SIBLING DEPENDENCES IN BRANCHING POPULATIONS

Per Broberg

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A-2361 Laxenburg, Austria

Foreword

The branching process got its start with a demographic question asked by Francis Galton, in answer to those who mourned that the great writers and statesmen of the past have so few descendants living today. Galton suspected that even in an increasing population most people will have no descendants, or none beyond two or three generations; most of the increase of the race occurs in relatively few lines of descent. We can say of people in slowly growing populations that either they will have thousands of descendants or they will have none; the chance that they will have just two generation after generation is remote.

In the ordinary branching process it is taken that each individual has a certain probability of generating another individual in each moment, and these probabilities are independent of one another. The parent has the same chance of bearing a child after having born 5 previously as she had at the outset.

What Per Broberg has done in the paper that follows is to allow for statistical dependency between siblings. He covers the case where a parent that has had several offspring is less likely to have one more. But it equally covers the case where having had a child shows that the person is fertile, and hence the probability of a further child is raised after the first birth. His results capture the asymptotic growth and fluctuations of such populations, that are followed to their ultimate theoretical condition of stability.

By making its assumptions more realistic, Per Broberg has increased the interest of the branching process for students of population.

Nathan Keyfitz
Leader, Population Program

ABOUT THE AUTHOR

Mr. Per Broberg comes from the Department of Mathematics, University of Göteborg, Sweden. He was a participant of the 1986 Young Scientist Summer Program.

Sibling dependences in branching populations

0. Introduction.

Both from the point of view of applications and from a more theoretical aspect it is of interest to model dependences between siblings in branching populations.Competition among siblings could give rise to negative correlation between the number of children that they beget.So called kin selection would have the same effect,see Horn (1981),whereas a stochastic family environment would entail positive correlations,(Broberg,1986).

A modified Bellman-Harris process model allowing for the siblings' life-spans and number of children being correlated appears in Crump and Mode (1969).The authors prove L_2 convergence of a normed process and exhibit properties of its limit. The topic has not, however, attracted any attention in the probabilistic literature since then.

The dependences treated here are such that the members of each sibling group of individuals, born into the population, reproduce and live according to a joint probability law depending on the sibling group size and the mother's birth pattern. These are inherited properties, but there is no other influence from ancestors. The idea of this paper is to embed such a process of sibling dependent individuals in an ordinary multi-type branching process, by regarding sibling groups as individuals, and to point at some conclusions that can be drawn concerning the growth, composition and extinction probability of the former.

1. Splitting populations.

1.1. Preliminaries.

To begin with we shall suppose that sibling groups act according to laws that depend only on their different sizes and that reproductions are of splitting type. Presuppose the existence of an individual life sample space U, endowed with the σ-algebra $\mathcal{U}$. The information provided by lives $u \in U$ may differ, but a life always tells about the reproduction point process $\eta(u,.)$, whose points of increase are the ages at child bearing. In this context we also define a stochastic sibling life span λ: $U \to \mathbb{R}_+$. This quantity satisfies the equality

$$\eta(u, \{\lambda(u)\}) = \eta(u, [0,\infty)),$$

i.e. $\lambda(u)$ is the age at splitting, if children are born.

Furthermore, assume that for each sibling group size, $k \in N$, there is given a joint probability distribution $P(k,.)$ on $(U^k,\mathcal{U}^k)$, the space of lives $u_1,\ldots, u_k$ pertaining to members of the sibling group.

Typically the numbering of siblings is arbitrary, so that the marginals $P(k,\{u_i \in A\})$ are independent of i. In this case we write

$$P(k, \{u_i \in A\}) = f(k, A), \quad \forall A \in \mathcal{U}, \ i \le k.$$

In some cases it is natural to assume a further homogeneity

$$P(k, \{u_i \in A\}) = Q(A)$$

for some Q, i.e. all lives are identically distributed, with a common distribution not dependent on the group sizes. Such a population will be called homogeneous. Certainly the above does not rule out lives of siblings being correlated.

For notational convenience we shall view $P(k,.)$ as a measure on $\Omega = \cup U^k$ equipped with the σ-algebra $\mathcal{A} = \sigma(\cup \mathcal{U}^k)$ = the minimal σ-algebra generated by sets $S \in \mathcal{U}^k$, $k \in N$, although $P(k,.)$ has its support on U^k.

The size of the sibling group will be interpreted as the type of the group.

Next, introduce the natural projections $u_i(\cdot)$, $i \in N$, by the requirement that $u_i(\omega)$ be the ith coordinate in ω, i.e if $\omega = (u_1,u_2,..)$ then $u_2(\omega)= u_2$. Section 2 will elaborate on these matters.

To avoid uninteresting notational complications we shall consider only populations initiated by one full group of siblings, or <u>micro individuals</u>. The sibling groups can now be regarded as <u>pseudo individuals</u> in an ordinary multi-type branching process, so that in the pseudopopulation an individual of some type (size) born at time 0 initiates the population.

Let the pseudoindividuals be enumerated in the conventional way, according to descent and family birth order, so that each group has its label

$$x \in I := \{0\} \cup \bigcup_{j=1}^{\infty} N^j.$$

Since the reproduction is splitting all individuals in a sibling group are born at the same time, which is taken to be the corresponding pseudoindividual's birth time. Set

$\eta_i(\omega,t) := \eta(u_i(\omega), [0,t])$ = the number of reproductions of the ith micro by age t.

In the new terminology a life $\omega = (u_1, \ldots, u_j) \in \Omega$, yields a reproduction point process on $N \times R_+$, determined by

$\xi(\omega, \{k\}\times[0,t]) = \sum_{i=1}^{j} 1\{k\}(\eta_i(\omega,t))$ = the number of micros that have begotten k children by time t since birth. Here $k \in N$, $t \in \mathbb{R}_+$, $1A$ is the indicator of set A.

At a minimum a pseudoindividual biography ω should tell us the life spans and reproductive histories of its constituent micros. Thus, a rudimentary life of a j-type pseudoindividual could look like the following:

$$\omega = \left\{ \{\lambda_i\}_{i=1}^{j}, \ \{\eta_i(\infty)\}_{i=1}^{j} \right\},$$

where the superindex refers to the siblings numbered from 1 to j, and $\eta_i(\infty) = \eta(u_i, [0,\infty))$..

1.1. The new population process.

The canonical pseudo population process (see Jagers & Nerman, 1984) can now be constructed on $N \times \Omega^I$, signifying the product of the space of possible types of the initial pseudoindividual and the space of all possible combinations of pseudoindividual lives.

As pointed out the splitting character of the underlying process renders it possible to define natural birth times of the pseudo-individuals, σ'_x corresponding to ω_x, $x \in I$, in the conventional recursive manner, Jagers and Nerman (1984) : $\sigma'_0 = 0$ and then

$$\sigma'_{xi} = \sigma'_x + \inf\{t: \xi(\omega_x, N \times [0,t]) \geq i\}, \inf \phi := \infty \quad (1).$$

Let ρ_x be the type of x and $j(i) \leq \rho_x$ be the number of the ith sibling giving birth at σ_{xij}; recall that siblings are numbered. Then we can write

$$\sigma'_{xi} = \sigma'_x + \lambda_x^{j(i)}, \quad \text{and}$$

$$\rho_{xi} = \eta(u_{j(i)}(\omega_x), [0,\infty)) =$$

$$= \text{the number of off-spring of sibling } j(i).$$

The actual construction of a canonical population process is postponed till next section, where it is made in a more general framework.

If we assume boundedness of micro reproductions the conversion of the process into a multi-type branching process in the sense of Mode (1971), i.e. a finite type space process, is straightforward. On the other hand, without this assumption a theory allowing a countable type space would be necessary. In the following suite of illustrating examples is thus imposed the condition of boundedness.

Example 1 Embedded generation counts.

Consider a branching population with binary reproduction, and suppose that there are two possible types of pseudo individuals: singletons (1) and twins (2), the micro individuals constituting the twins having dependent reproductions. If we let ζ_n count the number of micros in the nth generation, then with ζ_n^1 and ζ_n^2 the number of singletons and twins, respectively

$$\zeta_n = (\zeta_n^1 + 2\zeta_n^2) = (\zeta_n^1\ \zeta_n^2)\ \binom{1}{2}.$$

Let reproductive distributions be given by, for singeltons,

$$p_i = P(1,\ \eta_1(\infty) = i),\ p_0 + p_1 + p_2 = 1,$$

and for twins,

$$p_{ij} = P(2,\ \{\eta_1(\infty) = i,\ \eta_2(\infty) = j\} \cup \{\eta_1(\infty) = j,\ \eta_2(\infty) = i\}),$$

$$p_{00} + p_{10} + p_{11} + p_{20} + p_{21} + p_{22} = 1,$$

i.e. the joint distribution for the twins does not take into account the order between the two micros.

The matrix of expected reproduction is then

$$M = \begin{bmatrix} p_1 & p_2 \\ p_{21} + 2p_{11} + p_{10} & 2p_{22} + p_{21} + p_{20} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Furthermore, we can solve all sorts of multitype problems, like calculating probabilities of extinction and means, cf Mode (1971). It follows that, e.g. the mean of $\{\zeta_n\}$ initiated by a singelton is

$$E_1[\zeta_n] = (1\ \ 0)\ M^n\ \binom{1}{2}$$

If the Perron-Frobenius root r of M exceeds 1, $r > 1$, the asymptotically stable proportions are

$$s_i = \nu_i/(\nu_1 + \nu_2) = \text{a.s.} \lim_{n\to\infty} \zeta_n^i/(\zeta_n^1 + \zeta_n^2), \text{given non-extinction},$$

where $\nu = (\nu_1,\ \nu_2)$, is the left eigenvector of M corresponding to r. In our case

$$r = \frac{a+d}{2} + \sqrt{(a+d)^2/4 + cb - ad}$$

and

$$s_1 = (d-r-c)/(a+d-b-c-2r),$$

when this makes sense.

Example 2. Binary Fission.

Assume the following for twins in the preceeding example

$$\begin{cases} P(\text{one twin begets } 0 \text{ children, the other } 2) = p_1 \\ P(\text{both twins beget } 2 \text{ children}) = p_2 \\ P(\text{no reproduction}) = 1 - p_1 - p_2. \end{cases}$$

Since there is only one type of micros, the corresponding embedded macro process is actually a one-type Galton-Watson process, for definitions see Jagers (1975). It can be examined by counting the number of twins in generation n, z_n, instead of the count of micros ζ_n [cf. Example 1], which equal $2z_n$.

The probability generating function of the twin process initiated by one twin couple is

$$\varphi(s) = (1-p_1-p_2) + p_1 s + s^2 p_2, \quad s \in [0,1].$$

Let $m = p_1 + 2p_2$, so that $\{z_n\}$ is sub- or supercritical or critical when $m < 1$, $m > 1$ or $m = 1$ respectively. Assume that twins are of the same type. Fix the expectation $1 < m \le 2$, and observe that m determines the reproduction distribution of micros: the probability of splitting is $m/2$. What happens to the extinction probability when the dependence structure is varied? Certainly, there will be no extinction if $p_1 + p_2 = 1$, i.e. $p_1 = 2-m$ and $p_2 = m-1$. On the other hand, the extinction probability is highest when $p_2 = m/2$. The following calculations make that clear. Put $p := p_2$, $p \in (\frac{m-1}{2}, \frac{m}{2}]$, and consequently $p_1 = m-2p$. The generating function

$$\begin{aligned} \varphi_p(s) &= 1-(m-2p) - p + (m-2p)s + ps^2 = \\ &= 1 - m + ms + p(1-s)^2 \end{aligned}$$

is obviously increasing in p. The corresponding extinction probability $q(p)$, being the smallest non-negative solution of $\varphi_p(x) = x$, therefore also increases in p and attains its maximal value $\{1 + (1-m)^2\}/m$ for $p = m/2$.

Independent individual reproduction is obtained when

$$p_2 = p^2 \quad p_1 = 2p(1-p) \quad \text{for} \quad p = m/2.$$

This case has an intermediate extinction probability: the twin process can have both higher and lower probability of extinction. Furthermore, we note that the covariance of siblings, reproductions

$$\mathrm{Cov}[\eta_1(\infty), \eta_2(\infty)] = 2(m-p_1) - m^2 \geq 0$$

if an only if

$$p_1 \leq 2 \frac{m}{2} (1 - \frac{m}{2}).$$

This complies with the notion that positive covariance between siblings' reproductions should lead to a high extinction probability, due to large fluctuations in population size, as compared with the i.i.d. case. For more general comparisons of extinction probabilities cf Broberg (1986) □.

<u>Example 3.</u> <u>Convergence in L^2 of Normed Generation Counts.</u>

First some notation (detailed account in op.cit.) just for this example

ζ_n = the number of individuals of generation n.

$\xi_n(i)$ = the number of children of individual number i of the n:th generation.

$\zeta_n^* := \#\{ i; \xi_n(i) > 0 \; 1 \leq i \leq \zeta_n \}$ = the number of mothers in the nth generation.

$k(n,i) := \inf \{ k; \sum_{r=1}^{k} 1\{ \xi_n(r) > 0 \} = i \}$ = the individual number of the ith mother in the nth generation.

$\eta_n(i,j)$ = the number of children born by the j:th sibling in the i:th sibling group of the n:th generation.

Note $$\zeta_n = \sum_{i=1}^{\zeta_{n-1}} \xi_{n-1}(i) = \sum_{i=1}^{\zeta_{n-2}^*} \sum_{j=1}^{\xi_{n-2}(k(n-2,i))} \eta_{n-1}(i,j).$$

Moreover, set

$$\xi_i := \{\xi_i(j), 1 \leq j \leq \zeta_{i-1}\},$$

$$B_n := \sigma\{\xi_0, \ldots, \xi_{n-1}\}.$$

Assume

$$E[\xi_0] = m > 1$$

$$Var[\xi_0] = \sigma^2 < \infty \quad \text{and}$$

$$Cov[\eta_n(i,j),\ \eta_n(i,k)|B_n] = c \qquad \forall n,\ i \ \text{and} \ j \neq k \in N$$

if these individuals are born.

In order to prove the L^2- convergence of the martingale ζ_n/m^n calculate the variance of ζ_n using variance decomposition, independence between different sibling groups and the above:

$$Var[\zeta_n] = E[Var[\zeta_n|B_{n-1}]] + Var[E[\zeta_n|B_{n-1}]]$$

$$= E[\sum_{i=1}^{\zeta^*_{n-2}} Var[\sum_{j=1}^{\xi_{n-2}(k(n-2,i))} \eta_{n-1}(i,j)|B_{n-1}]] + m^2 Var[\zeta_{n-1}]$$

$$= E[\sum_{i=1}^{\zeta^*_{n-2}} (\xi_{n-2}(k(n-2,i))\sigma^2 + \xi_{n-2}(k(n-2,i))(\xi_{n-2}(k(n-2,i))-1)c)]$$

$$+ m^2 Var[\zeta_{n-1}] = m^{n-2}(m\sigma^2 + (\sigma^2 + m^2 - m)c)$$

$$+ m^2 Var[\zeta_{n-1}].$$

By induction

$$Var[\zeta_n] = (m\sigma^2 + (\sigma^2 + m^2 - m)c)\frac{(m^{2n-2} - m^{n-2})}{m - 1} .$$

Hence the convergence of ζ_n/m^n in L^2. □

1.3. Counting micros by characteristics.

Characteristics analogous to the ones in ordinary branching processes in the sense of Jagers (1975) may now be defined with respect to the pseudo process. By a random characteristic we shall understand any real valued process $\{\chi(a),\ a \in R_+\}$ defined on $N_+ \times \Omega^I$ [op.cit.]. Throughout χ is taken to vanish for negative arguments.

For each $x \in I$, the shift-like operator S_x maps $(\rho_0,\ \{\omega_y,\ y \in I\})$ to $(\rho_x,\ \{\omega_{xy},\ y \in I\})$, thereby rendering x an ancestor with

$$\chi_x(a) = \chi((\rho_x,\ \{\omega_{xy};\ y \in I\},\ a) = \chi \circ S_x(a).$$

Population size can be measured by

$$z_t^\chi = \sum_{x,\sigma_x \leq t} \chi_x(t - \sigma_x) \qquad (2),$$

where $\chi_x(a)$ is interpreted as the score of x at age a, and z_t^χ as the total score at time t .

Below are some examples of characteristics exhibiting ways of measuring the population of microindividuals.

a. The number of micros born up to t;

use $\chi(\rho_0, \{\omega_x, x \in I\}, a) = \begin{cases} \rho_0, & a \geq 0 \\ 0, & a < 0 \end{cases}$

to calculate z_t^χ.

b. The number of living micros:

Use $\chi(a) = \sum_{i=1}^{\rho_0} 1[0, \lambda_i](a)$; the first argument suppressed.

c. The number of micros, that have begotten k (micro) children:

$$\chi(a) = \sum_{i=1}^{\rho_0} 1\{k\}(\eta_i(a)).$$

Example 2 (cont.). Now let also the life spans of twins be dependent:

$$L(t_1, t_2) = P(\lambda_1 \leq t_1, \lambda_2 \leq t_2).$$

The marginal distribution is

$$L_i(t) = P(\lambda_i \leq t) = L(t), \quad i = 1,2.$$

Denote by z_t the number of micros alive at time t.

For the expected number of individuals alive at time t, $E[z_t|z_0 = 2]$, dependences pose no problem, thanks to the linearity of expectations regarded as operators. We just use the characteristic $\chi'(a) = 1[0,\lambda](a)$ and let $z_t^{\chi'} \equiv \sum_{x \in I} \chi'(t-\sigma_x')$ be a process starting from one individual, and procede as for an ordinary one-type branching process according to the marginal distribution of our twin process. Then $E[z_t^\chi|z_0 = 2] = 2E[z_t^{\chi'}]$. Introducing $p = p_1/2 + p_2$, $\mu(t) = E[\eta[0,t]] =$ $= 2L(t)p$ and $\nu(t) = \sum_{k=0}^{\infty} \mu^{*k}(t)$ (the renewal function), where $*$ stands for convolution, we obtain

$$E\left[z_t^{\chi'}\right] = (E[\chi_0'] * \nu)(t)$$

and in particular

$$E[z_t] = 2 \sum_{k=0}^{\infty} (2p)^k (L^{*k} * (1-L))(t). \qquad \square$$

Example 4. In order to hint at a possible extension of the above setting, we consider a binary splitting process where twins are dependent and the individuals can be of two kinds. Again, regard twins as pseudo-individuals, who will now be of three types: both twins of first or second kind, or one of each. From the joint probability laws $P(k,.)$, $k = 1, 2, 3$ (where k is now referring to type), we can obtain a reproductive matrix as in Mode (1971) $M = \{\mu(i,j)\}$ and

$$E[z_n | \text{ start from a mixed pair}] = (0 \quad 1 \quad 0) \; M^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} 2. \qquad \square$$

2. More general populations.

2.1. The model.

It is possible to remove the restriction that micros reproduce by splitting, and to permit the distances in time between sibling births and their mother's birth and even her entire biography to influence the reproduction of the sibling group. This can be done by the inclusion of the micro-mothers' entire lives in the types of sibling groups. The type of a pseudoindividual then becomes an element $u \in U$, the individual life sample space, endowed with the σ-algebra $\mathcal{U}$. In the sequel only the information provided by u through the reproduction point process $\eta(u,.)$, will be included in the type for notational reasons, but the results easily extend to the more general formulation. For the sake of convenience choose $\Gamma = \mathcal{N}(\mathbb{R}_+)$ = pointprocesses on $\mathbb{R}_+$. However, the theory holds true for the more general formulation.

Denote the ages at childbearing (the points in $\eta(u,.)$) of a micro life $u \in U$ by

$$\tau_1(u) \le \tau_2(u) \le \dots \in [0, \infty]$$

- the convention being that $\tau_k(u) = \infty$ if $\eta(u, [0, \infty)) < k$. These ages determine birth times in the usual way: $\sigma_0 = 0$ and $\sigma_{xi} := \sigma_x + \tau_x^i$, where τ_x^i is the age of x at the birth of its ith child.

Denote by $\gamma(A)$ for, $A \subseteq \mathbb{R}_+$, the number of points in A of the point process γ.

Assume given a set of joint probability measures $\tilde{P}(\gamma,\cdot)$ on $U^{\gamma(\mathbb{R}_+)}$. These describe the reproduction and dependence structure.

We are now going to construct a branching population $(U_0^I, \mathcal{U}_0^I, P)$ with the required features.

Choose some $\delta \notin U$ and put $U_0 := U \cup \{\delta\}$. Define a measure on U_0^∞ through $P(\gamma, A\times\{\delta,\delta,\dots\}) := \tilde{P}(\gamma, A)$ for $A \in \mathcal{U}^{\gamma(\mathbb{R}_+)}$.

Set $\Gamma := \mathcal{N}(\mathbb{R}_+), \Omega := U_0^\infty, \mathcal{U}_0 := \sigma(\mathcal{U} \cup \{\delta\})$, i.e. the smallest σ-algebra containing $\mathcal{U} \cup \{\delta\}$, and $\mathcal{A} := \mathcal{U}_0^\infty$. Denote by $\mathcal{G}$ the vague Borel algebra of sets in Γ, see Kallenberg (1983) p.12.

Introduce the individual space $I_1 := I \setminus \{0\}$. Make the construction through the mapping

$$\varphi : \{u_x, x \in I_1\} \longrightarrow \{\ \omega_x, x \in I\}$$

, where $\omega_x := \{u_{xi}, i=1,2,..\}$. This is a mapping from $U_0^{I_1}$ to Ω^I. It has the inverse φ^{-1}: Write $p_i(\omega_x) = u_{xi}$, the ith coordinate of ω_x; put

$$\varphi^{-1}\{\omega_x, x \in I\} := \{\ p_i(\omega_x),\ xi \in I_1\} = \{u_x,\ x \in I_1\}.$$

$\mathcal{A}^I$ is the smallest σ-algebra which makes all coordinate projections $\omega_x(\{\omega_y,\ y \in I\}) = \omega_x$ measurable. Furthermore, the product σ-algebra $\mathcal{A}$

ensures that the projections $p_i:\Omega \longrightarrow U_0$ has the same property. The inverse mapping φ^{-1} consists of compositions of the above mappings and thus we may infer $\varphi^{-1}\mathcal{A}^I \subseteq \mathcal{U}_0^{I_1}$, i.e. φ is measurable.

Then we need the natural coordinate projections $u_x:U_0^{I_1} \longrightarrow U_0$ such that

$u_x(\{u_y,y\in I\})=u_x$.

Define the types ρ: $\Omega^I \longrightarrow \Gamma$ through

$\rho_x(\{\omega_y,y\in I\}):=\eta(u_x(\varphi^{-1}\{\omega_y,y\in I\}))=\eta(u_x)$ $x\neq 0$.

Given $\{P(\gamma',\cdot),\gamma'\in\Gamma\},\{\rho_x,x\in I\}$ and $\gamma=\rho_0$, the construction in Nerman (1984) then yields a unique branching population $(\Gamma\times\Omega^I,\mathcal{G}\times\mathcal{A}^I,P_\gamma)$, called the macro or pseudo population process.

Then $\{u_x,\ x\in I_1\}$ obeys the law $P^\gamma = P_\gamma\varphi^{-1}$ for any point process γ describing the start. In the homogeneous case let us add the outcome u_0 distributed according to the marginal distribution Q. Given u_0 $P^{\eta(u_0,[0,\infty))}$ is the law for $\{u_x,\ x\in I_1\}$. This defines a law $P:=P(du_x,\ x\in I)=Q(du_0)P^{\eta(u_0,\mathbb{R}_+)}(du_y,\ y\in I)$ on $(U_0^I,\mathcal{U}_0^I)$.

Remark. Alternatively we could have started with γ equal to Dirac measure in zero and then renumbered the coordinates u_x so that u_1 was taken to be u_0, $u_{1,1}$ regarded as u_1 etc to obtain P as above.

From now on simply write U instead of U_0.

Let us make the convention that a sibling group is considered to be born at the time of its micro mother's birth. Observe that this is another convention than the one made for the splitting case.Define birthtimes for the macro process by

$\sigma_x' := \sigma_x \circ \varphi^{-1}$,if $\eta_x(\mathbb{R}_+)\circ\varphi^{-1} > 0$,

$\sigma_x' := \infty$ otherwise.

The macro process has a reproduction point process

$\xi(\rho_0,\{u_i,i\in N\},A\times B):=\#\{i;\tau_i(\rho_0)\in A,\eta(u_i)\in B\}$

The expectation of this quantity with respect to P_γ is the expected reproduction measure

$$\mu(\gamma, A \times B) = E_\gamma[\xi(A \times B)]$$
$$= \sum_{\tau_i(\gamma)\in A} Q_\gamma(B) = \gamma(A)Q_\gamma(B),$$

which in the homogeneous case, where individual lives have the marginal distribution Q, amounts to

$$\gamma(A)Q(B).$$

Remark. The general concept of homogeneity is defined in obvious parallell to the splitting case.

In Taib (1987) the reader will find a rigorous proof of the fact that quite generally imbedding schemes yield new branching processes.By multi-type versions of the methods employed in that reference,one could prove the new process to possess the branching property, see e.g Athreya and Kaplan (1978),by showing independence of daughter processes conditioned on what has happened up to the first pseudo generation.

First we consider non-individual characteristics χ: $U^I \longrightarrow \mathbb{R}_+$. Later on it will be necessary to only permit individual characteristics, which are such that they only depend on the life of one individual, the one corresponding to their index. Using the shift-like mapping S_x: $\{u_y, y \in I_1\} \longrightarrow \{u_{xy}, y \in I\}$ define $\chi_x := \chi\circ S_x$, $x \in I_1$. Then there is a multi-type characteristic χ', which performs the same counting as $z_t^\chi = \sum_{x\in I_1} \chi(t-\sigma_x)$. To ease the notation identify $z^{\chi'}_{\rho_0,t}$ and $z^{\chi'}_t$ (which obey the law P_{ρ_0}). For the macro process let T_x be the mapping from $\{\omega_y, y \in I\}$ to $\{\omega_{xy}, y \in I\}$.

Lemma 1. Let the micro process start in some arbitrary fashion. Put $u:=\{u_x, x\in I_1\}$ and

$$\chi'(\rho_x\circ\varphi(u), T_x\circ\varphi(u), t) := \sum_{i=1}^{\infty} \chi(\{u_{xiy}, y\in I\}, t - \tau_i(\{u_{xy}, y\in I\})). \quad (3)$$

Then $z_t^{\chi} = z_t^{\chi'} o\varphi$.

Remark.

The definition of χ' subsumes that χ vanishes for negative age arguments.

Proof.
$$z_t^{\chi'} o\varphi(u) = \sum_{x\in I} \chi'(\rho_x o\varphi(u), T_x o\varphi(u), t-\sigma_x' o\varphi(u))$$
$$= \sum_{x\in I} \sum_{i=1}^{\infty} \chi(\{u_{xiy}, y\in I\}, t - \tau_i(\{u_{xy}, y\in I\}) - \sigma_x(u))$$
$$= \sum_{x\in I} \sum_{i\in N} \chi(\{u_{xiy}, y\in I\}, t - \sigma_{xi}(u)) = z_t^{\chi}(u) ,$$

since if $xi\in I$ is the ith sibling in group $x \in I$ then

$$\tau_i(\{ u_{xy}, y\in I\}) + \sigma_x' o\varphi(u) = \sigma_{xi}.$$

□

The new type space can be quite complex. In the homogeneous case however, when marginal distributions of micro lives are identical, obeying the law $Q(\cdot)$ on U, and independent of the micro mother's life, the situation becomes simpler.

It is desirable that the process be Malthusian; the concept of Malthusianness for multitype processes presupposes the existence of an $\alpha \in \mathbb{R}$ and an eigenfunction $h: \Gamma \to (0, \infty)$ so that $\forall\gamma \in \Gamma$

$$h(\gamma) = \int_{\Gamma} \int_0^{\infty} h(\gamma')e^{-\alpha t}\mu(\gamma, d\gamma \times dt).$$

(Sufficient conditions for Malthusianess in general type spaces are exhibited in Jagers (1983).) In the present homogeneous case we can however be explicit about the eigenfunction. Suppose that for $\alpha \in \mathbb{R}_+$

$$E[\hat{\eta}(\alpha)] := E[\int_0^{\infty} e^{-at}\eta(u, dt)] = 1.$$

Use the conventional notation $\hat{f}(\alpha)=\int_0^{\infty} e^{-\alpha t}f(dt)$.

Now the process is Malthusian with α above as Malthusian parameter and the eigenfunction

$$h(\gamma) = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} = \int_0^{\infty} e^{-\alpha t}\gamma(dt)= \hat{\gamma}(\alpha).$$

To see this note that

$$\int_\Gamma \int_0^\infty h(\eta)e^{-\alpha t}\mu(\gamma,\ d\eta \times dt) = \int_\Gamma \int_0^\infty \hat{\eta}(\alpha)Q(d\eta)e^{-\alpha t}\gamma(dt) = E[\hat{\eta}(\alpha)]\hat{\gamma}(\alpha)$$

$$= h(\gamma)$$

(remember $E[\hat{\eta}(\alpha)] = 1$), as required.

In theorems to come we restrict attention to the subset

$$\Gamma' = \{\gamma:\ \hat{\gamma}(\alpha) < \infty\}$$

of types. Since

$$\int \hat{\gamma}(\alpha)Q(d\gamma) = E[\hat{\eta}(\alpha)] = 1,$$

$$Q(\Gamma') = 1.$$

For simplicity identify Γ and Γ'.

<u>Example 5.</u> Some simple sufficient conditions assuming Malthusianess in the not necessarily homogeneous case follow below; for more general results see Jagers (1983). If the reproduction measure has the special form

$$\mu(\gamma,\ dt \times d\gamma') = \gamma(dt)K(\gamma,\ \gamma')\lambda(d\gamma'),$$

$$Q_\gamma(d\gamma') = K(\gamma,\ \gamma')\lambda(d\gamma')$$

such that

$$\hat{\gamma}(\alpha)K(\gamma,\ \gamma') \in L^2(\Gamma \times \Gamma,\ \lambda \times \lambda),\quad \alpha > 0,$$

then, the operator

$$T_\alpha:\ L^2(\lambda) \to L^2(\lambda)$$

$$T_\alpha f(\gamma) = \int_{R_+} \int_\Gamma f(\gamma')e^{-\alpha t}\gamma(dt)K(\gamma,\gamma')\lambda(d\gamma')$$

$$= \int_\Gamma f(\gamma')\hat{\gamma}(\alpha)K(\gamma,\gamma')\lambda(d\gamma')$$

is compact, Reed and Simon (1972) p.206. Restrict the operator to bounded functions and suppose the spectral radius to be 1; in some cases one can show that the spectral radius $r(\alpha)$ is greater than one for $\alpha \geq \alpha_0$ and then show continuity in α.

If, furthermore, a communication property is assumed:

$$\lambda(G) > 0,\ \lambda(\Gamma\backslash G) > 0 \Rightarrow \int_{\Gamma\backslash G} \int_G K(s,t)\lambda(ds)\lambda(dt) > 0,$$

then, as in Jagers (1983), one may invoke Jentzsch's theorem to conclude the existence of a unique normalized strictly positive eigenfunction.

2.2. First Moments.

Suppose that each micro individual born into the population has a label $x \in I$ and denote its life by u_x and its birth time by σ_x. Let χ be a non-negative product-measurable process defined on $U \times \mathbb{R}$, which vanishes for negative values of the second argument (the age-argument), and consider

$$(4) \qquad \sum_{x \in I} \chi(u_x, t-\sigma_x),$$

which measures the total score of all micros at time t.

Taking the expectation of this quantity, interchanging the order of summation and integration and applying the standard conditioning arguments, shows that this mean is unaffected by changes in sibling dependence structure. Thus the classical single type formula is valid:

$$E[\sum_{x \in I} \chi(u_{x'}, t-\sigma_{x'})] = \int_0^\infty E[\chi(t-v)] \sum_{k=0}^\infty \mu^{*k}(dv),$$

where $\mu(dt) = E[\eta(dt)]$, and the first argument in χ has been dropped.

In the rest of this section we will, unless anything else indicated, consider a homogeneous process, which is supercritical $(\alpha > 0)$.

It is of interest to see to what extent asymptotic results from classical one-type theory carry over to this new setting, where dependences are allowed. Part of the answer for first moments lies in

Theorem 1. Provided either (i) $E[\chi]$ is continuous almost everywhere,

$\mu(dt) = E[\eta(dt)]$ is non-lattice, $\beta = \int_0^\infty ve^{-\alpha v}\,\mu(dv) < \infty$

(α the Maltusian parameter), and $\sum_{k=0}^{\infty} \sup_{k\le a\le k+1} e^{-\alpha a}E[\chi(a)] < \infty$

or

(ii) $\mu(dt)$ has a non-trivial Lebesgue component, $\beta < \infty$ and $e^{-\alpha t}E[\chi(t)]$ is bounded, Lebesgue integrable and tends to zero as $t \to \infty$,

then

$$e^{-\alpha t}E[z_t^\chi] \to E[\hat{\chi}(\alpha)]/\alpha\beta.$$

Remark.

In case (ii) the convergence takes place even in total variation.

Proof. For proofs: (i) Jagers (1975) and Çinlar (1975). (ii) Nummelin (1978). □

Later on we shall need the following Lemma,where it is assumed that the process starts with a general γ,not necessarily Dirac measure in zero.

Lemma 2. In both the settings (i) and (ii)

$$\lim_{t\to\infty} e^{-\alpha t}E_\gamma[z_t^{\chi'}] = \hat{\gamma}(\alpha)E[\hat{\chi}(\alpha)]/\alpha\beta$$

for χ' as above and $\gamma \in \Gamma$.

Proof. Write $z^\chi_{t-\tau_i(\gamma)}(i)$ for the micro process stemming from an individual born at time $\tau_i(\gamma) := \inf\{t;\gamma[0,t] \ge i\}$, Jagers and Nerman (1984). Observe that by Lemma 1 and the definition of E

$$e^{-\alpha t}E_\gamma[z_t^{\chi'}] = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} E[z^\chi_{t-\tau_i(\gamma)}(i)\circ\varphi^{-1}]e^{-\alpha(t-\tau_i(\gamma))} \quad (5),$$

since each term is bounded. Finally apply Theorem 1. □

Corollary 1. If $e^{-\alpha t}E[z_t^{\chi}] \to E[\hat{\chi}(\alpha)]/\alpha\beta$ and $t \longrightarrow E[z_t^{\chi}]$ is bounded on finite intervals, then

$$\sup_{\gamma,t} e^{-\alpha t}E_{\gamma}[z_t^{\chi}]/h(\gamma) < \infty.$$

Proof. From the convergence of the hypothesis

$$\sup_{t} e^{-\alpha t}E[z_t^{\chi}] = c < \infty,$$

and hence by (5)

$$e^{-\alpha t}E_{\gamma}[z_t^{\chi'}]/\hat{\gamma}(\alpha) \le \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} c/\hat{\gamma}(\alpha) = c. \qquad \square$$

Since the pseudoindividuals of generation n stem from microindividuals in generation $n-1$, we have the following expected reproduction measure for the nth generation of a population starting off with an pseudo ancestor of type γ,which not necessarily equals $1\{0\}$:

$$\mu^{(n)}(\gamma,\ dt \times d\gamma') = \gamma * \mu^{*(n-1)}(dt) \times Q(d\gamma'),\ n \ge 1,$$

where

$$\mu(dt) = E[\eta(dt)] \quad \text{as before.}$$

Collecting terms and taking into account the non-random start we obtain the expected population measures

$$\nu(\gamma, dt \times d\gamma') = \gamma * \sum_{n=0}^{\infty} \mu^{*n}(dt) \times Q(d\gamma'),$$

μ^{*o} as usual signifying unit mass at the origin. With $\nu = \sum_{n=0}^{\infty} \mu^{*n}$ this reduces to

$$\gamma * \nu(dt) \times Q(d\gamma').$$

2.3. Second moments.

We shall now explore the asymptotics of $e^{-2\alpha t}\mathrm{Var}_{\gamma}[z_t^{\chi'}]$, χ' as in the previous section.

The starting point will be

Lemma 3.3 in Nerman (1984). Suppose that $m^{\chi'}_{\gamma,t} = E_\gamma[z^{\chi'}_t]$ is finite and that χ' is individual, i.e. only depending on the life indicated by its index. Then for $\gamma \in \Gamma$, $t \in \mathbb{R}_+$

$$\mathrm{Var}_\gamma[z^{\chi'}_t] = \int_0^\infty \int_\Gamma g_2(\gamma', t-v)\nu(\gamma, dv\times d\gamma') \quad (6),$$

where

$$g_2(\gamma,s) = \mathrm{Var}[\chi'_0(s) + \int_0^\infty \int_\Gamma m^{\chi'}_{\gamma,s-v}\, \xi_0(\rho_0, dv\times d\gamma')].$$

Unlike the means, the variances will obviously be affected by changes in the dependence structure of the sibling lives. However, traditional conditions on the micro reproductions η and the micro characteristics χ still yield variance convergences as in the case of independent reproductions:

Theorem 2

If characteristics are individual and

1. $\sup_v e^{-2\alpha v}\mathrm{Var}[\chi(v)] = c_1 < \infty$
2. $\lim_{v\to\infty} e^{-2\alpha v}\mathrm{Var}[\chi(v)] = 0$
3. $e^{-\alpha t} m^{\chi'}_{\gamma,t} \to \hat{\gamma}(\alpha)E[\hat{\chi}(\alpha)]/\alpha\beta$,

   in either of the settings (i) or (ii), $\gamma \in \Gamma$, and
4. $E[\hat{\eta}^2(\alpha)] < \infty$.

Then for any $\gamma \in \Gamma$

$$\lim_{t\to\infty} e^{-2\alpha t}\mathrm{Var}_\gamma[z^{\chi'}_t] = (E[\hat{\chi}(\alpha)]/\alpha\beta)^2\{\bar{v}^2\hat{\gamma}(2\alpha)/(1-\hat{\mu}(2\alpha))\},$$

with

$$v^2(\gamma) = \mathrm{Var}_\gamma[\bar{\xi}_0],$$

$$\bar{\xi}_0 = \int_0^\infty \int_\Gamma \hat{\gamma}(\alpha)e^{-\alpha v}\xi_0(\rho_0, dv\times d\gamma),$$

and

$$\bar{v}^2 = \int_\Gamma v^2(\gamma)Q(d\gamma)$$

Remark. Note

$$\bar{\xi}_0 = \sum_{i=1}^{\infty} e^{-\alpha\tau_i(\gamma)} \hat{\eta}_i(\alpha)$$

and

$$v^2(\gamma) = \hat{\gamma}(2\alpha)\mathrm{Var}[\hat{\eta}(\alpha)] + \sum_{i\neq j} e^{-\alpha(\tau_i(\gamma)+\tau_j(j))} \mathrm{Cov}[\hat{\eta}_i(\alpha), \hat{\eta}_j(\alpha)]$$

So if, for instance,

$$c = \mathrm{Cov}[\hat{\eta}_i(\alpha), \hat{\eta}_j(\alpha)], \ i \neq j,$$

then we have

$$\bar{v}^2 = \mathrm{Var}[\hat{\eta}(\alpha)]E[\hat{\gamma}(2\alpha)] + c\ E[\sum_{i\neq j} e^{-\alpha(\tau_i(\gamma)+\tau_j(\gamma))}] =$$

$$= \mathrm{Var}[\hat{\eta}(\alpha)]E[\hat{\gamma}(2\alpha)] + c\ E[(\hat{\gamma}^2(\alpha) - \hat{\gamma}(2\alpha))].$$

And finally, invoking homogeneity:

$$\bar{v}^2 = c\ E[\hat{\gamma}^2(\alpha)] + (\mathrm{Var}[\hat{\gamma}(\alpha)] - c)E[\hat{\gamma}(2\alpha)]$$

Corollary 2. Under the conditions of Theorem 2

$$\lim_{t\to\infty} e^{-2\alpha t}\mathrm{Var}[z_t^{\chi}] = (E[\hat{\chi}(\alpha)]/\alpha\beta)^2\ \bar{v}^2/(1-\hat{\mu}(2\alpha))$$

Proof of the Corollary. The process starts with one ancestor born at time zero: γ has mass one at the origin and so $\hat{\gamma}(\alpha) = 1$. □

Proof of the theorem. A slight modification of the proof of Theorem 6.1 in Nerman (1984) validates the statement. The technique is that of repeated dominated convergence.

First note that for g_2 as in (6) we can write

$$e^{-2\alpha s} g_2(\gamma,s) \leq e^{-2\alpha s} 2(\mathrm{Var}[\chi_0'(s)] + \mathrm{Var}_\gamma[\int_0^\infty \int_\Gamma m^{\chi'}_{\gamma,s-v} \xi(\rho_0,\ dvxd\gamma')]) \quad (7).$$

We shall show that the right member is dominated by $2(c_1 + c_3^2)\hat{\gamma}^2(\alpha)$ for appropriate constants c_1 and c_3. With $K_t^{\chi} = E[\chi(t)]$ Jensen's inequality yields

$$[e^{-\alpha s}(\chi_0'(s) - E_\gamma[\chi_0'(s)])]^2 =$$

$$= [\hat{\gamma}(\alpha) \sum_{i=1}^{\infty} (\chi(u_i, s-\tau_i(\gamma)) - K^{\chi}_{s-\tau_i(\gamma)}(e^{-\alpha\tau_i(\gamma)}/\hat{\gamma}(\alpha)) e^{-\alpha(s-\tau_i(\gamma))}]^2 \le$$

$$\le \hat{\gamma}^2(\alpha) \sum_{i=1}^{\infty} \{e^{-\alpha\tau_i(\gamma)}(\chi(u_i, s-\tau_i(\gamma)-K^{\chi}_{s-\tau_i(\gamma)}))\}^2 (e^{-\alpha\tau_i(\gamma)}/\hat{\gamma}(\alpha)).$$

By this and condition 1 the first term on the right side of (7) is dominated by (t_i for $\tau_i(\gamma)$)

$$2\hat{\gamma}^2(\alpha)E_{\gamma}(\sum_{i=1}^{\infty}(\chi'(u_i, s-t_i) - K^{\chi'}_{s-t_i})e^{-\alpha(s-t_i)}(e^{-\alpha t_i}/\hat{\gamma}(\alpha)))^2$$

$$\le 2\hat{\gamma}^2(\alpha) \int_0^{\infty}(e^{-\alpha t}/\hat{\gamma}(\alpha))\mathrm{Var}[\chi'(s-t)e^{-\alpha(s-t)}]\gamma(dt)$$

$$\le 2\hat{\gamma}^2(\alpha)c_1.$$

According to Condition 3 and Corollary 1 there exists a constant

$$c_3 = \sup_{\gamma\in\Gamma', t} e^{-\alpha t} m^{\chi}_{\gamma,t}/h(\gamma) < \infty.$$

Thus

$$2e^{-2\alpha s}\mathrm{Var}_{\gamma}[\int_0^{\infty}\int_{\Gamma} m^{\chi}_{\gamma,s-v}\ \xi(\rho_0,\ dvxd\gamma')]$$

$$\le 2\ c_3^2\ \mathrm{Var}_{\gamma}[\int_0^{\infty}\int_{\Gamma} \hat{\gamma}'(\alpha)e^{-\alpha v}\ \xi(\rho_0,\ dvxd\gamma')].$$

Using Jensen's inequality on

$$(\hat{\gamma}(\alpha) \sum_{i=1}^{\infty} (e^{-\alpha t_i}/\hat{\gamma}(\alpha))\hat{\eta}^i(\alpha))^2,$$

we can dominate this by

$$2c_3^2E_{\gamma}(\int_0^{\infty}\int_{\Gamma}\hat{\gamma}'(\alpha)e^{-\alpha v}\xi_0(\rho_0,\ dvxd\gamma'))^2$$

$$= 2c_3^2E_{\gamma}(\sum_{i=1}^{\infty} e^{-\alpha t_i}\int_0^{\infty}e^{-\alpha t}\eta_i(dt))^2$$

$$= 2c_3^2E_{\gamma}(\sum_{i=1}^{\infty}(e^{-\alpha t_i}/\hat{\gamma}(\alpha))\hat{\eta}_i(\alpha))^2\hat{\gamma}^2(\alpha)$$

$$\le 2c_3^2\sum_{i=1}^{\infty}(e^{-\alpha t_i}/\hat{\gamma}(\alpha))E[\hat{\eta}^2(\alpha)]\hat{\gamma}^2(\alpha)$$

$$= 2c_3^2E[\hat{\eta}^2(\alpha)]\hat{\eta}^2(\alpha).$$

Our next claim is that $(\hat{\gamma}'(\alpha))^2$ is integrable with respect to $e^{-2\sigma v}\nu(\gamma,\ dvxd\gamma')$. But this follows from

$$\int_0^\infty \int_\Gamma h^2(\gamma')e^{-2\alpha v}\nu(\gamma,\ dvxd\gamma')$$

$$= \int_0^\infty \int_\Gamma (\hat{\gamma}'(\alpha))^2 e^{-2\alpha v}\gamma*\nu(dv)Q(d\gamma')$$

$$= E[\hat{\eta}^2(\alpha)] \int_0^\infty e^{-2\alpha v}\gamma*\nu(dv)$$

$$= E[\hat{\eta}^2(\alpha)] \quad \hat{\gamma}(2\alpha)/(1-\hat{\mu}(2\alpha)) < \infty,$$

recall that $\int_0^\infty e^{-2\alpha t}\mu(dt) = \hat{\mu}(2\alpha) < 1$ and

$$\hat{\nu}(2\alpha) = (1 - \hat{\mu}(2\alpha))^{-1}$$

Hence the Dominated Convergence Theorem may be applied: From Condition 2 and the Cauchy-Schwartz inequality

$$\lim_{t\to\infty} e^{-2\alpha t}\mathrm{Var}_\gamma[\chi_0(t) + \int_0^\infty \int_\Gamma m^\chi_{\gamma',t-v}\xi_0(\rho_0,\ dvxd\gamma')]$$

$$= \lim_{t\to\infty} \mathrm{Var}_\gamma[\int_0^\infty \int_\Gamma e^{-\alpha(t-v)}m^\chi_{\gamma',t-v}e^{-\alpha v}\xi_0(\rho_0,\ dvxd\gamma')],$$

which by Condition 3 equals

$$\mathrm{Var}_\gamma[\int_0^\infty \int_\Gamma h(\gamma')(E[\hat{\chi}(\alpha)]/\alpha\beta)e^{-\alpha v}\xi_0(\rho_0,\ dvxd\gamma')]$$

$$= (E[\hat{\chi}(\alpha)]/\alpha\beta)^2\mathrm{Var}_\gamma[\bar{\xi}_0],$$

in the notation

$$\bar{\xi}_0 = \int_0^\infty \int_\Gamma h(\gamma')e^{-\alpha v}\xi_0(\rho_0,\ dvxd\gamma').$$

Integrating the right hand side with respect to $\nu(\gamma,\ dvxd\gamma')$ gives the limit:

$$\lim_{t\to\infty} e^{-2\alpha t}\mathrm{Var}_\gamma[z^\chi_{\rho_0,t}] =$$

$$= (E[\hat{\chi}(\alpha)]/\alpha\beta)^2 \int_0^\infty \int_\Gamma \mathrm{Var}_{\gamma'}[\bar{\xi}_0]e^{-2\alpha v}\nu(\gamma,\ dvxd\gamma')$$

$$= (E[\hat{\chi}(\alpha)]/\alpha\beta)^2 \int_\Gamma \mathrm{Var}_{\gamma'}[\bar{\xi}_0]Q(d\gamma')\hat{\gamma}(2\alpha)/(1-\hat{\mu}(2\alpha))$$

This ends the proof. □

Remark. Theorem 6.2 in Nerman (1984) could have been invoked to prove a similar theorem. Nerman's results follow from a study of the embedded Markov renewal sequence (M,T) generated by

$$\mu_\alpha(\gamma,\ d\gamma_1\times dt) = \hat{\gamma}_1(\alpha)e^{-\alpha u}\mu(\gamma,\ d\gamma_1\times dt)/\hat{\gamma}(\alpha)$$
$$= \hat{\gamma}_1(\alpha)e^{-\alpha u}\gamma(dy)Q(d\gamma_1)/\hat{\gamma}(\alpha).$$

M is in our case recurrent with respect to Q (and indeed i.i.d.) with stationary measure $\hat{\gamma}(\alpha)Q(d\gamma)$, as required.

Furthermore, the integrals (written in our notation)

$$\int_\Gamma [\int_0^\infty u(e^{-\alpha u}/\hat{\gamma}(\alpha))\gamma(du)]\hat{\gamma}(\alpha)Q(d\gamma) = \int_0^\infty ue^{-\alpha u}\mu(du) = \beta$$

and

$$\int_\Gamma \hat{\gamma}(\alpha)Q(d\gamma)/\hat{\gamma}(\alpha) = Q(\Gamma) = 1,$$

which should be finite, are certainly so under the conditions of the previous theorem.

Then Nerman (1984) postulates the existence of a probability measure

φ

on $\mathbb{R}_+ \times \Gamma$, a constant $0 \leq c < 1$, and a measurable $f: \Gamma \to [0,1]$ such that

$$\int_\Gamma f(\gamma)\hat{\gamma}(\alpha)Q(d\gamma) > 0,$$

$$\sum_{n=1}^\infty c^{n-1}\mu_\alpha^{*n}(\gamma,\cdot) \geq f(\gamma)\varphi(\cdot)$$

and such that the measure

$\int_\Gamma f(\gamma)\varphi(\cdot\times d\gamma)$ is spread out, i.e. some convolution power of it has a nontrivial Lebesgue component. This condition poses some problems, since the measures here are not necessarily spread out.

The natural condition $P_\gamma(y_t' < \infty) = 1$ $\forall\gamma \in \Gamma$, $\forall t$, where y_t' = total number of macros born before t, is satisfied.

However last but not least,a condition needed for the convergence of $e^{-\alpha t}\nu(\gamma,\ [0,t] \times \Gamma')$, stemming from theorem. 5.2 of Nerman (1984), namely $\inf_\gamma \hat{\gamma}(\alpha) > 0$, became dispensable here.

Example 6. Let us examine the extreme case of total reproductive similarity between siblings, i.e. they reproduce simultaneously. This means that

$$\mathrm{Var}_\gamma\Big[\sum_{i=1}^{\infty} e^{-\alpha t_i}\hat{\eta}^i(\alpha)\Big] = \hat{\gamma}^2(\alpha)\mathrm{Var}[\hat{\eta}(\alpha)].$$

Thus according to the preceding theorem

$$e^{-2\alpha t}\mathrm{Var}_\gamma[z_t^\chi] \to (E[\hat{\chi}'(\alpha)]/\alpha\beta)^2$$
$$\times \mathrm{Var}[\hat{\eta}(\alpha)] + E[\hat{\eta}^2(\alpha)]\hat{\gamma}(2\alpha)/(1 - \hat{\mu}(2\alpha))),$$

for any χ satisfying the conditions of the theorem (and $\beta < \infty$).

If γ places all mass at zero, then the above turns into

$$e^{-2\alpha t}\mathrm{Var}_\gamma[z_t^\chi] \to (E[\hat{\chi}(\alpha)]/\alpha\beta)^2\mathrm{Var}[\hat{\eta}(\alpha)]$$
$$\times E[\hat{\eta}^2(\alpha)]/(1 - \hat{\mu}(2\alpha),$$

which should be compared with the corresponding convergence in a population of independently reproducing individuals to the limit

$$(E[\hat{\chi}(\alpha)]/\alpha\beta)^2\mathrm{Var}[\hat{\eta}(\alpha)]/(1 - \hat{\mu}(2\alpha)).$$

A Martingale.

As in the preceding section only the homogeneous supercritical Malthusian case is dealt with. Following Nerman (1984) we define a process on the macro space by

$$w_t = \sum_{x\in I(t)} e^{-\alpha\sigma'_x} h(\rho_x) ,$$
$$I(t) = \{xj:\ \sigma'_x \le t < \sigma'_{xj} < \infty\},$$

the coming generation at time t. With

$X(n)$:= the number of the nth individual to appear,

$$\mathcal{F}_n := \sigma(\rho_0, \omega_{X(1)}, \ldots, \omega_{X(n)})$$

and

$$\mathcal{F}_t := \mathcal{F}_{y'_t} = \sigma\Big(\bigcup_{n=0}^{\infty} \mathcal{F}_n \cap \{\sigma'_{X(n)} \le t\}\Big),$$

it follows that w_t is a martingale with respect to $\mathcal{F}_t$ (op.cit.). Hence

$$E_\gamma[w_t] = E_\gamma[w_0] = \hat{\gamma}(\alpha).$$

w_t can be expressed as

$$e^{-\alpha t}z_t^\chi \quad ,$$

by means of the characteristic

$$\chi'(\rho_0, a) = e^{\alpha a} \int_\Gamma \int_a^\infty e^{-av} \hat{\gamma}(\alpha) \xi_0(\rho_0, dvxd\gamma').$$

With the above χ' and the notation from the previous section, we can write

$$e^{-2\alpha t} g_2(\gamma, t) = \mathrm{Var}_\gamma[\chi_0'(t) e^{-\alpha t} +$$

$$+ \int_0^t \int_\Gamma e^{-\alpha(t-v)} m^{\chi'}_{\gamma', t-v} e^{-\alpha v} \xi_0(\rho_0, dvxd\gamma')]$$

$$= \mathrm{Var}_\gamma[\chi_0'(t) e^{-\alpha t} + \int_0^t \int_\Gamma h(\gamma') e^{-\alpha v} \xi_0(\rho_0, dvxd\gamma')]$$

$$\le \mathrm{Var}_\gamma[\int_0^\infty \int_\Gamma h(\gamma') e^{-\alpha v} \xi_0(\rho_0, dvxd\gamma')] =$$

$$= \mathrm{Var}_\gamma[\xi_0] \le E_\gamma[\bar{\xi}_0^2] \le h^2(\gamma) E[\hat{\eta}^2(\alpha)],$$

see proof of preceding theorem. Again assume $E[\hat{\eta}^2(\alpha)] < \infty$. Then

$$\mathrm{Var}_\gamma[w_t] = e^{-2\alpha t} \mathrm{Var}[z_t^{\chi'}] = \int_0^t \int_\Gamma \mathrm{Var}_{\gamma_1}[\bar{\xi}_0]$$

$$e^{-2\alpha v} \nu(\gamma, dvxd\gamma_1) \le$$

$$\int_0^\infty \int_\Gamma h^2(\gamma') E[\hat{\eta}^2(\alpha)] e^{-2\alpha v} \nu(\gamma, dvxd\gamma') \le$$

$$\le E[\hat{\eta}^2(\alpha)]\ E[\hat{\eta}^2(\alpha)] \hat{\gamma}(2\alpha)/(1 - \hat{\mu}(2\alpha)) < \infty, \ \forall \gamma \in \Gamma.$$

The L_2- convergence of w_t is now given by standard results for martingales with bounded variance,cf eg Corollary 2.2 in Hall and Heyde (1980) .

<u>Theorem 3.</u> If $E[\hat{\eta}^2(\alpha)] < \infty$,there exists a w_∞ such that $w_t \xrightarrow{L_2(P_\gamma)} w_\infty$ and $E_\gamma[w_\infty] = \hat{\gamma}(\alpha)$, $\forall \gamma \in \Gamma$.

<u>The Convergence of the Process in L_2.</u>

Very much in the vein of Jagers and Nerman (1984) we can derive the following

<u>Theorem 4.</u> Suppose that the conditions of Theorem 1 are satisfied. Then

$\forall \gamma \in \Gamma$

$$z^{\chi'}_{t,\gamma}\ e^{-\alpha t} \xrightarrow{L_2} w_\infty\ E[\hat{\chi}(\alpha)]/\alpha\beta.$$

Here w_∞ as in the previous section, and $z^{\chi'}_{t,\gamma}$ denotes $z^{\chi'}_t$ with a pseudoancestor of type γ.

Proof. Defining $K = E[\hat{\chi}(\alpha)]/\alpha\beta$, we arrive at

$$E_\gamma[(e^{-\alpha t}z^\chi_t - Kw_\infty)^2] \le 2E_\gamma[e^{-\alpha t}z^\chi_t - Kw_t)^2] + 2K^2E_\gamma[(w_t - w_\infty)^2].$$

Theorem 2 shows $E_\gamma[(w_t - w_\infty)^2] \to 0$, $t \to \infty$. Since $E_\gamma[e^{-\alpha t}z^\chi_t - Kw_t] = e^{-\alpha t}m^\chi_{\gamma,t} - \hat{\gamma}(\alpha)K \to 0$, as $t \to \infty$, it is enough to prove that

$$\lim_{t\to\infty} \mathrm{Var}_\gamma[e^{-\alpha t}z^\chi_t - Kw_t] = 0.$$

Denote the characteristic used in the proof of Theorem 2 by $\tilde{\chi}'(\rho_0, w_0, a) = e^{\alpha a} \int_\Gamma \int_Q^\infty \hat{\gamma}'(\alpha)e^{-\alpha v}\xi_0(\rho_0,\ dvxd\gamma')$. Recall the relation

$$\mathrm{Var}[\eta_1 - \eta_2] = 2\mathrm{Var}[\eta_1] + 2\mathrm{Var}[\eta_2] - \mathrm{Var}[\eta_1 + \eta_2],$$

and the preceding two theorems to realize that

$$\lim_{t\to\infty} \mathrm{Var}_\gamma[z^\chi_t e^{-\alpha t} - Kw_t] = 2 \lim_{t\to\infty} e^{-2\alpha t}\mathrm{Var}_\gamma[z^\chi_t]$$

$$+ 2 \lim_{t\to\infty} e^{-2\alpha t}\mathrm{Var}_\gamma[KZ^{\tilde{\chi}}_t] - \lim_{t\to\infty} e^{-2\alpha t} \times$$

$$\mathrm{Var}_\gamma[z^{\chi+K\tilde{\chi}}_t] = 0.$$

N.B. $z^{K\tilde{\chi}}_t = Kz^{\tilde{\chi}}_t$ and $z^{\chi+\tilde{\chi}} = z^\chi_t + z^{\tilde{\chi}}_t$. □

Corollary 3. Under the conditions of Theorem 3

$$z^\chi_t o\varphi \xrightarrow{L_2} w_\infty E[\hat{\chi}(\alpha)]/\alpha\beta$$

Proof. Specialize to the case when γ puts mass one at the origin. □

Convergence in Probability and in L_1.

This section sets out to prove convergence in probability of the process under quite general conditions, namely those stated in Theorem 1. Adding one condition we also obtain convergence in L_1.

Before getting on to Theorems 4a and 4b we have to make a couple of things clear.

Following Nerman (1984) define the truncated coming generation

$$I(t,s) = \{xk, k\in N;\ \sigma'_x \le t < \sigma'_{xk} \le t+s\}$$

and the corresponding sum of reproductive values

$$w_{t,s} = \sum_{x\in I(t,s)} \hat{\rho}_x(\alpha)\exp(-\alpha\sigma'_x).$$

This quantity is useful as an underestimate of the martingale w_t. The next result states that more precisely.With y'_t=the total number of macros born up to time t:

<u>Proposition 1.</u> In the homogeneous case assume $\beta < \infty$. Then if $\rho_0 = \gamma$

$$\{w_\infty > 0\} = \{y'_t \to \infty\} = \{\limsup_{t\to\infty} w_{t,s} > 0\}$$

a.s. , $s \ge s(\gamma)$ for some $s(\gamma)$.Furthermore $P_\gamma(\{y'_t \longrightarrow \infty\} \cup \{w_t = 0\}) = 1$.

<u>Proof.</u> From the definitions

$$\{y'_t \to \infty\} \supseteq \{w_\infty > 0\} \supseteq \{\limsup_{t\to\infty} w_{t,s} > 0\}.$$

Put

$$\mathcal{F}_\infty = \lim_{k\to\infty} \mathcal{F}_k.$$

From

$$\nu(\gamma,\ \Gamma\times[0,t]) < \infty \quad \forall t$$

clearly

$$P_\gamma(\sigma'_{X(k)} \longrightarrow \infty) = 1,$$

whence

$$P_\gamma(\{w_\infty > 0\}\cup\{y'_t \not\longrightarrow \infty\} \cup \{\sigma_{X(k)} \not\longrightarrow \infty\} \mid \mathcal{F}_n)$$

tends to

$$1\{w_\infty > 0\} \cup \{\ y'_t \not\longrightarrow \infty\ \}$$

,as $n \longrightarrow \infty$,according to Levy's theorem,both unions being measurable $\mathcal{F}_\infty$,see e.g. Chung (1974).

The latter probability exceeds

$$P_\gamma(\{\limsup_{t\to\infty} w_{t,s} > 0\}\cup\{y'_t \not\longrightarrow \infty\}\,|\,\mathcal{F}_n).$$

Since also

$$\{\limsup_{t\to\infty} w_{t,s} > 0\}\cup\{y'_t \not\longrightarrow \infty\}$$

is measurable $\mathcal{F}_\infty$, the same theorem gives that the last mentioned probability tends to

$$1\{\limsup_{t\to\infty} w_{t,s} > 0\}\cup\{y'_t \not\longrightarrow \infty\}, \text{as } n \longrightarrow\infty.$$

And this will be shown to equal one a.s. Introducing the truncated variable $w^c_{t,s} := w_{t,s} \wedge c, c>0$, and invoking a version of Fatou's Lemma (1.6.8 (b) in Ash, 1972) shows

$$E_\gamma[\limsup_{t\to\infty} w_{t,s}] \geq E_\gamma[\limsup_{t\to\infty} w^c_{t,s}] \geq \limsup_{t\to\infty} E_\gamma[w^c_{t,s}].$$

That the right hand side is bounded away from zero for c and s appropriately chosen becomes clear from consideration of the expected difference between w_t and $w_{t,s}$. Passing to the limit using Theorem 1 and writing

$$\chi(a) = \int_\Gamma \int_{a+s}^{\infty} e^{-\alpha v}\, \hat{\gamma}'(\alpha)\xi_0(\rho_0, dv \times d\gamma'):$$

$$\lim_{t\to\infty} E_\gamma[w_t - w_{t,s}] = \lim_{t\to\infty} e^{-\alpha t}E_\gamma[z^\chi_t] =$$

$$= E_\gamma[\hat{\chi}(\alpha)]/\alpha\beta = \hat{\gamma}(\alpha) \int_s^\infty (u-s)e^{-\alpha u}\mu(du) \leq$$

$$\hat{\gamma}(\alpha) \int_s^\infty u e^{-\alpha u}\mu(du),$$

which does not exceed $\hat{\gamma}(\alpha)/2$ for s greater than some s_0. Recall that

$$\lim_{t\to\infty} E_\gamma[w_t] = \hat{\gamma}(\alpha).$$

Thus

$$E_\gamma[\limsup_{t\to\infty} w_{t,s}] \geq \hat{\gamma}(\alpha)/2,$$

implying that

$$P_\gamma(\limsup_{t\to\infty} w_{t,s} > 0) > 0.$$

And from the recurrence with respect to Q, noted in the Remark following the proof of Theorem 2, the process $\rho_{X(n)}$ will a.s. revisit the set

$$\{\gamma_1 : P_{\gamma_1}[\limsup_{t\to\infty} w_{t,s} > 0] > \delta(\gamma) > 0\}$$

(for some $\delta(\gamma), \gamma$ the starting type) an infinite number of times on $\{y'_t \longrightarrow \infty\}$.

Hence a.s., with $T_x: \{\rho_0, \omega_y, y \in I\} \longrightarrow \{\rho_x, \omega_{xy}, y \in I\}$,

$$\lim_{n\to\infty} P_\gamma(\{\limsup_{t\to\infty} w_{t,s} > 0\}\cup\{y'_t \not\to \infty\}|\mathcal{F}_n)$$

$$\geq \limsup_{n\to\infty} P_\gamma(\limsup_{t\to\infty} w_{t,s} \circ T_{X(n+1)} > 0|\mathcal{F}_n)$$

$$= \limsup_{n\to\infty} P_{\rho_{X(n+1)}}(\limsup_{t\to\infty} w_{t,s} > 0) > \delta(\gamma) > 0.$$

Consequently the indicator exceeds $\delta(\gamma)$, in which case it must equal unity a.s. □

Much as in Jagers and Nerman (1984) Theorem 5.8, we can prove

<u>Theorem 4.a.</u> Suppose that the characteristic χ satisfies the conditions of Theorem 1. Then for any starting type $\gamma \in \Gamma$

$$e^{-\alpha t} z_t^{\chi'} \circ \varphi(u) \longrightarrow E[\hat{\chi}(\alpha)] w_\infty \circ \varphi(u)/\alpha\beta$$

in probability, as $t \to \infty$.

<u>Proof.</u> The object is to show that for any $\epsilon > 0$ and s large enough

$$\limsup_{t\to\infty} P_\gamma(|e^{-\alpha(t+2s)} z_{t+2s}^{\chi'} - E[\hat{\chi}(\alpha)] w_\infty/\alpha\beta| > \epsilon) < \epsilon.$$

Again following Jagers and Nerman (1984) define a truncated characteristic:

Let χ'_c be defined through (2) with χ replaced by

$$\chi_c = c \wedge \chi\, 1_{[0,c]}, \quad c > 0;$$

recall the truncated coming generation

$$I(t,s) = \{xk,\ \sigma'_x \leq t < \sigma'_{xk} \leq t+s\}$$

and the corresponding sum of reproductive values

$$w_{t,s} = \sum_{x\in I(t,s)} \hat{\rho}_x(\alpha)\exp(-\alpha\sigma'_x).$$

For $s > c > 0$ we write

$$P_1(t) = \exp(-\alpha(t+2s))(z^{\chi'}_{t+2s} - z^{\chi'_c}_{t+2s}),$$

$$P_2(t) = \exp(-\alpha(t+2s))(z^{\chi'}_{t+2-\sigma'_x}(x) - \sum_{x\in I(t,s)} z^{\chi'_c}_{t+2s-\sigma'_x}(x)),$$

$$P_3(t) = |\exp(-\alpha(t+2s)) \sum_{x\in I(t,s)} (z^{\chi'_c}_{t+2s-\sigma'_x}(x) - m^{\chi'_c}_{t+2s-\sigma'_x})|,$$

$$P_4(t) = |\exp(-\alpha(t+2s)) \sum_{x\in I(t,s)} m^{\chi'_c}_{t+2s-\sigma'_x} - E[\hat{\chi}_c(\alpha)]w_{t,s}/\alpha\beta|,$$

$$P_5(t) = |E[\hat{\chi}_c(\alpha)]w_{t,s}/\alpha\beta - E[\hat{\chi}(\alpha)]w_t/\alpha\beta|,$$

$$P_6(t) = |E[\hat{\chi}_c(\alpha)]w_t/\alpha\beta - E[\hat{\chi}_c(\alpha)]w_\infty/\alpha\beta|,$$

and

$$P_7 = |E[\hat{\chi}_c(\alpha)]w_\infty/\alpha\beta - E[\hat{\chi}(\alpha)]w_\infty/\alpha\beta|.$$

Clearly

$$|\exp(-\alpha(t+2s))z^{\chi'}_{\rho_0,t+2s} - E[\hat{\chi}(\alpha)]w_\infty/\alpha\beta$$

$$\le P_1(t) + P_2(t) + P_3(t) + P_4(t) + P_5(t) + P_6(t) + P_7.$$

For each $i = 1,2,\ldots, 7$ and any $\epsilon > 0$ it will be shown that s and c may be chosen large enough to make

$$\limsup_{t\to\infty} P_\gamma(P_i(t) > \epsilon) < \epsilon.$$

This will either be done directly or via Markov's inequality, the inequality

$$P_\gamma(p_1(t) +\ldots+ p_7 > \epsilon) \le$$

$$P_\gamma(p_1(t) > \epsilon/7) +\ldots+ P_\gamma(p_7 > \epsilon/7)$$

concluding the argument, Jagers and Nerman (1984). Here we shall only deal with $p_3(t)$ thoroughly. Briefly: p_7 and $p_1(t)$ are taken care of by monotone convergence, $p_5(t)$ and $p_2(t)$ by finiteness and asymptotic equality of w_t and $w_{t,s}$, $p_4(t)$ by Theorem 1.

Now, let $N(t,s) = \#I(t,s)$.

Before we can invoke the Strong Law of Large Numbers we need to know that

$$\{N(t,s) \to \infty\} = \{y'_t \to \infty\} \text{ a.s.}$$

This is however clear from

$$\{y'_t \longrightarrow \infty\} = \{\limsup_{t\to\infty} w_{t,s} > 0\} \supseteq \{N(t,s) \to \infty\} \supseteq \{y'_t \longrightarrow \infty\}.$$

The convergence of $p_3(t)$ will follow from

$$\limsup_{t\to\infty} P_\gamma(|\sum_{x'\in I(t,s)} (z^{\chi'_c}_{t+2s-\sigma_x} - m^{\chi'_c}_{t+2s-\sigma_x})/N(t,s)| >$$
$$> \epsilon \;|\{y'_t \to \infty\}) = 1 \quad \forall\, \epsilon > 0,$$

by an application of the Dominated Convergence Theorem and as a consequence of

$$\limsup_{t\to\infty} P_\gamma(|\sum_{x'\in I(t,s)} (z^{\chi'_c}_{\rho_x,t+2s-\sigma'_x}(x) - m^{\chi'_c}_{t+2s-\sigma_x})/N(t,s)| >$$
$$\epsilon|\mathcal{F}_{y'_t}) = 0$$

a.s. on $\{y'_t \to \infty\}$.

Conditioned on $\mathcal{F}_{y'_t}$, $I(t,s)$, $N(t,s)$ and $\sigma'_x \in I(t,s)$ are all known, while $z^{\chi'_c}_{\rho_x,u}(x) = z^{\chi'_c}_{\rho_0,u} \circ S_x$, $x \in I(t,s)$, are independent.

The conditional distributions of $z^{\chi'_c}_{\rho_x,t+2s-\sigma'_x}(x)$, $x \in I(t,s)$, are stochastically dominated by $c\, y_{2s}$, where y_{2s} are the number of micros born up to time $2s$. Furthermore $E[z^{\chi'_c}_{\rho_x,t+2s-\sigma'_x}|\mathcal{F}_{y_t}] = m^{\chi'_c}_{\rho_x,t+2s-\sigma'_x}$, and finally an invocation of Lemma 5.7 of op.cit., which is a Weak Law of Large Numbers, settles the matter. □

A modification of the proof of Theorem 5.3 in Jagers and Nermam (1984) yields

Theorem 4.b. Under the condition

$$(x \log x) \quad E[\hat{\eta}(\alpha)\log^+(\hat{\eta}(\alpha))] < \infty,$$

the convergence in the previous theorem takes place in L_1.

Remark.

As easy corollaries follow statements analogous to theorems 4.a and 4.b for the process z^χ_t starting from one ancestor.

Multi-type Processes with Sibling Dependences.

The above scheme can be extended to the case of a finite-type process. For the sake of illustration choose a type space with just two types. We can perform the same imbedding as before, the type of the pseudo-individual now being both the birth times and the types of the siblings: $\Gamma = \mathcal{N}(\mathbb{R}_+ \times \{1,2\})$. In order to prove the process to be Malthusian we need to exhibit an eigenfunction, ie $h:\Gamma \longrightarrow \mathbb{R}$ and an $\alpha \in \mathbb{R}$:

$$h(\gamma) = \int_\Gamma \int_R e^{-\alpha t} h(\eta)\mu(\gamma, dt \times d\eta) \qquad (8).$$

First some notation. Let for $i=1,2$

γ^i := the reproduction point process corresponding to births of individuals of type i borne by a mother with life γ.

Thus $\gamma = (\gamma^1, \gamma^2)$ and $\gamma[0,t] = \gamma^1[0,t] + \gamma^2[0,t]$.

Assume given probability measures $Q(i,A), i=1,2, A \in \mathcal{B}(\mathcal{N}(\mathbb{R}_+ \times \{1,2\}))$ = the vague Borel σ-algebra, see Section 2.1

Next put

$$\mu(\gamma, dt \times d\eta) := \{\gamma^1(dt)Q(1,d\eta) + \gamma^2(dt)Q(2,d\eta)\}.$$

Now try out the function

$$h(s,\gamma) = w(1)\hat{\gamma}^1(s) + w(2)\hat{\gamma}^2(s),$$

for some weights $w(1)$ and $w(2)$, such that $w(1)+w(2)=1$.

Insert this candidate in (8) :

$$\int_\Gamma \int_R e^{-st} h(s,\eta)\mu(\gamma, dt \times d\eta)$$

$$= \int_\Gamma \int_R e^{-st}(w(1)\hat{\eta}^1(s) + w(2)\hat{\eta}^2(s))\{\gamma^1(dt)Q(1,d\eta) + \gamma^2(dt)Q(2,d\eta)\}$$

Denoting

$$\mu_i^j(t) := \int_\Gamma \eta^j[0,t)Q(i,d\eta)$$

and

$$\hat{\mu}_i^j(s) := \int_\Gamma \hat{\eta}^j(s)Q(i,d\eta),$$

we can write the last member in obvious matrix notation (row vectors and T for transpose) as

$$w\,\hat{\mu}(s)\,\hat{\gamma}(s)^T.$$

Suppose that the matrix $\mu(t)$ is non-lattice,the matrix $\mu(\infty)$ irreducible and furthermore that μ is Malthusian,ie for some $s=\alpha$ (the Malthusian parameter) $\hat{\mu}(s)$ has spectral radius one and for all i and j

$$\int_0^\infty u\,e^{-\alpha u}\,\mu_i^j(du) < \infty.$$

Put

$$\kappa := \sum_{i,j=1}^{2} w^*(i)w(j)\int_0^\infty u\,e^{-\alpha u}\,\mu_i^j(du) < \infty,$$

where w and w^* denote the left and right eigenvectors of $\hat{\mu}(\alpha)$,see Asmussen and Hering (1983) p. 398.By letting w be the left eigen vector we are done.(This amounts to putting

$$w(2) := \hat{\mu}_1^1(\alpha)\hat{\mu}_2^2(\alpha)/\{1+\hat{\mu}_1^1(\alpha)\hat{\mu}_2^1(\alpha)-\hat{\mu}_1^2(\alpha)\hat{\mu}_2^1(\alpha)-\hat{\mu}_2^2(\alpha)\}.)$$

As in the one-type case a renewal argument yields the convergence of normed expectations.Go back to the k-type case and note that the above reasoning obviously holds true for any finite number of types.Define

$$m_{i,t}^{\chi} := E_i[z_{\rho_0,t}^{\chi}],$$

the expectation taken for $\rho_0 = 1\{0\}x\{i\}$.If the $e^{-\alpha t}E_i[\chi(t)]$ are directly Riemann integrable,then invoking (8.2) in Asmussen and Hering (1983),one may conclude

$$\lim_{t\to\infty} e^{-\alpha t}m_{i,t}^{\chi} = \kappa^{-1}w(i)\sum_{j=1}^{k} w^*(j)\int_0^\infty e^{-\alpha t}E_j[\chi(t)]dt.$$

Further results can be patterned after preceding Sections.For instance,using the martingale $\sum_{x\in I(t)} e^{-\alpha\sigma'_x} h(\rho_x)$ and convergence of normed variances one may establish convergence in L_2.Exactly the same kind of Dominated Convergene arguments apply given a new set of conditions analoguous to the old ones.

Generalizations to more general type spaces are feasible but will be attempted elsewhere.

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