax guaranteed estimate may then be specified, which coincides, in the case of quadratic integral constraints, with the regularizator introduced by A.N. Tikhonov for treating ill-posed inverse problems of mathematical physics.

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A. B. Kurzhanski and A. Yu. Khapalov

Introduction

This report deals with the problem of guaranteed estimation of the state of a distributed system on the basis of available measurements. The disturbances in the initial distribution, in the system inputs and in the available observations are assumed to be unknown in advance. No statistical information on these is given and it is only the restrictions on the possible realizations of these functions that are taken to be available. The inverse problem which arises here therefore reduces to the description of the "informational set" of all solutions that are consistent with the system equation, the available measurement and the constraints on the uncertainties. A minimax guaranteed estimate $U^o(\theta , \bullet)$ similar to the "Chebyshev center" of a given set may then be specified, which coincides, in the case of quadratic integral constraints, with the regularizator introduced by A.N. Tikhonov for treating ill-posed inverse problems of mathematical physics, [4]. The evolution of the sets $U(\theta , \bullet)$ and of the estimates $u^o(\theta , \bullet)$ is also specified.

The description given in the sequel is related to parabolic systems.

1. The Estimation Problem

In a bounded domain Ω of the space $\mathbf{R}^n$ consider a distributed field described as the solution to the problem

$$\frac{\partial u\ (x\ ,\ t)}{\partial t} = L\ u(x\ ,\ t) + f(x\ ,\ t) \tag{1.1}$$

$$t \in \mathbf{T}_\theta = [0\ ,\ \theta]\ ,\ x \in \Omega \subseteq \mathbf{R}^n$$

$$u(x\ ,\ o) = u_o(x)\ ,\ u(x\ ,\ t)\ |_{\partial\Omega} = 0 \tag{1.2}$$

Here $\partial\Omega$ is the boundary of Ω

$$L = \sum_{i,j=1}^{\infty} \frac{\partial}{\partial x_i}\ (a_{ij}(x)\ \frac{\partial}{\partial x_j}) + c(x)\ ,$$

is a nondegenerate symmetric elliptic operator with given coefficients $a_{ij}(x)$, $c(x)$. Assuming $f\ (\bullet) \in L_2\ (T_\theta\ ;\ L_2(\Omega))$, $u_o\ (\bullet) \in L_2(\Omega)$ we will consider $u = u(x\ ,\ t)$ as a weak solution of problem (1.1), (1.2), treating it as an element of the space $L_2(T_\theta\ ;\ H_0^1(\Omega))$ so that its traces $u(\bullet\ ,\ \tau)$ over the cross-sections Ω_τ of $(\Omega \times T_\theta)$ are the elements of $L_2(\Omega)$ that vary continuously in t in the metric of $L_2(\Omega)$ [1, 2].

Here

$$H_0^1\ (\Omega) = \left\{\varphi \mid \varphi \in L_2(\Omega)\ ,\ \frac{\partial \varphi}{\partial x_i} \in L_2(\Omega)\ \ ,\ \varphi \mid_{\partial\Omega} = 0\right\}$$

is a Sobolev space, $L_2(T\ ,\ B)$ is the space of square integrable functions that map T onto B.

It is further assumed that the parameters of the system (1.1) are such (see [2, 3]) that there exists a unique solution to the problem (1.1), (1.2), which may be represented in the form

$$u(\bullet\ ,\ t) = G(t)\ u_0(\bullet) + \int_0^t G(t-\sigma) f(\bullet\ ,\ \sigma)\ d\sigma\ ,$$

where

$$G(t) \in \mathbb{L}(L_2(\lambda)\ ;\ L_2(\Omega))$$

is a strongly continuous semigroup in $L_2(\Omega)$, defined by the unbounded operator L.

$$G(t)\ u_0(\bullet) = \int_\Omega G(x\ ,\ y\ ,\ t) u_0(y)\ dy\ ,$$

Here $G(x\ ,\ y\ ,\ t)$ is a Greene function for problem (1.1), (1.2), i.e.

$$\frac{\partial G(x\ ,\ y\ ,\ t)}{\partial x} = LG(x\ ,\ y\ ,\ t)\ ,\ x\ ,\ y \in \Omega\ ,\ t \in T_\theta\ ,$$

$$G(x\ ,\ y\ ,\ t) \mid_{\partial(\Omega\ \times\ \Omega)} = 0\ ,\ G(x\ ,\ y\ ,\ 0) = \delta(x-y)\ .$$

The "input functions" $f(x\ ,\ t), u_o(x)$ are taken to be unknown in advance. However, it is presumed that they satisfy some preassigned constraints which will be specified below.

It is understood that the solution $u(x,t)$ is inaccessible for direct measurement. The available information on $u(x,t)$ is given through a "measurement equation".

$$y(t) = \mathbf{G}(t)\ u(\bullet\ ,\ t) + \eta(t)\ ,\ t \in \mathbf{T}_\theta \tag{1.3}$$

where $y(t)$ is the available measurement observation $(y \in \mathbf{R}^m)$, $\mathbf{G}(t)$ is a linear (nonstationary) operator that maps $L_2(\Omega)$ into $\mathbf{R}^m$, $\eta(t)$ is the measurement "noise" $(\eta(\bullet) \in L_2^m\ (\mathbf{T}_\theta)\ ,\ y(\bullet) \in L_2^m(T_\theta))$. The *operator* $\mathbf{G}(t)$ describes the structure of the observations.

The problem is to estimate $u(x\ ,\ \theta)$ - the solution at instant θ on the basis of the observation $y(\bullet)$. It is assumed that the only information on the unknown "inputs" and "noise" $f\ ,\ u_o\ ,\ \eta$ is the restriction of these functions to a given preassigned set

$$V \subseteq L_2\ (T_\theta\ ;\ L_2(\Omega)) \times L_2(\Omega) \times L_2^m\ (T_\theta)\ :$$

$$\omega\ (\bullet) = \{f(\bullet)\ ,\ u_o(\bullet)\ ,\ \eta(\bullet)\} \in V\ . \tag{1.4}$$

Therefore the problem is to determine the solution or solutions $u(x, t)$ that satisfy relations (1.1) - (1.4) for a prescribed "measurement" $y(\bullet)$. (In general the solution $u(x, t)$ to (1.1) - (1.4), $y(t)$ given, is obviously nonunique). This leads us to the following.

Definition 1.1 The ***informational domain*** $U(\theta, y(\bullet))$ of the states $u(x, \theta)$ of system (1.1), (1.2) that are consistent with measurement $y(t)$ of (1.3) and with restriction (1.4) is the set of all those functions $u(x, \theta)$ for each of which there exists a triplet $w*(\bullet) = \{f*(\bullet), u_o*(\bullet), y*(\bullet)\}$ that satisfies (1.4) and generates a pair $u*(x, \theta), y*(t)$ (due to (1.1) - (1.3)) that satisfies the equalities $u*(x, \theta) = u(x, \theta), y*(t) \equiv y(t), t \in T_\theta$.

It is clear that set $U(\theta, y(\bullet))$ always includes the unknown actual state $u(x, \theta)$ of the system. Therefore we are to specify set $U(\theta, y(\bullet))$ and its evolution in θ. The set $U(\theta, y(\bullet))$ is convex if V is convex. With $U(\theta, y(\bullet))$ convex it also makes sense to determine a "guaranteed" estimate $u^o(x, \theta)$ of the actual state $u(x, \theta)$ according to the relation

$$\sup \{ || u(\bullet) - u^o(\bullet, \theta) || \; | u(\bullet) \in U(\theta, y(\bullet))\} = \tag{1.5}$$
$$\inf \{ \sup\{ || u(\bullet) - \omega(\bullet) || \; | u(\bullet) \in U(\theta, y(\bullet))\} \mid \omega(\bullet) \in L_2(\Omega)\}$$

(for a prescribed norm $|| \bullet ||$). Element $u^o(\bullet, \theta)$ is known as the *Chebyshev center* for $U(\theta, y(\bullet))$.

The solution to the problems of the above may be specified more explicitly for specific types of sets V.

2. The Solutions for Integral Constraints

Assume the set V is defined by a quadratic integral functional

$$\int_\Omega (u_o(x) - u_o*(x))^2 \, m(x) \, dx +$$
$$+ \int_o^\theta \int_\Omega (f(x, t) - f*(x, t)^2 \, k(x, t) \, dx \, dt + \tag{2.1}$$
$$+ \int_o^\theta (\eta(t) - \eta*(t))' \, N(t) \, (\eta(t) - \eta*(t)) \, dt \le \mu^2$$

where the functions $m(x) > 0$, $k(x, t) > 0$, the matrix $N(t) \geq 0$ and the triplet $\omega*(\bullet) = \{f*(\bullet) \; u*_o(\bullet), \eta*(\bullet)\}$ are given in advance. The solution may now be calculated explicitly.

Theorem 2.1 *Under the constraint* (2.1) *on* $\omega(\bullet) = \{f(\bullet), u(\bullet), \eta(\bullet)\}$ *the set* $U(\theta, y(\bullet))$ *is an ellipsoid in the sense that its support function*

$$\rho(\varphi(\bullet) \mid U(\theta, y(\bullet))) = \sup\{ < \varphi(\bullet), u(\bullet) > \mid \tag{2.2}$$

$$| \; u(\bullet) \in U(\theta , y(\bullet))\}$$

$$= (\mu^2 - h^2(\theta))^{1/2} \left(P(\theta , \varphi(\bullet)) - \int_\Omega \int_\Omega \varphi(x) \, B(x , y , \theta) \, \varphi(y) \, dx \, dy\right)^{1/2}$$

$$+ \int_\Omega u^o(\theta , x) \, \varphi(x) \, dx$$

for any element $\varphi(\bullet)$ *of the subset* $\Phi \subseteq L_2(\bullet)$ *that defines the weak solutions.*

Here $< \varphi(\bullet) , u(\bullet) >$ stands for the scalar product in $L_2(\bullet)$. Taking $|| \varphi(\bullet) ||^2 = < \varphi(\bullet) , \varphi(\bullet) >$ in (1.5) it is possible to see that the function $u^0(x , \theta)$ in (2.2) is the solution to problem (1.5).

The evolution of $u(\theta , y(\bullet))$ in θ may now be described by the evolution of the functions $u^o(\theta , x)$, $h^2(\theta)$, $B(x , y, , \theta)$ and $P(\theta , \varphi(\bullet))$ in θ. These are described by the following relations

$$\frac{\partial u^o(x , \theta)}{\partial \theta} = L u^o(x , \theta) + (y(\theta) - \mathbf{G}(\theta) \, u^o(\bullet , \theta) - \tag{2.3}$$

$$- \eta*(\theta))' \, N(\theta) \, (f(x , \theta , \theta) - \mathbf{G}(\theta) \, B(\bullet , x , \theta)) + f*(x , \theta) ,$$

$$x \in \Omega , \theta > 0 ; u^o(x , \theta) \, |_{x \in \partial \Omega} = O , u^o(x , \theta) \, |_{\theta = 0} = u_o*(x)$$

$$\frac{dh^2(\theta)}{d\theta} = (y(\theta) - \mathbf{G}(\theta) \, u^o(\bullet , \theta) - \eta*(\theta))' \, N(\theta) \times \tag{2.4}$$

$$\times \, (y(\theta) - \mathbf{G}(\theta) \, u^o(\bullet , \theta) - \eta*(\theta)) , \; h^2(o) = 0 \quad \theta > 0$$

$$\frac{\partial B(x , y , \theta)}{\partial \theta} = L_x \, B(x , y , \theta) + L_y \, B(x , y , \theta) + \tag{2.5}$$

$$+ (\, f(x , \theta , \theta) - \mathbf{G}(\theta) \, B(x , \bullet , \theta))' \, N(\theta) \, (\, f(y , \theta , \theta) -$$

$$- \mathbf{G}(\theta) \, B(\bullet , y , \theta)) \; ; x , y , \in \Omega , \theta > 0$$

$$B \, (x , y , \theta) \, |_{\theta = 0} = B \, (x , y , \theta) \, |_{\partial(\Omega \times \Omega)} = 0$$

Here $L_x = L$, L_y is defined similarly (with x substituted for y), $P(\theta , \varphi(\bullet))$, $f(x , \theta , \theta)$ are computed by means of the following formulae:

$$P(\theta , \varphi(\bullet)) = \int_\Omega \left[\int_\Omega \mathbf{G}(x , y , \theta) \, \varphi(y) \, dy \right]^2 m^{-1}(x) \, dx + \tag{2.6}$$

$$\int_0^\theta \int_\Omega \left[\int_\Omega G(x , y , \theta - t) \, \varphi(y) \, dy \right] k^{-1} \, (x , t) dx \, dt ,$$

$$f(x , t , \theta) = \int_\Omega G(x , y , \theta) \, (\mathbf{G}(t) \, G(\bullet , y , t)) m^{-1}(y) \, dy + \tag{2.7}$$

$$+ \int_0^t \int_\Omega G(x\,,\,y\,,\,\theta - \tau)\ (\mathbf{G}(t)\ G(\bullet\,,\,y\,,\,t - \tau))k^{-1}\,(y\,,\,\tau)\ dy\ d\tau$$

Formulae (2.3), 2.5) - (2.7) are similar to those for infinite - dimensional stochastic filtering [5,6], however here the analogy ends, since relations (2.2) and (2.4) are only specific for the deterministic approach discussed in this paper. This approach yields some further relations that are important for estimating the solutions of distributed systems.

The relations of the above allow to derive some formulae for approximating $U(\theta\,,\,y(\bullet))$ in case of "separate" constraints

$$\int_\Omega (u_o(x) - u_o{*}(x))^2\ m_s(x)\ dx \le 1$$

$$\int_0^\theta \int_\Omega (f(x\,,\,t) - f{*}(x\,,\,t))^2\ k_s(x\,,\,t)\ dx\ dt \le 1 \tag{2.8}$$

$$\int_0^\theta (\eta(t) - \eta{*}(t))'\ N_s(t)\ (\eta(t) - \eta{*}(t))\ dt \le 1$$

$$(m_s(x) > 0\,,\ k_s(x\,,\,t) > 0\,,\ N_s(t) \ge 0)$$

The solution to the problem with constraint (2.8) is reduced to the previous problem with restriction (2.1) where $m(x)\,,\,k(x\,,\,t)\,,\,N(t)$ are substituted for $\alpha\ m_s(x)\,,\,\beta\ k_s(x\,,\,t)\,,\,\gamma\ N_s(t)$. Assuming $\varsigma = \{\alpha\,,\,\beta\,,\,\gamma\,;\,\alpha > 0\,,\,\beta > 0\,,\,\gamma > 0\,\}$; we will denote the respective informational domain as $U_\varsigma(\theta\,,\,y(\bullet))$. The result is then given by **Lemma 2.1.** *Under the constraint (2.8) the following relations are true*

$$U(\theta\,,\,y(\bullet)) = \bigcap\ \{U_\varsigma\,(\theta\,,\,y(\bullet))\ |\ \alpha + \beta + \gamma = 1\,\}$$

$$\rho(\varphi(\bullet)\ |\ U(\theta\,,\,y(\bullet)) = \inf\,\{\rho(\varphi(\bullet)\ |\ U_\varsigma(\theta\,,\,y(\bullet)))\ |\ \alpha + \beta + \gamma = 1),\ \forall\ \varphi(\bullet) \in \Phi$$

A specific subproblem is to find the "worst case" measurement $y_w(\bullet)$ which will be defined as such for which the domain $U(\theta\,,\,y_w(\bullet))$ would be the "largest" possible. For restriction (2.1) we observe that in formula (2.2) the operators $P(\theta\ \varphi(\bullet))$, $B(x\,,\,y\,,\,\theta)$ do not depend upon $y(\bullet)$. This reduces the problem to finding the measurement $y_w(\bullet)$ for which $\mu^2 - h^2(\theta)$ would be the maximal possible.

Lemma 2.2. *For restriction* (2.1) *the measurement* $y_\omega(\bullet)$ *that ensures the existence of a function* $b(\bullet)$ *such that*

$$\rho\ (\varphi(\bullet)\ |\ U(\theta\,,\,y_\omega(\bullet))) + \langle \varphi\,(\bullet)\,,\,b(\bullet) \rangle \ge \rho\ (\ \varphi(\bullet)\ |\ U(\theta\,,\,y(\bullet)))$$

for any $\varphi(\bullet) \in \Phi$ *and any feasible* $y(\bullet)$ *is the one generated by the triple* $\omega{*}(\bullet) = \{f{*}(\bullet)\,,\,u_o{*}(\bullet)\,,\,\eta{*}(\bullet)\}$ *due to equations* (1.1) - (1.3).

On the other hand the "best" measurement is the one where $U(\theta\,,\,y(\bullet))$ reduces to a singleton. For example, suppose $f(x\,,\,t) \equiv f^*(x\,,\,t)$,

$$\omega(\bullet) = \{u_o(\bullet)\,,\,\eta(\bullet)\}$$

Then the "best" observation may be constructed as follows

Denote

$$Y = \{y(\bullet) \mid \exists\; u_o(\bullet) \in L_2(\Omega) \Longrightarrow$$
$$y(t) = \mathbf{G}(t)\; S(t)\; u_o(\bullet)\;,\; t \in T_\theta\}\;,$$

where

$$S(t)\; u(\bullet) = \int_\Omega G(x\;,\; z\;,\; t)\; u_o(z)\; dz$$

Suppose

$$f(\bullet) \in L_2^m\;(T_\theta)$$

Introducing a product

$$< f(\bullet)\;,\; f(\bullet) >_N = < f(\bullet)\;,\; N(\bullet)\; f(\bullet)>$$

we may pass to a standard representation

$$f(\bullet) = f_Y(\bullet) + f_Y^\perp(\bullet)$$

where

$$f_Y(\bullet) \in Y$$
$$<f_Y(\bullet)\;,\; f_Y^\perp(\bullet) >_N = 0$$

Lemma 2.3 *For restriction* (2.1), $(f(\bullet) \equiv f^*(\bullet))$, *assume the available observation* $y(\bullet) = \tilde{y}(\bullet)$ is such that

$$<(\; \tilde{y}(t) - \mathbf{G}(t) \int_0^t G(t-\sigma)\; f(\bullet\;,\;\sigma)d\sigma)^\perp\;,\; (\; \tilde{y}(t) - \mathbf{G}(t) \int_0^t G(t-\sigma)\; f(\bullet\;,\;\sigma)d\sigma)^\perp >_N = \mu^2\;.$$

Then the set $U(\theta\;,\; \tilde{y}(\bullet)$ *is a singleton.*

3. The Solutions for Instantaneous ("Geometric") Constraints.

A more complicated solution arises when the given restrictions on $f\;,\; u_o\;,\; \eta$ are of the following type

$$f(x\;,\;t) \in \mathbf{P} \subseteq \mathbf{R}\;,\; u_o(x) \in Q \subseteq \mathbf{R}_1\;, \quad \eta(t) \in R \subseteq \mathbf{R}^m \qquad (3.1)$$

where P , Q are given intervals, R is a given convex compact set in $\mathbf{R}^m$. The relations of § 2 may again be used for approximating the solution.

Having fixed the triplets $\omega*(\bullet) = \{f*(\bullet)\;,\; u_o*(\bullet)\;,\; \eta*(\bullet)\}$ and $\wedge(\bullet) = \{k(\bullet)\;,\; m(\bullet)\;,\; N(\bullet)\}$ we will denote the respective solution of (2.3) (due to (2.3) - (2.7)) as $u^o(x\;,\;\theta \mid \omega*(\bullet)\;,\; \wedge(\bullet))$. The class of triplets $\omega*(\bullet)$ that satisfy (3.1) will be denoted as $\tilde{\Omega}$ and the $\wedge(\bullet)$'s are to be taken from

$$\wedge_+ = \{\wedge(\bullet) : k(\bullet) > 0 \, , \, m(\bullet) > 0 \, , \, N(\bullet) \geq 0\} \, .$$

For the set $U(\theta \, , \, y(\bullet))$ consistent with constraints (3.1) we have

Theorem 3.1 *The following relations are true*

$$U(\theta \, , \, y(\bullet)) \subseteq \bigcap_{\wedge(\bullet)} \bigcup_{\omega(\bullet)} \{u^o(\bullet \, , \, \theta \mid \omega(\bullet) \, , \, \wedge(\bullet)) \mid \omega(\bullet) \in \tilde{\Omega} \, , \, \wedge(\bullet) \in \wedge_+ \}$$

$$\rho(\varphi(\bullet) \mid U(\theta \, , \, y(\bullet))) = \inf \{\sup\{ < \varphi (\bullet) \, , \, u^o \, (\bullet \, , \, \theta \mid \omega(\bullet) \, , \, \wedge(\bullet)) > \mid$$
$$\mid \omega(\bullet) \in \tilde{\Omega} \, \} \mid \wedge(\bullet) \in \wedge_+\}$$

Therefore the support functional $\rho(\varphi(\bullet) \mid U(\theta \, , \, y(\bullet))$ may be calculated by minimizing a *multiple integral*

$$J(\varphi(\bullet) \, , \, \wedge(\bullet)) = \sup\{< \varphi(\bullet) \, , \, u^o \, (\bullet \, , \, \theta \mid \omega(\bullet) \, , \, \wedge(\bullet)) > \mid \omega(\bullet) \in \tilde{\Omega}\}$$

over all $\wedge(\bullet) \in \wedge^+$.

The projection of the set $U \, (\, \theta \, , \, y(\bullet))$ over a prescribed "direction" $\varphi(\bullet) \in \Phi$ may now be calculated as follows

$$- \inf \{ \, J(- \, \varphi(\bullet) \, , \, \wedge(\bullet) \,) \mid \wedge(\bullet) \in \wedge_+ \} \leq \, < \varphi(\bullet) \, , \, u^o(\bullet \, , \, \theta) > \tag{3.2}$$
$$\leq \, \inf \{ \, J(\, \varphi(\bullet) \, , \, \wedge(\bullet) \,) \mid \wedge(\bullet) \in \wedge_+\}$$
$$\forall \, u^o(\bullet \, , \, \theta) \in U(\theta \, , \, y(\bullet))$$

Remarks

(1) Relations similar to those of the above may be derived for a number of mixed boundary value problems with uncertainty also in the boundary values.

(2) An example of a typical measurement operator $\mathbf{G}(t)$ is the following

$$\mathbf{G}(t)u(\bullet \, , \, t) = \int_\Omega h(x \, , \, t) \, u(x \, , \, t)dx$$

$$h(x \, , \, t) \in L_2^m \, (T_\theta \, ; \, L_2(\Omega))$$

with $h(x \, , \, t)$ given.

Another example (when $u(x \, , \, t)$ belongs to an adequate class of functions) is

$$\mathbf{G}(t) \, u(\bullet \, , \, t) = \text{col} \, [u(\bar{x}_1(t) \, , \, t) \, , \ldots , \, u(\bar{x}_k(t) \, , \, t)] \quad , \quad (t \in (O \, , \, \theta))$$

where the measurements are taken at specified points $\bar{x}_i(t)$. The latter case may be considered particularly for problem (1.1), (1.2) with $x \in \Omega \subset R^1$, since $H_0^1 \, (\Omega) \subset C \, (\bar{\Omega})$ [1-3].

(3) The nature of relations (3.2) is such that the substitution of *any element* $\wedge(\bullet) = \wedge^* \, (\bullet)$ into $J(\varphi(\bullet) \, , \, \wedge(\bullet))$ already gives us a guaranteed estimate;

$$- J(- \varphi(\bullet) , \wedge^* (\bullet)) \leq < \varphi(\bullet) , u^0(\bullet , \theta) > \leq J(\varphi(\bullet) , \wedge^* (\bullet))$$

The respective numerical procedure may therefore combine a random selection scheme for $\wedge^* (\bullet)$ with the calculation of a multiple integral J.

4. The Regularization of the Solution to an Inverse Problem

Consider system (1.1) - (1.3) with $f(x , t) \equiv 0 , \eta(t) \equiv 0$. The initial distribution $u^0(x)$ is taken to be unknown.

Problem 4.1 With measurement $y(t) , t \in T_\theta$, given, specify the state $\bar{u}(x , \theta)$ of system (1.1) - (1.3), $(f(x , t) \equiv 0 , \eta(t) \equiv 0)$ at time θ.

Here $y(\bullet) \in L_2^m(T_\theta)$, $\mathbf{G}(t)$ of (1.3) is a "measurement operator" specified according to Remark (2) of § 3.

If one rewrites the resolving relations in an operator form, we have

$$y(t) = \mathbf{G}(t) \; S(t - \theta) \; \bar{u}(\bullet , \theta) \tag{4.1}$$

where

$$S(t) \; u(\bullet) = \int_\Omega G(x , z , t) \; u_o(z) \; dz ,$$

$y(\bullet)$ is the known measurement and $\bar{u}(\bullet , \theta)$ is the unknown distribution to be specified.

Assumption 4.1

(a) The inverse operator $(\mathbf{G}(t) \; S(t - \theta))^{-1}$ is defined for every $y(\bullet) \in Y$,

(b) There exists a number $K > 0$ such that the following inequality is true

$$||(\mathbf{G}(t) \; S(t - \theta)^{-1} y(\bullet) \; ||_{L_2(\Omega)} \leq K \; || \; y(\bullet) \; ||_{L_2^m \; (T_\theta)}$$

$$\forall \; y(\bullet) \in Y$$

Conditions (a) ensures only the invertibility of the operator $H(t) = \mathbf{G}(t) \; S(t - \theta)$ (this may be considered as a "quasiobservability" property). However conditions (a), (b) taken together ensure that there exists a bounded inverse operator $H^{-1}(t)$ defined on Y and therefore that the domains $U(\theta , y(\bullet))$ of § 2 are bounded (this may be considered as a "genuine" observability).

The class of operators $H(t)$ that satisfy Assumption 4.1 is nonvoid [5]. Thus, for the one-dimensional heat equation with pointwise measurement $y(t) = u(\bar{x} , t) , 0 \leq \bar{x} \leq l$, it requires that $\bar{x} \; / \; l$ is an irrational point of a special type [5], [6].

Nevertheless the inverse problem of specifying $u(x , \theta) \in L_2(\Omega)$ from $y(\bullet) \in L_2^m \; (T_\theta)$ is an *ill-posed* problem as small perturbations of $y(\bullet)$ (taken in $L_2^m \; (T_\theta)$) may yield $z(\bullet) = y(\bullet) + \eta(\bullet) \bar{\in} Y$.

A numerically stable regularized solution for Problem 4.1 may be achieved by using the state-space "filtering" approach of § § 1, 2.

Indeed one may assign to Problem 4.1 a "perturbed" problem of "filtering".

Denote

$$Z_\epsilon = \{z(\bullet) \mid || z(\bullet) - y(\bullet) ||_{L_2^m(T_\theta)} \leq \epsilon\}$$

and assume that constraint (2.1) is now transformed into the inequality $(\delta > 0)$

$$\alpha \int_\Omega u_o^2(x)\, dx + \alpha^{-1} \int_o^\theta \eta' \eta(t)\, dt \leq 1 + \delta \tag{4.2}$$

With $\epsilon^2 \alpha^{-1} \leq \delta$ and α sufficiently small the ellipsoidal set $U_{\alpha\,,\,\delta}(\theta\,,\, z(\bullet))$ is non-void for any $z(\bullet) \in Z_\epsilon$.

Denote $\underline{T}$ to be the unit map; define map K as

$$K\, u_o(\bullet) = y(\bullet)$$

due to the relation

$$y(t) = \mathbf{G}(t)\, S(t)\, u_o(\bullet)\ (K : L_2(\Omega) \longrightarrow L_2^m(T_\theta))$$

and denote

$$T = (\frac{1}{\alpha} KK^* + \alpha\, \underline{T})^{-1}$$

assuming K^* to be the adjoint map for K

Lemma 4.1 *The support function for* $U_{\alpha\,,\,\delta}(\theta\,,\, z(\bullet)$ *is specified as follows* $(\forall\, \varphi(\bullet) \in \Phi)$

$$\rho\, (\varphi(\bullet) \mid U_{\alpha,\delta}(\theta\,,\, z(\bullet))) = < \varphi(\bullet)\,,\, u^*_{\alpha,\epsilon}(\bullet\,,\,\theta \mid z(\bullet)) \tag{4.3}$$

$$+ (1 + \delta - <z(\bullet)\,,\, (\frac{1}{\alpha} KK^* + \alpha\underline{T})^{-1} (z(\bullet) >)^{\frac{1}{2}} (\frac{1}{\alpha} <\varphi(\bullet)\,,\, S(\theta)\, S^*(\theta)\, \varphi(\bullet)> -$$

$$- \frac{1}{\alpha} < \varphi(\bullet)\,,\, S(\theta)\, K^*\, T\, K\, S^*\,(\theta)\, \varphi(\bullet) >)^{\frac{1}{2}}\,.$$

The element

$$u^*_{\alpha,\,\epsilon}\,(\bullet\,,\,\theta \mid z(\bullet)) = S(\theta)\, u^*_{o\,,\,\alpha\,,\,\epsilon}(\bullet \mid z(\bullet))$$

where

$$u^*_{o,\alpha,\epsilon}(\bullet \mid z(\bullet)) = K^*\,(\frac{1}{\alpha} KK^* + \alpha\, \underline{T})^{-1}\,(z(\bullet)\,,\ z(\bullet) \in z_\epsilon \tag{4.4}$$

Observe that the elements $u^*_{\alpha,\epsilon}(\bullet, \theta \mid z(\bullet))$, $u^*_{o,\alpha,\epsilon},(\bullet \mid z(\bullet))$ do not depend upon δ.

Lemma 4.2 *Assuming $\epsilon \to 0$, $\alpha \to 0$ with $\epsilon^2 \alpha^{-1} \to 0$, the following convergence relations are true*

$$\| u^*_{\alpha,\epsilon}(\bullet, \theta \mid z(\bullet)) - \bar{u}(\bullet, \theta) \|_{L_{2(\Omega)}} \to 0 \tag{4.5}$$

The transformation $u_{\alpha,\epsilon}(\bullet, \theta \mid z(\bullet))$ from $L_2^m(T_\theta)$ into $L_2(\Omega)$ generated by the centres for the ellipsoidal "informational" domains of the perturbed problem is therefore a regularizing operator for the solution of the inverse problem 4.1 (under Assumption 4.1) in the sense of Lemma 4.2. This regularizing solution *coincides with the Tikhonov regularizator* for problem 4.1 [4, 6].

Remark 4.1

(a) The solution to the regularizing perturbed are obtained through partial equations (2.3), (2.5) - (2.7) that may be discretized into a class of systems of ordinary differential equations relevant to the solution of the conventional linear-quadratic problem of control whose numerical solutions had been thoroughly studied.

(b) The solutions to problems of § § 2, 3 may be discretized into similar problems for ordinary linear differential equations whose solutions had been described for example in [7 - 9].

(c) The specification of the input $u_o(x)$ for a given measurement $y(\bullet)$ under Assumption 4.1 may be achieved through a modification of the procedure of the above.

In the absence of Assumption 4.1, the solution may be facilitated by some additional restriction on $u_o(\bullet)$ given "a priori". However rather than estimating a unique state $u(\bullet, \theta)$ we are now bound to estimate *the whole set* $\bar{U}(\bullet, \theta)$ of states $\{\bar{u}(\bullet, \theta)\}$ consistent with the measurement $y(\bullet)$ and the "a priori" restriction on $u_o(\bullet)$. The regularizator of the above then formally gives us *one* of the admissible solutions to the problem.

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