# STABLE APPROXIMATIONS OF SET-VALUED MAPS

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August 1987 (Manuscript completed August 1986)
WP-87-74



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# FOREWORD

A good descriptive model of a dynamical phenomenon has inherent stability of its solution, by that one means that small changes in data will result only in "small" changes in the solution. It is thus a criterion that can, and should, be used in the evaluation of dynamical models. This report, that develops approximation results for set-valued functions, provides stability criteria based on generalized derivatives. It also provides estimates for the region of stability.

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## INTRODUCTION

Let us consider two Banach spaces $X$ and $Y$ and a set-valued map $F$ from $X$ to $Y$. An element $y_0 \in Y$ being given, we consider a solution $x_0 \in X$ to the inclusion

$$(*) \quad y_0 \in F(x_0) \ .$$

We shall approximate such a solution $x_0$ by solutions $x_n \in X_n$ to the inclusions

$$(*)_n \quad y_n \in F_n(x_n)$$

where $X_n$ and $Y_n$ are Banach spaces, $F_n$ are set-valued maps from $X_n$ to $Y_n$ and $y_n$ are given.

We extend Lax's celebrated assertion that "consistency and stability imply convergence" (see e.g., Aubin (1972)) still holds true for solving very general inclusions, since we assume only that the graphs of the set-valued maps $F$ and $F_n$ are closed. Namely, we prove that if $X_n$, $Y_n$ are approximations of $X$ and $Y$, if $y_n$ "approximates" $y_0$ and if the $F_n$ are "consistent with $F$", an adequate "stability property" of the set-valued maps $F_n$ implies the convergence of some solutions $x_n$ to $(*)_n$ to $x_0$. We shall also derive an estimate of the error between $x_n$ and $x_0$, which is of the same order as the error between $y_0$ and $y_n$ and the consistency error between $F$ and $F_n$.

In the process, we obtain an adaptation of the Banach-Steinhauss Theorem to closed convex processes, the set-valued analogues of continuous linear operators.

The tool used to define the "stability" of the set-valued maps $F_n$ is the "contingent derivative" introduced in Aubin (1981) (see Aubin-Ekeland, (1984), Chapter 7). Stability of the $F_n$'s means, roughly speaking, that the norms of the inverses of the contingent derivatives of the $F_n$ are uniformly bounded. The techniques used in the proof are the ones used for proving inverse function theorems for set-valued maps (Aubin (1982), (1984), Aubin-Ekeland (1984), Aubin-Frankowska (1987), Frankowska (1986)). They are based on Ekeland's Theorem.

## 1. STABILITY AND CONSISTENCY IMPLY CONVERGENCE

Let $X$ be a Banach space. We consider a family of Banach spaces $X_n$ and operators $p_n \in \mathcal{L}(X_n, X)$ which are right invertible. We denote by $r_n \in \mathcal{L}(X, X_n)$ a right-inverse of $p_n$. The family $(X_n, p_n, r_n)$ is a *convergent approximation* of $X$ if

$$\begin{cases} i) & \| p_n r_n \|_{\mathcal{L}(X,X)} \leq \text{cste} \\ ii) & \forall x \in X,\ p_n r_n x \text{ converges to } x \text{ when } n \to \infty \end{cases} \tag{1.1}$$

If a Banach space $X_0$ is contained in $X$ with a stronger topology, we denote by

$$e^X_{X_0}(p_n r_n) := \| 1 - p_n r_n \|_{\mathcal{L}(X_0, X)}$$

the "error function". The Banach spaces $X_n$ are supplied with the norm $\| x_n \|_n := \| p_n x_n \|_X$.

We then consider convergent approximations $(X_n, p_n, r_n)$ and $(Y_n, q_n, s_n)$ of the Banach spaces $X$ and $Y$.

We also consider set-valued maps $F$ from $X$ to $Y$ and $F_n$ from $X_n$ to $Y_n$. We denote by $\Phi\,(x_0, y_0; F_n)$ the lack of consistency of $F_n$ at $(x_0, y_0)$, defined by:

$$\Phi\,(x_0, y_0; F_n) := \inf_{x_n \in X_n} \left( \| x_0 - p_n x_n \| + d(y_0, q_n F_n(x_n)) \right). \tag{1.2}$$

We say that $F_n$ are *consistent* with $F$ at $(x_0, y_0)$ if $\Phi\,(x_0, y_0, F_n) \to 0$.

As announced in the introduction, the definition of "stability" we suggest involves the concept of "contingent derivative".

Let us begin by defining the concept of contingent cone to $K$ at $x \in K$, introduced by G. Bouligand in the 30's.

We say that $v \in X$ belongs to the "contingent cone" $T_K(x)$ to $K$ at $x$ if and only if

$$\liminf_{h \to 0+} \frac{d(x + hv, K)}{h} = 0 \quad . \tag{1.3}$$

It is a closed cone (not necessarily convex), equal to $X$ whenever $x$ belongs to the interior of $K$, which coincides with the tangent space when $K$ is a smooth manifold and with the tangent cone of convex analysis when $K$ is a convex subset. (See Aubin-Ekeland (1984), Chapter 7, for more details)

When $F$ is a set-valued map from $X$ to $Y$, the "contingent derivative" $DF(x_0, y_0)$ at a point $(x_0, y_0)$ of the graph of $F$ is the set-valued map from $X$ to $Y$ defined by

$$\begin{array}{l} v \text{ belongs to } DF(x_0, y_0)(u) \text{ if and only if } (u, v) \text{ belongs} \\ \text{to the contingent cone to the graph of } F \text{ at } (x_0, y_0) \quad . \end{array} \tag{1.4}$$

In other words

$$\text{Graph } DF(x_0, y_0) := T_{\text{Graph } F}(x_0, y_0) \quad . \tag{1.5}$$

Set-valued maps whose graph are cones are positively homogeneous: they are actually called "processes" (see Rockafellar (1967), (1970)). Hence contingent derivatives are "closed processes".

One can also prove that $v$ belongs to $DF(x_0, y_0)(u)$ if and only if

$$\liminf_{\substack{h \to 0 \\ u' \to u}} d\left(v, \frac{F(x_0 + hu') - y_0}{h}\right) = 0 \ . \tag{1.6}$$

We are now ready to define "stable families" of set-valued maps $F_n$.

DEFINITION 1.1 *Let $(x_0, y_0)$ belong to the graph of $F$ and suppose that the approximations $(X_n, p_n, r_n)$ and $(Y_n, q_n, s_n)$ of $X$ and $Y$ are given. We say that a family of set-valued maps $F_n : X_n \to Y_n$ is* stable *around $(x_0, y_0)$ if there exist constants $c > 0$, $\eta > 0$ and $\alpha \in ]0, 1[$ such that, for all $(x_n, y_n) \in$ Graph $F_n$ satisfying*

$$\| p_n x_n - x_0 \| + \| q_n y_n - y_0 \| < \eta \ ,$$

*for all $v_n \in Y_n$, there exist $u_n \in X_n$ and $w_n \in Y_n$ satisfying*

$$\begin{cases} i) \quad v_n \in DF_n(x_n, y_n)(u_n) + w_n, \\ ii) \quad \| p_n u_n \|_X \le c \| q_n v_n \|_Y, \\ iii) \quad \| q_n w_n \|_Y \le \alpha \| q_n v_n \|_Y. \end{cases} \qquad (1.7)$$

STABILITY THEOREM 1.1 *Let $X$ and $Y$ be Banach spaces and $(X_n, p_n, r_n)$, $(Y_n, q_n, s_n)$ two families of convergent approximations.*

*Let us consider set-valued maps $F$ from $X$ to $Y$ and $F_n$ from $X_n$ to $Y_n$ with closed graphs.*

*Let $x_0$ be a solution to the inclusion*

$$(*) \quad y_0 \in F(x_0) \ .$$

*Suppose the set-valued maps $F_n$ are consistent with $F$ and stable around $(x_0, y_0)$. If $q_n y_n$ converges to $y_0$, then there exist solutions $x_n$ to the inclusions*

$$(*)_n \quad y_n \in F_n(x_n)$$

*such that $p_n x_n$ converges to $x_0$.*

*Furthermore, there exists a constant $l > 0$ such that, for all $y_n$, $\hat{y}_n$ and $\hat{x}_n \in F_n^{-1}(\hat{y}_n)$ satisfying $q_n y_n \to y_0$, $q_n \hat{y}_n \to y_0$ and $p_n \hat{x}_n \to x_0$, we have*

$$d(\hat{x}_n, F_n^{-1}(y_n)) \le l \| q_n y_n - q_n \hat{y}_n \| \ . \qquad (1.8)$$

*In particular, we deduce that*

$$d(x_0, p_n F_n^{-1}(y_n)) \leq l \| y_0 - q_n y_n \| + (l+1)\, \Phi\,(x_0, y_0; F_n) \ . \tag{1.9}$$

REMARK *Stability is necessary*

When the vector spaces $X_n$ are finite dimensional, condition (1.8) is actually equivalent to the stability of the $F_n$. Indeed, let $v_n \in Y_n$ be fixed and set $y_n := \hat{y}_n + hv_n$ for all $h > 0$.

By (1.8), there exists $x_h \in F_n^{-1}(y_n + hv)$ such that $\| x_h - \hat{x}_n \|_n \leq l(1+\epsilon)h \| v_n \|_n$. Hence $u_h := (x_n - \hat{x}_n)/h$ is bounded by $l(1+\epsilon)\| v \|$.

Since the dimension of $X_n$ is finite, a subsequence (again denoted) $u_h$ converges to some $u$, a solution to $v_n \in DF_n(x_n, y_n)(u)$ and $\| u \| \leq l(1+\epsilon)\| v_n \|_n$. Hence the $F_n$'s are stable. □

REMARK By taking $y_n = s_n y_0$ and $F_n := s_n F p_n$, we obtain the estimates $\| y_0 - q_n y_n \| = \| y_0 - q_n s_n y_0 \|$, and

$$\Phi(x_0, y_0; F_n) \leq \| x_0 - p_n r_n x_0 \| + \| y_0 - q_n s_n y_0 \| + \sup_n \| q_n s_n \| d(y_0, F(p_n r_n x_0)) \ .$$

The right-hand side converges to 0 when $F$ is lower semicontinuous. □

REMARK *First Stability Criteria*

The set-valued maps $F_n$ are stable when, for instance, their contingent derivatives $DF_n(x_n, y_n)$ are surjective and when the norms of their inverse $DF_n(x_n, y_n)^{-1}$ are uniformly bounded. The norm of $DF_n(x_n, y_n)^{-1}$ is defined by

$$\| DF_n(x_n, y_n)^{-1} \| := \sup_{\| q_n v_n \| = 1} \ \inf_{u_n \in DF_n(x_n, y_n)^{-1}(v_n)} \| p_n u_n \| \ . \tag{1.10}$$

The question arises whether an extension of the Banach-Steinhauss Theorem could provide stability criteria.

For that purpose we need to introduce the set-valued analogues of continuous operators, which are the set-valued maps whose graphs are closed convex cones (instead of closed vector spaces). They are called "closed convex processes". A map $A$ with closed graph is a closed convex process if and only if

$$\begin{cases} \forall x \in X, \forall \lambda > 0, \; A(\lambda x) = \lambda A(x) \\ \forall x, y \in X, \; A(x) + A(y) \subset A(x + y) \end{cases} \tag{1.11}$$

Contingent derivatives are not always closed convex processes. When the spaces are finite dimensional, the lower semicontinuity of $(x, y) \mapsto$ Graph $DF(x, y)$ at $(x_0, y_0)$ implies that $DF(x_0, y_0)$ is a closed convex process (see Aubin-Clarke (1977)).

When the contingent derivative is not a closed convex process, we can consider closed convex processes contained in it.

For instance, we could work with the asymptotic derivative, introduced by Frankowska (1983), (1985). If $A$ is a closed process from $X$ to $Y$, the set-valued map $A_\infty$ from $X$ to $Y$ is defined by

$$A_\infty(x_0) := \bigcap_{x \in X} (A(x_0 + x) - A(x)) \tag{1.12}$$

Since the graph of $A_\infty$ is a Minkowski difference (or the asymptotic cone of Graph $A$), it is a closed convex cone. Hence $A_\infty$ is a closed convex process contained in $A$. Consequently, the "asymptotic contingent derivative" $D_\infty F(x, y)$ defined by

$$D_\infty F(x, y)(u) := \bigcap_{v \in X} (DF(x, y)(u + v) - DF(x, y)(v)) \tag{1.13}$$

is a closed convex process contained in the contingent derivative. It also contains always the derivative $CF(x, y)$, whose graph is the Clarke tangent cone to the graph of $F$ at $(x, y)$, introduced in Aubin (1982) (see also Aubin-Ekeland, (1984), Chapter 7).

In any case, let us consider a family of closed convex processes $A_n$ from $X_n$ to $Y_n$ such that

$$\text{Graph } A_n \subset \text{Graph } DF_n(x_n, y_n) \tag{1.14}$$

UNIFORM BOUNDEDNESS THEOREM 1.2 *Let us assume that the closed convex processes* $A_n$ *are surjective and satisfy, for all* $(x_n, y_n) \in \text{Graph } A_n \cap ((x_0, y_0) + \eta B)$,

$$\forall\, v \in Y, \text{ there exists } u_n \in A_n^{-1}(s_n v) \text{ such that } \sup_n \| p_n u_n \| < +\infty \quad . \tag{1.15}$$

*Then the family of set-valued maps* $F_n$ *is stable.*

PROOF We consider the functions $\rho_n$ and $\rho$ defined by

$$\rho_n(v) := \inf_{u_n \in A_n^{-1}(s_n v)} \| p_n u_n \|$$

and

$$\rho(v) := \sup_n \rho_n(v) \quad .$$

Since $A_n$ is a convex process and $s_n$ is linear, we deduce that $\rho_n$ is convex and positively homogeneous (sublinear). Since each set-valued map $A_n^{-1} s_n$ is a closed convex process whose domain is the whole space, the function $\rho_n$ is continuous, thanks to the Robinson-Ursescu's (Robinson (1976), Ursescu (1975)) theorem, an extension of the Banach Closed Graph Theorem. Then the function $\rho$ is lower semicontinuous, convex and positively homogeneous. Assumption (1.15) implies that it is finite. We thus deduce from Baire's Theorem that it is continuous, and thus, that there exists a constant $c > 0$ such that

$$\rho(v) \le c \| v \|$$

i.e. that for all $v \in Y$, there exists $u_n \in A_n^{-1}(s_n v)$ such that $\| p_n u_n \| \le c \| q_n s_n v \|$. By taking $v = q_n v_n$, we deduce that the family of $F_n$'s is stable. □

REMARK If the Banach spaces $X_n$ are reflexive, we do not need the Robinson-Ursescu Theorem, since it is easy to check that the function $\rho_n$ is lower semicontinuous, and thus, continuous. □

We also mention another useful consequence of the Uniform Boundedness Theorem.

THEOREM 1.3 *Let us consider a metric space $U$. Banach spaces $X$ and $Y$, and a set valued-map associating to each $u \in U$ a closed convex process $u \mapsto A(u): X \to Y$. Let us assume that the familly of convex processes $\{A(u),\ u \in U\}$ is bounded, in the sense that*

$$\forall\, x \in X,\ \sup_{u \in U}\ \inf_{y \in A(x)} \| y \| < \infty.$$

*Then the following are equivalent*

*i) the set-valued map $u \mapsto Graph\, A(u)$ is lower semicontinuous,*

*ii) the set-valued map $(u, x) \mapsto A(u)(x)$ is lower semicontinuous.*

PROOF Condition *ii*) implies condition *i*), even when the family $\{A(u)\}$ is not bounded. For proving the converse, consider a sequence of elements $(u_n, x_n)$ converging to $(x, u)$ and choose an arbitrary element $y$ in $A(u)(x)$ We have to approximate it by elements $y_n \in A(u_n)(x)$.

Since $u \mapsto Graph\, A(u)$ is lower semicontinuous, we can approximate $(x, y)$ by elements $(\hat{x}_n, \hat{y}_n) \in Graph\, A(u_n)$. By Theorem 1.2, applied to the family $\{A(u_n)^{-1}\}$, there exists a constant $l > 0$ such that

$$\| A(u_n) \| := \sup_{x \in X}\ \inf_{y \in A(u_n)(x)} \| y \| \leq l$$

Hence we can choose $z_n \in A(u_n)(x_n - \hat{x}_n)$ such that $\| z_n \| \leq l \| x_n - \hat{x}_n \| (1 + \epsilon)$. Therefore $y_n := \hat{y}_n + z_n$ does belong to $A(u_n)(x_n)$ and converges to $y$ because $z_n$ converges to 0 and $\hat{y}_n$ to $y$. □

REMARK *Dual stability criteria*

Closed convex processes, as continuous linear operators, can be transposed. Let $A$ be a set-valued map from $X$ to $Y$. Its transpose $A^*$ from $Y^*$ to $X^*$ is the closed convex process defined by

$p \in A^*(q)$ if and only if

$$\forall x \in X, \forall y \in A(x), <p, x> \leq <q, y> \ .$$

In other words, $p$ belongs to $A^*(q)$ if and only if $(p, -q)$ belongs to the polar cone of Graph $A$. (See Rockafellar (1967), (1970), Aubin-Ekeland (1984)).

Many properties of transposition of continuous linear operators can be extended to closed convex processes. For instance, $q$ belongs to $(\mathrm{Im}\, A)^-$ if and only if $0 \in A^*(-q)$:

$$(\mathrm{Im}\, A)^- = - A^{*-1}(0) \ .$$

Therefore, if the vector space $Y$ is finite dimensional, $A$ is surjective if and only if the kernel $A^{*-1}(0)$ of its transpose is reduced to 0.

We also check that in this case

$$\| A^{-1} \| \leq \sup_{p \in A^*(B_*)} \| p \|$$

where $B_*$ is the unit ball of $Y^*$. It is easy to deduce from Theorem 1.2 the following

COROLLARY 1.1 *Let us consider closed convex processes $A_n$ contained in $DF_n(x_n, y_n)$ for all $(x_n, y_n)$ in* Graph $F_n \cap ((x_0, y_0) + \eta B)$. *Let us assume that their transpose $A_n$ satisfy*

$$\begin{cases} i) & \forall n, A_n^{*-1}(0) = \{0\} \\ ii) & \sup_n \sup_{\|f\|_{X^*} \leq 1} \sup_{q_n \in A_n^{*-1}(p_n^* f)} \| s_n^* q_n \|_{Y^*} =: c < +\infty \end{cases}$$

*Then the family of $F_n$'s is stable.*

REMARK *Graph and pointwise convergence of set-valued maps.*

We consider now the case when $X_n = X$ and $Y_n = Y$ for all $n$.

Let $F_n$ be a family of set-valued maps from $X$ to $Y$. We can define the convergence of the set-valued maps $F_n$ either from the convergence of their graphs (graph convergence) or from the convergence of their values $F_n(x_n)$ (pointwise convergence).

We recall the following definitions of the Kuratowski upper and lower limits of a sequence of subsets $K_n$ of a Banach space $K_n$.

$$\limsup_{n\to\infty} K_n := \bigcap_{\substack{\epsilon>0\\ N>0}} \bigcup_{n\geq N} (K_n + \epsilon B)$$

$$\liminf_{n\to\infty} K_n := \bigcap_{\epsilon>0} \bigcup_{N>0} \bigcap_{n\geq N} (K_n + \epsilon B)$$

We denote by $F^\sharp$ the set-valued map defined by

$$\text{Graph } F^\sharp := \limsup_{n\to\infty} \text{Graph } F_n$$

and by $F^\flat$ the set-valued map defined by

$$\text{Graph } F^\flat := \liminf_{n\to\infty} \text{Graph } F_n \ .$$

The following relations follow directly from the definitions

$$F^\sharp(x) = \limsup_{x_n\to x} F_n(x_n)$$

$$= \bigcap_{\substack{\epsilon>0\\ N>0\\ \eta>0}} \bigcup_{\substack{n>N\\ x_n\in(\text{Dom } F_n)\cap(x+\eta B)}} (F_n(x_n) + \epsilon B) \ .$$

It is also easy to check that

$$F^\flat(x) \supset \liminf_{x_n\to x} F_n(x_n)$$

$$= \bigcap_{\epsilon>0} \bigcup_{\substack{N>0\\ \eta\geq 0}} \bigcap_{\substack{n>N\\ x_n\in(\text{Dom } F_n)\cap(x+\eta B)}} (F_n(x_n) + \epsilon B) \ .$$

The Stability Theorem (applied to the maps $F_n^{-1}$ instead of the maps $F_n$) implies the equality of $F^\flat$ and $F^\sharp$.

PROPOSITION 1.1 *Let us assume that the set-valued maps $F_n^{-1}$ are stable around $(y_0, x_0) \in \text{Graph } F^\flat$. Then $y_0$ belongs to $\liminf_{x_n\to x_0} F_n(x_n)$.* □

This point of view, that leads to replacing pointwise by graph convergence was already found to be advantageous in the "epigraphical" setting, i.e., for the set-valued functions $x \mapsto f(x) + \mathbf{R}_+$ where f is an extended real valued function defined on the space $X$. The results reported in the literature are mostly of topological nature, cf. Salinetti and Wets (1976), Dolecki, Salinetti and Wets (1983); for more about epi-convergence and graph convergence consult Attouch (1984). In a subsequent paper, we develop the applications of these results to epigraphical maps, and show how they can be used to obtain approximation and stability results of a quatitative nature for variational problems.

## 2. THE LINEAR CASE WITH CONSTRAINTS

We shall deduce the above theorem from a simpler statement. We consider two Banach spaces $Z$ and $Y$, a continuous linear operator $A \in \mathcal{L}(Z, Y)$ and a subset $K$ of $Z$. We consider the problem (a linear equation with constraints)

find $x_0 \in K$ a solution to $Ax = y_0$.

REMARK By taking $Z := X \times Y$, $K :=$ Graph $F$, $A := \pi_Y$, the projection from $X \times Y$ to $Y$, we observe that inclusion (*) is a particular case of this problem . □

We approximate this problem by introducing

$$\begin{cases} i) & \text{convergent approximations } (Z_n, p_n, r_n) \text{ and } (Y_n, q_n, s_n) \text{ of the spaces } Z \text{ and } Y \\ ii) & \text{subsets } K_n \subset Z_n \\ iii) & \text{continuous linear opertors } A_n \in \mathcal{L}(Z_n, Y_n) \end{cases}$$

We use the following approximate problems:

$$\text{find } x_n \in K_n, \text{ a solution to } Ax_n = y_n \ . \tag{2.1}$$

The "convergence" of $y_n$ to $y_0$, of $K_n$ to $K$ at $x_0$ and of $A_n$ to $A$ is measured by the following

$$\begin{cases} i) & \| y_0 - q_n y_n \| \\ ii) & d(x_0, p_n K_n) \\ iii) & c(A, A_n) := \sup_n \sup_{\|p_n y_n\| \le 1} \| A p_n u_n - q_n A u_n \| \end{cases} \tag{2.2}$$

DEFINITION 2.2 *We shall say that these approximations* $(K_n, A_n)$ *are "stable" if and only if there exist constants* $c > 0$, $\eta > 0$ *and* $\alpha \in ]0, 1[$ *such that for all n, for all* $x_n \in K_n$ *satisfying* $\| p_n x_n - x_0 \| \le \eta$ *and for all* $v_n \in Y_n$, *there exist* $u_n \in X$ *and* $w_n \in Y_n$ *satisfying*

$$\begin{cases} i) \quad u_n \in T_{K_n}(x_n), \quad A u_n = v_n + w_n \\ ii) \quad \| p_n u_n \| \le c \| q_n v_n \| \text{ and } \| q_n w_n \| \le \alpha \| q_n v_n \| \end{cases} \tag{2.3}$$

THEOREM 2.1 *Let us assume that the subsets* $K_n$ *are closed. Assume that the approximations are stable. Then, if* $\| y_0 - q_n y_n \|$, $d(x_0, p_n K_n)$ *and* $c(A, A_n)$ *converge to 0, there exist solutions* $x_n \in K_n$ *to* $A_n x_n = y_n$ *which converge to* $x_0$. *Furthermore, there exists a constant l such that, for all* $\hat{x}_n \in K_n$ *such that* $p_n \hat{x}_{\hat{n}}$ *converges to* $x_0$, *we have*

$$\begin{cases} d(\hat{x}_n, (A_n^{-1}(y_n) \cap K_n) \le l \| q_n y_n - q_n A_n \hat{x}_n \| \\ \le \quad l( \| y_0 - q_n y_n \| + c(A, A_n)( \|x_0\| + \|x_0 - p_n \hat{x}_n \|) \\ \qquad + l \| A \| \| x_0 - p_n \hat{x}_n \|) \ . \end{cases} \tag{2.4}$$

*In particular,*

$$d(x_0, p_n(A_n^{-1}(y_n) \cap K_n))$$

$$\le l( \| y_0 - q_n y_n \| + c(A, A_n)( \| x_0 \| + d(x_0, p_n K_n)) + (1 + l \| A \|) d(x_0, p_n K_n) \ .$$

PROOF of THEOREM 2.1 Supplied with the metric $d(x_n, \bar{x}_n) := \| p_n x_n - p_n \bar{x}_n \|$, $K_n$ is complete. We apply Ekeland's theorem to the continuous function $V_n$ defined on $K_n$ by

$$V_n(x_n) := \|y_n - A_n x_n\|_n = \| q_n y_n - q_n A_n x_n \|_Y$$

Let $\epsilon < (1 - \alpha)/c$ be chosen.

We take $\hat{x}_n \in K_n$ such that $\| x_0 - p_n \hat{x}_n \|$ converges to 0.

Therefore, Ekeland's theorem implies the existence of $\bar{x}_n \in K_n$ satisfying

$$\begin{cases} i) \quad V_n(\bar{x}_n) + \epsilon \| p_n \bar{x}_n - p_n \hat{x}_n \| \leq V_n(\hat{x}_n) \\ ii) \quad \forall x_n \in K_n, \ V_n(\bar{x}_n) \leq V_n(x_n) + \epsilon \| p_n \bar{x}_n - p_n x_n \| \end{cases} \tag{2.6}$$

The first inequality implies that

$$\begin{cases} \| x_0 - p_n \bar{x}_n \| \leq \frac{1}{\epsilon} V_n(\hat{x}_n) + \| x_0 - p_n \hat{x}_n \| \\ \leq \frac{1}{\epsilon} \| q_n y_n - q_n A_n \hat{x}_n \| + \| x_0 - p_n \hat{x}_n \| =: E_\epsilon(n) \end{cases} \tag{2.7}$$

The error $E_\epsilon(n)$ converges to 0 since

$$E_\epsilon(n) \leq \frac{1}{\epsilon}( \| y_0 - q_n y_n \| + \| A x_0 - A p_n \hat{x}_n \| + \| A p_n \hat{x}_n - q_n A_n \hat{x}_n \| + \| x_0 - p_n \hat{x}_n \|$$

$$\leq \frac{1}{\epsilon}( \| y_0 - q_n y_n \| + c(A, A_n) \| p_n \hat{x}_n \| + (1 + \frac{\| A \|}{\epsilon}) \| x_0 - p_n \hat{x}_n \|$$

and since $\| p_n \hat{x}_n \| \leq \| x_0 \| + \| x_0 - p_n \hat{x}_n \| \leq 2 \| x_0 \|$. Consequently, for $n$ large enough, the $p_n \bar{x}_n$ belong to $B(x_0, \eta)$. By the stability assumption, we can associate with $v_n := y_n - A_n \bar{x}_n$ elements $u_n \in T_{K_n}(\bar{x}_n)$ and $w_n \in Y_n$

$$\begin{cases} i) \quad y_n - A_n \bar{x}_n = A u_n + w_n \\ ii) \quad \| p_n u_n \| \leq c \| q_n (y_n - A_n \bar{x}_n) \|, \ \| q_n w_n \| \leq \alpha \| q_n (y_n - A_n \bar{x}_n) \| \end{cases} \tag{2.8}$$

By the very definition of the contingent cone, we assign to any $h > 0$ (which will converge to 0) elements

$$x_n := \bar{x}_n + h u_n + O(h) \in K_n \tag{2.9}$$

where $O(h)$ converges to 0 with $h$.

By taking such an $x_n$, from the second inequality of (2.6), we obtain

$$\| q_n(y_n - A_n\bar{x}_n) \| = V_n(\bar{x}_n) \le V_n(x_n) + \epsilon \| p_n\bar{x}_n - p_n x_n \|$$

$$\le \| q_n(y_n - A_n\bar{x}_n - hA_n u_n - hAO(h)) \|$$

$$+ \epsilon h( \| p_n u_n \| + \| p_n O(h) \|)$$

$$\le (1 - h) \| q_n(y_n - A_n\bar{x}_n) \| + h( \|q_n w_n \| + \| q_n A O(h) \|)$$

$$+ \epsilon h( \|p_n u_n \| + \| p_n O(h) \| )$$

This implies that

$$\| q_n(y_n - A_n\bar{x}_n) \| \le \| q_n w_n \| + \| q_n A O(h) \| + \epsilon( \| p_n u_n \| + \| p_n O(h) \|$$

$$\le (\alpha + \epsilon c) \| q_n(y_n - A_n\bar{x}_n) \| + \| q_n A O(h) \| + \epsilon \| p_n O(h) \|.$$

By letting $h$ converge to 0, we obtain

$$\| q_n(y_n - A_n\bar{x}_n) \| \le (\alpha + \epsilon c) \| q_n(y_n - A_n\bar{x}_n) \| \quad .$$

Since $\alpha + \epsilon c < 1$, this implies that $\bar{x}_n \in K_n$ and $A_n\bar{x}_n = y_n$.

Therefore,

$$d(\hat{x}_n, (A_n^{-1}(y_n) \cap K_n)) \le \| p_n\hat{x}_n - p_n\bar{x}_n \|$$

$$\le \frac{1}{\epsilon} \| y_0 - q_n y_n \| + \frac{1}{\epsilon} c(A, A_n)( \| x_0 \| + \| x_0 - p_n\hat{x}_n \|) + \frac{1}{\epsilon} \| x_0 - p_n\hat{x}_n \|$$

Since this inequality is true for any $\epsilon < (1 - \alpha)/c$ we can let $\epsilon$ converge to $(1 - \alpha)/c$, so that

$$d(\hat{x}_n, p_n(A_n^{-1}(y_n) \cap K_n))$$

$$\le l \| y_0 - q_n y_n \| + lc(A, A_n)( \| x_0 \| + \| x_0 - p_n\hat{x}_n \| ) + l \| A \| \, \| x_0 - p_n\hat{x}_n \| \quad .$$

By taking $\hat{x}_n \in K_n$ such that $\| x_0 - p_n\hat{x}_n \| \le d(x_0, p_n K_n)\ (1 + \beta)$ and letting $\beta$ converge to 0, we obtain the estimate (2.5). □

PROOF of the STABILITY THEOREM 1.1 We take $Z := X \times Y$, $K = \text{Graph}\, F$ and $A := \pi_Y$, $Z_n = X_n \times Y_n$, $K_n := \text{Graph}\, F_n$ and $A_n := \pi_{Y_n}$. We observe that $c(A, A_n) = 0$ since, for all $u_n = (x_n, y_n)$,

$$A(p_n \times q_n)(x_n, y_n) - q_n A_n(x_n, y_n) = q_n y_n - q_n y_n = 0$$

The stability of the set-valued maps $F_n$ is just the same as the stability of their graphs with respect to the projections $\pi_Y$ and $\pi_{Y_n}$.

If $(\hat{x}_n, \hat{y}_n)$ is in the graph of $F_n$, we deduce that $\| q_n y_n - q_n A_n(\hat{x}_n, \hat{y}_n) \| = \| q_n y_n - q_n \hat{y}_n \|$. Finally, we can estimate the distance between $(x_0, y_0)$ and the image of the graph of $F_n$ by $p_n \times q_n$ in the following way.

$$d((x_0, y_0), (p_n \times q_n)\, \text{Graph}\, F_n) =$$

$$= \inf_{\substack{x_n \in X_n \\ y_n \in F_n(x_n)}} (\| x_0 - p_n x_n \| + \| y_0 - q_n y_n \|) =$$

$$= \inf_{x_n \in X_n} (\| x_0 - p_n x_n \| + d(y_0, q_n F_n(x_n)) =$$

$$=: \Phi(x_0, y_0; F_n)$$

Indeed, $(p_n \times q_n)\, \text{Graph}\, F_n = \text{Graph}\,(q_n F_n p_n^{-1})$, where the domain of $q_n F_n p_n^{-1}$ is $p_n X_n$. On $p_n X_n$, one has $q_n F_n p_n^{-1} p_n x_n = q_n F_n x_n$.

Hence Theorem 1.1 follows from Theorem 2.1. □

## 3. A STABILITY CRITERION

We devote this section to criteria implying that a family of subsets $K_n$ is stable. For simplicity, we consider the case when $Z_n := Z$, $Y_n := Y$, $p_n := Id$, $q_n := Id$ and $A_n := A$. It is time to recall that the Kuratowski lim inf

$$\liminf_{n \to \infty} K_n = \bigcap_{\epsilon > 0} \bigcup_{N > 0} \bigcap_{n \geq N} (K_n + \epsilon B) \tag{3.1}$$

is the set of $x$'s such that $x = \lim_{n \to \infty} x_n$ where $x_n \in K_n$.

The stability assumption (2.3) implies implicitly that $x_0$ belongs to the lim inf of the subsets $K_n$.

We consider now the lim inf of the contingent cones

$$T(x_0) := \liminf_{K_n \ni x_n \to x_0} T_{K_n}(x_n) = \bigcap_{\epsilon > 0} \bigcup_{N, \eta} \bigcap_{\substack{n \geq N \\ x_n \in K_n \cap (x + \eta B)}} T_{K_n}(x_n + \epsilon B), \tag{3.2}$$

and we address the following question: under which conditions does the "pointwise surjectivity assumption"

$$A\, T(x_0) = Y \tag{3.3}$$

imply the stability of the $K_n$. The next result answers this question when the dimension of $Y$ is finite, unfortunately.

PROPOSITION 3.1 *Assume that $T(x_0)$ is convex and that $A\, T(x_0) = Y$. Let us assume that there exists a space $H \supset Y$ such that the injection from $Y$ to $H$ is compact. There exists a constant $c > 0$ such that, for all $\alpha \in ]0, 1[$, there exist $\eta > 0$ and $N \geq 1$ with the following property:*

$$\forall v \in Y, \forall n \geq N, \forall x_n \in K_n \cap (x_0 + \eta B) \ ,$$

*there exist solutions $u_n \in T_{K_n}(x_n)$ and $w_n \in Y$ to*

$$A u_n = v + w_n, \ \| u_n \|_Z \leq c \| v \|_Y, \ \| w_n \|_H \leq \alpha \| v \|_Y \tag{3.4}$$

REMARK When $Y$ is finite dimensional, we can take $H = Y$. □

PROOF Let $S$ denote the unit sphere of $Y$, which is relatively compact in $H$. Hence there are $p$ elements $v_i$ such that the balls $v_i + (\alpha/2)B_H$ cover $S$. Since $T(x_0)$ is convex and $A\,T(x_0) = Y$, Robinson–Ursescu's Theorem implies the existence of a constant $\lambda > 0$ such that we can associate with any $v_i \in S$ an $u_i \in T(x_0)$ satisfying $\| u_i \|_Z \leq \lambda$. By the very definition of $T(x_0)$, we can associate with $\alpha \in ]0, 1[$ integers $N_i$ and $\eta_i > 0$ such that $\forall n \geq N_i$, $\forall x_n \in K_n \cap (x_0 + \eta_i B)$, there exist $u_n^i \in T_{K_n}(x_n)$ satisfying

$$\| u_i - u_{n_i} \|_Z \leq \alpha/2 \| A \|_{\mathcal{L}(Z, H)} .$$

Let $N := \max_{1 \leq i \leq p} N_i$ and $\eta := \min_{1 \leq i \leq p} \eta_i$. We take $n \geq N$ and $x_n \in K_n \cap (x_0 + \eta B)$. Let $v$ belong to $Y$. There exists $v_i \in S$ such that

$$\| v_i - \frac{v}{\| v \|_Y} \|_H \leq \frac{\alpha}{2} \quad .$$

Set $v_n = \| v \|_Y u_n^i$ and $w_n = v - A v_n$. We see that $v_n \in T_{K_n}(x_n)$, that

$$\| v_n \|_Z = \| v \|_Y \| u_n^i \|_Z \leq \| v \|_Y (\lambda + \| u_i - u_n^i \|_Z)$$

$$\leq \| v \|_Y (\lambda + \alpha/2 \| A \|_{\mathcal{L}(Z, H)}) \leq c \| v \|_Y$$

(where $c := \lambda + 1/2 \| A \|_{\mathcal{L}(Z, H)}$) and that

$$\| w_n \|_H = \| v - A \| v \|_Y u_n^i \|_H$$

$$= \| v \|_Y ( \| \frac{v}{\|v\|_Y} - v_i + A(u_i - u_n^i) \|_H$$

$$\leq \| v \|_Y (\frac{\alpha}{2} + \| A \|_{\mathcal{L}(Z, H)} \| u_i - u_n^i \| )_Z \leq \alpha \| v \|_Y$$

This proves our claim. □

This result justifies a further study of the lim inf of contingent cones to $T_{K_n}(x_n)$.

We introduce the cone $C_{\vec{K_n}}(x_0)$ of elements $v$ such that

$$\lim_{\substack{h \to 0+ \\ K_n \ni x_n \to x_0}} \frac{d(x_n + hv,\, K_n)}{h} = 0 \tag{3.5}$$

When all the $K_n$'s are equal to $K$, then $\liminf_{n \to \infty} K_n = K$ and $C_{\vec{K_n}}(x_0)$ coincides with the Clarke tangent cone to $K$ at $x_0$.

It is clearly a closed convex cone: indeed, let $v_1$, and $v_2$ belong to $C_{\vec{K_n}}(x_0)$, $x_n \in K_n$ a sequence converging to $x_0$ and $h_n \to 0^+$. There exists a sequence $v_{1n}$ converging to $v_1$ such that $x_n^1 := x_n + h_n v_{1n}$ belongs to $K_n$ for all $n$. Since $x_n^1$ also converges to $x_0$, there exists a sequence $v_{2n}$ converging to $v_2$ such that $x_n^1 + h_n v_{2n} \in K_n$ for all $n$. Hence $x_n + h_n(v_{1n} + v_{2n}) = x_n^1 + h_n v_{2n} \in K_n$ for all $n$ and $v_{1n} + v_{2n}$ converges to $v_1 + v_2$. Then $v_1 + v_2$ belong to $C_{\vec{K_n}}(x_0)$.

A slight modification of a result of Aubin–Clarke (1977) implies the following relations between $T(x_0)$ and $C_{\vec{K_n}}(x_0)$.

PROPOSITION 3.3 *Assume that $Z$ is reflexive and that the subsets $K_n$ are weakly closed. Then*

$$\liminf_{K_n \ni x_h \to x_0} T_{K_n}(x_n) \subset C_{\vec{K_n}}(x_0) \tag{3.6}$$

PROOF Let $v$ belong to lim inf $T_{K_n}(x_n)$. Then, for any $\epsilon > 0$, there exists $N$ such that

$$d(v,\, T_{K_n}(y_n)) \le \epsilon \quad \text{when } n \ge N \text{ and } y_n \in K_n \cap (x_0 + \eta B)$$

Let us set $g_n(t) := d(x_n + tv,\, K_n)$. By Proposition 4.1.3, p178 of Aubin–Cellina (1984),

$$\liminf_{h \to 0+} \frac{1}{h} \left(d_{K_n}(x_n + tv + hv) - d_{K_n}(x_n + tv)\right) \le d(v,\, T_{K_n}(y_n))$$

where $y_n \in K_n$ is a best approximation of $x_n + tv$. Let $x_{0n} \in K_n$ denote a best approximation of $x_0$. Since

$$\| y_n - x_0 \| \leq \| y_n - (x_n + tv) \| + \| x_n - x_0 + tv \|$$

$$\leq \| x_n - (x_n + tv) \| + \| x_n - x_0 + tv \|$$

$$\leq 2t \| v \| + \| x_n - x_0 \| \leq \eta$$

when $x_n \in (x_0 + (\eta/2)B) \cap K_n$ and $t \leq \eta/4 \| v \|$, we deduce that the function

$$g_n(t) := d_{K_n}(x_n + tv)$$

which is almost everywhere differentiable, satisfies

$$g'_n(t) \leq \epsilon \text{ for all } n \geq N \text{ and } t \leq \eta/4 \| v \|$$

By integrating from 0 to $h$, we deduce that

$$\frac{d_{K_n}(x_n + hv)}{h} = \frac{g_n(h) - g_n(0)}{h} \leq \epsilon$$

for all $h \leq \eta/4 \| v \|$, $n \geq N$ and $x_n \in K_n \cap (x_0 + \eta/2B)$. □

The converse is true when the dimension of $Z$ is finite or when the subsets $K_n$ are convex. More generally, we introduce the following "weak contingent cones" $T_K^\sigma(x)$ define in the following way:

$v$ belongs to $T_K^\sigma(x)$ if and only if there exist a sequence $h_n \to 0+$ and a sequence $w_n$ converging weakly to $v$ such that $x_n + h_n v_n$ belongs to $K$ for all $n$.

We see at once that

$$T_K(x) \subset T_K^\sigma(x)$$

and that they coincide when the dimension of $Z$ is finite or when $K$ is convex: indeed, in this case, $T_K(x)$ and $T_K^\sigma(x)$ are the closure and the weak closure of the *convex* cone spanned by $K - x$, which thus are equal.

We then obtain the following trivial inclusion:

PROPOSITION 3.2 *Assume that Z is reflexive, then*

$$C_{\overrightarrow{K_n}}(x_0) \subset \liminf_{K_n \ni x_n \to x_0} T^g_{K_n}(x_0)$$

PROOF Assume that $v$ belongs to $C_{\overrightarrow{K_n}}(x_0)$. Then, for all $\epsilon > 0$, there exist $\eta > 0$, $N$ and $\beta > 0$ such that, for all $h \leq \beta$, $n \geq N$ and $x_n \in K_n \cap (x_0 + \eta B)$,

$$d(x_n + hv, K_n) \leq \epsilon h$$

Let us fix such an $n \geq N$ and $x_n \in K_n \cap (x_0 + \eta B)$. Let $y_n^h \in K_n$ such that $\| x_n - y_n^h + v \| \leq 2\epsilon h$ and set $v_n^h := (y_n^h - x_n)/h$. Since $\| v_n^h - v \| \leq 2\epsilon$ and since the space is reflexive, a subsequence of $v_n^h$ converges weakly to some $v_n \in v + 2\epsilon$. Such a $v_n$ belongs to $T^g_{K_n}(x_n)$. Hence $d(v, T^g_{K_n}(x_n))$ converges to 0. □

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