ON THE ANTI-MONOTONICITY OF DIFFERENTIAL MAPPINGS CONNECTED WITH GENERAL EQUILIBRIUM PROBLEM

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FOREWORD

This paper is concerned with the anti-monotonicity of differential mappings connected with general equilibrium problems. These results can be used for the investigation of different game theory problems, for example Nash equilibria for noncooperative n-person games. Such an approach gives possibility to construct recurrent algorithms for finding the equilibria point.

This research was conducted within the framework of the Adaptation and Optimization Project in the System and Decision Sciences Program.

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ABSTRACT

Let X be a subset of a Hilbert space H and $\Psi: X \times X \rightarrow R$, $\Psi(x, x) = 0$ for all $x \in X$. Let $G(x) = \partial_y \Psi(x, y)|_{y=x}$ denote generalized dierential with respect to the second argument at the point (x, x). We shall be concerned with the properties of the function Ψ sucient to ensure the anti-monotonicity of the map $G(x)$. It will be shown that for the anti-monotonicity of the map $G(x)$ it is sucient to assume convexity-concavity of the function Ψ. In the case of the weakly convex-concave function Ψ the map $G(x)$ is anti-monotone under some conditions on the remainder terms. In the case of the quasi convex-concave function Ψ, the condition similar to the anti-monotonicity condition hold.

Some properties of the weakly convex functions used in article will be proved.

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1. INTRODUCTION

The mathematical problems discussed in this article were stimulated by the investigations of simulation model for international oil trade (SMIOT) developed at the International Institute for Applied Systems Analysis [1].

Briey the main idea of this model is the following. There is a market of a single homogeneous product, which consists of some sellers (exporters) and a single buyer (importer). Let $i = 1, \ldots, n$ be the exporters, $f^i(z)$ be the marginal cost of which any exporter i produces the amount z of the product for marketing and $r(z)$ be the price at which the importer would agree to buy the amount z of the product. If x; denotes the amount of the product sold by exporter i, then the revenue $\varphi_i(x)$ of the exporter i, can be expressed as follows:

$$\varphi_i(x) = r(x_1 + \cdots + x_n) - \int_0^{x_i} f^i(z)dz$$

Let s be a number of a time point, x_i^s be the amount of the product sold by exporter i at time s. The dynamics of the model is given by the relation

$$x_i^{s+1} = \max \left\{ 0, x_i^s - \rho_s \frac{\partial \varphi_i(x^s)}{\partial x_i} \right\}, s = 0, 1, \ldots$$

where ρ_s, $s = 0, 1, \ldots$ are the positive scalar values.

In more general form

$$x^{s+1} = \pi_X(x^s - \rho_s g(x^s)), s = 0, 1, \ldots \quad (1)$$

where $\pi_X(\cdot)$ denotes the operation of projection on feasible set

$X = \{x \in R^n; 0 \le x_i, i = 1, \ldots, n\}$;

$$g(x) = \left[\frac{\partial \varphi_1(x0}{x_l}, \ldots, \frac{\partial \varphi_n(x)}{x_n}\right]$$

is a dierential map of preference.

The aim of the study is to formulate assumptions on the functions $r(z)$, $f^i(z)$, $i = 1, \ldots, n$ such that the process (1) has a stable cycle, converges to a Nash equilibrium point or converges to some point. The results of this article allow us to formulate conditions on the functions $r(z)$, $f^i(z)$, $i = 1, \ldots, n$ insuring the convergence of process (1) to a Nash equilibrium point. These questions were discussed in papers [3] – [5] and in more general situations in [6], [7], etc.

We study some conditions on the payo functions sucient for the anti-monotonicity of mapping $g(x)$ or others dierential mappings. Hence we can formulate the conditions under which many iterative algorithms for search equilibrium points are applicable (see, for example, [8] – [14]).

STATEMENT OF THE PROBLEM Let X be a subset of a Hilbert space H and $<\cdot, \cdot>$ denote the inner product. We say the multivalued mapping G is anti-monotone on X if $<g(x) - g(y), y - x> \ge 0$ for all $x, y \in X$, $g(x) \in G(x)$, $g(y) \in G(y)$.

Let $\Psi: X \times X \rightarrow R$ be a quasi or weakly [15] convex-concave function, $\Psi(x, x) = 0$ for all $x \in X$. Let $G(x) = \partial_y \Psi(x, y)|_{y=x}$ denote generalized dierential of the function Ψ with respect to the second argument at the point (x, x).

We shall be concerned with the properties of the function Ψ sucient to ensure the anti-monotonicity of the map $G(x)$. It will be shown that for the anti-monotonicity of the map $G(x)$, it is sucient to assume convexity-concavity of the function Ψ. In the case of the weakly convex-concave function Ψ the map $G(x)$ is anti-monotone under some conditions on the remainder terms. Some properties of the weakly convex functions used in article will be proved. Investigation of such questions can be motivated by the problem of nding the points $x^* \in X$ dened by variational inequality

$$<g(x^*), x^* - x> \ge 0 \quad \text{for all } x \in X , \tag{2}$$

where $g(x^*) \in G(x^*)$.

If the multivalued map $G(x)$ is anti-monotone or strictly anti-monotone we can use results stated in [8] – [14], etc. for solving the problem (2).

In the case of the quasi convex-concave function $\Psi(x, y)$ and $\Psi(x, x) = 0$ for $x \in X$ the weaker condition then the anti-monotonicity holds:

$$\text{if } \max_{y \in X} \Psi(x^*, y) = 0 \text{ then}$$

$$<g(x), x^* - x> \geq 0 \text{ for all } x \in X, g(x) \in G(x) ,$$

where $G(x)$ is a dierential of quasi concave function $\Psi(x, x)$ with respect to second argument.

Let us consider some problems which can be reduced to the variational inequality (2).

EXAMPLE 1 *Nash equilibria for noncooperative n-person games.* Let X be a convex closed bounded subset of the production $H_1 \times \cdots \times H_n$ of a Hilbert spaces H_i, $i = 1, \ldots, n$. A point $x_i \in H_i$ is a strategy of i-th player $i = 1, \ldots, n$ and $\varphi_i(x) = \varphi_i(x_1, \ldots, x_n)$ is his payo function. The element $(x_1, \ldots, x_{i-1}$ $y_i, x_{i+1}, \ldots, x_i)$ is denoted by (y_i / x). The point $x^* = (x_1^*, \ldots, x_n^*) \in X$ is referred to as the Nash equilibrium of n-person game if for $i = 1, \ldots, n$

$$\varphi_i(x^*) = \max_{y_i} \{\varphi_i(y_i / x^*) : (y_i^* / x) \in X\} .$$

Let us introduce the function $\Psi(x, y)$:

$$\Psi(x, y) = \sum_{i=1}^{n} (\varphi_i(y_i / x) - \varphi_i(x)), \; y = (y_1, \ldots, y_n) .$$

It is not dicult to see that $\Psi(x, x) \equiv 0$, $x \in X$. We suppose that the functions $\varphi_i(x)$, $i = 1, \ldots, n$ are continuous on X. The point $x^* \in X$ is dened as the normalized equilibrium point if

$$\max_{y \in X} \Psi(x^*, y) = 0 . \tag{3}$$

LEMMA 1 (See for example [16]). The normalized equilibrium point is the equilibrium point, the reverse is true if $X = X_1 \times \cdots \times X_n$, $X_i \subset H_i$.

The condition (2) is a necessary optimality condition for the problem (3), for this reason the problem of nding Nash equilibrium is reduced to the problem (2), EXAMPLE 2 An equilibrium point is dened in [17] as follows. Let X be a convex closed subset of an Euclidean space R^n, the functions

$$\Phi_0(x, y), \Phi_1(x, y), \ldots, \Phi_n(x, y)$$

be concave with respect to $y \in X$ for each $x \in X$ and continuous with respect to x, y on $X \times X$. Let us denote

$$X(x) = \{y \in X : \Phi_i(x, y) \geq 0, \; i = 1, \ldots, n\} \; .$$

The problem consists in nding the point x^* such that

$$x^* \in X(x^*); \; \max \{\Phi_0(x^*, y) : y \in X(x^*)\} = \Phi_0(x^*, x^*) \; .$$

If the condition $X(x) = X$ is true for all $x \in X$, then the problem is reduced to problem (3), where $\Psi(x, y) = \Phi_0(x, y) - \Phi_0(x, x)$. It is easy to see $\Psi(x, x) \equiv 0$ for all $x \in X$. The necessary condition (2) can be used in this case too.

EXAMPLE 3 Let us consider one more problem which can be reduced to (3). We assume the X is a subset of a Hilbert space H (or more general space), and $\Psi : X \times X \rightarrow R$ is a function satisfying $\sup_{y \in X} \Phi(y, y) \leq 0$. The problem consists in nding the point $x^* \in X$ such that

$$\sup_{y \in X} \Phi(x^*, y) \leq 0 \; . \tag{4}$$

We suppose $\Psi(x, y) = \Phi(x, y) - \Phi(x, x)$. If the point $x^* \in X$ satises (3), then it satises (4) too. For this reason the necessary condition (2) can be used in this case as well.

Theorems concerning the existence of problem (4) solutions were formulated in [18]. In the same book there are references on the original papers related to this problem.

2. WEAKLY CONVEX FUNCTIONS

In this section basic properties of weakly convex functions [15] are investigated. The family of weakly convex functions includes smooth and convex functions and is closed with respect to the summation and pointwise maximum. We give new denition of this family useful for applications. It will be shown that this denition and the denition given by E. Nurminski [15] are equivalent.

DEFINITION 1 *Let X be a convex subset of a Hilbert space H. A continuous function $f : X \to R$ is called weakly convex on X if for all $x \in X$, $y \in X$, $0 \le \alpha \le 1$ the following inequality holds.*

$$\alpha f(x) + (1 - \alpha) f(y) \ge f(\alpha x + (1 - \alpha) y) + \alpha(1 - \alpha) r(x, y) \ , \tag{5}$$

where the remainder $r : X \times X \to R$ satises

$$\frac{r(x, y)}{\|x - y\|} \to 0 \quad \text{if } x \to z, y \to z \tag{6}$$

for all $z \in X$.

The set $\partial f(x)$ is called a dierential of a weakly convex function $f(x)$ at a point $x \in X$ on X if for all $g(x) \in \partial f(x)$

$$f(y) - f(x) \ge \langle g(x), y - x \rangle + r(x, y) \tag{7}$$

for all $x, y \in X$.

We say that a function $f(x)$ is weakly concave on X, if $-f(x)$ is weakly convex on X. A dierential of the weakly concave function $f(x)$ is dened as a dierential of the weakly convex function $-f(x)$ taken with sign minus.

THEOREM 1 *Let X be an open convex subset of a Hilbert space H and the function $f(x)$ is weakly convex on X. Then the set $\partial f(x)$, $x \in X$ is non-empty, convex, closed bounded and*

$$f'(x, p) = \max \{\langle g, p \rangle : g \in \partial f(x)\} \ , \tag{8}$$

where $f'(x, p)$ is a derivative of the function $f(x)$ at a point x along a direction p.

PROOF We start with the following lemma.

LEMMA 2 *For any $x \in X$, $p \in H$ a derivative*

$$f'(x, p) = \lim_{\lambda \downarrow 0} \frac{f(x + \lambda p) - f(x)}{\lambda}$$

exists and is nite.

PROOF First of all we prove an existence of the nite or innite derivative $f'(x, p)$ for any $x \in X$, $p \in H$. For $0 \le \lambda_2 < \lambda_1$ inequality (5) implies

$$\frac{\lambda_2}{\lambda_1} f(x + \lambda_1 p) + \frac{\lambda_1 - \lambda_2}{\lambda_1} f(x) \ge f(x + \lambda_2 p) + \frac{\lambda_2(\lambda_1 - \lambda_2)}{\lambda_1^2} r(x + \lambda_1 p, x) \ ,$$

consequently

$$\frac{f(x+\lambda_1 p)-f(x)}{\lambda_1} \ge \frac{f(x+\lambda_2 p)-f(x)}{\lambda_2} + \frac{\lambda_1-\lambda_2}{\lambda_1}\cdot\frac{r(x+\lambda_1 p, x)}{\lambda_1} . \quad (9)$$

The last inequality implies the existence of the derivative $f'(x, p)$ because

$$\frac{r(x+\lambda p, x)}{\lambda} \longrightarrow 0 \quad \text{for } \lambda \downarrow 0 .$$

The derivative $f'(x, p)$ can not take the value $+\infty$. Let us prove that the derivative $f'(x, p)$ bounded below. For $\varepsilon > 0$, $\lambda > 0$ the inequality (5) implies

$$\frac{\lambda}{\lambda+\varepsilon} f(x-\varepsilon p) + \frac{\varepsilon}{\lambda+\varepsilon} f(x+\lambda p) \ge f(x) + \frac{\lambda\varepsilon}{(\lambda+\varepsilon)^2} r(x-\varepsilon p, x+\lambda p) .$$

After the equivalent transformation

$$\frac{f(x+\lambda p)-f(x)}{\lambda} \ge \frac{f(x)-f(x-\varepsilon p)}{\varepsilon} + \frac{1}{\lambda+\varepsilon} r(x-\varepsilon p, x+\lambda p) .$$

If ε and λ are suciently small, then from (6) obtain

$$\frac{r(x-\varepsilon p, x+\lambda p)}{\lambda+\varepsilon} > \delta = \text{const} ,$$

consequently

$$f'(x, p) \ge \frac{f(x)-f(x-\varepsilon p)}{\varepsilon} + \delta .$$

LEMMA 3 *The derivative $f'(x, p)$ is a convex positive-homogeneous function continuous at the point 0 with respect to p.*

PROOF Let us prove a positive-homogeneity of the function $f'(x, p)$. According to the denition of a derivative for $\alpha > 0$

$$f'(x, \alpha p) = \lim_{\lambda \downarrow 0} \frac{f(x+\lambda\alpha p)-f(x)}{\lambda}$$

$$= \alpha \lim_{\lambda \downarrow 0} \frac{f(x+\alpha\lambda p)-f(x)}{\alpha\lambda} = \alpha f'(x, p)$$

For $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 = 1$ the inequality (5) implies

$$\frac{f(x+\lambda(\lambda_1 p_1 + \lambda_2 p_2)) - f(x)}{\lambda} =$$

$$\frac{f(\lambda_1(x+\lambda p_1) + \lambda_2(x+\lambda p_2)) - \lambda_1 f(x) - \lambda_2 f(x)}{\lambda} \le$$

$$\lambda_1 \frac{f(x+\lambda p_1)-f(x)}{\lambda} + \lambda_2 \frac{f(x+\lambda p_2)-f(x)}{\lambda} - \frac{\lambda_1\lambda_2 r(x+\lambda p_1, x+\lambda p_2)}{\lambda}$$

Passing to limit $\lambda \downarrow 0$

$$f'(x, \lambda_1 p_1 + \lambda_2 p_2) \le \lambda_1 f'(x, p_1) + \lambda_2 f'(x, p_2) \ .$$

Prove that the function $f'(x, p)$ is continuous with respect to p at the point 0 for any $x \in X$. To check this it is sucient to prove boundedness of the function $f'(x, p)$ in a neighborhood of the point 0 (see for example [19]).

Passing to limit $\lambda_2 \downarrow 0$ in (9) obtain

$$\frac{f(x+\lambda_1 p)-f(x)}{\lambda_1} \ge f'(x, p) + \frac{r(x+\lambda_1 p, x)}{\lambda_1} \ .$$

The condition (6) implies that there exists a neighborhood $U(0)$ of the point 0 such that $|\frac{r(x+\lambda_1 p, x)}{\lambda_1}| < \delta = const$ for all $p \in U(0)$. Since $f(x)$ is continuous, then we can suppose that the function $f(x+\lambda_1 p)$ is bounded on $U(0)$ with respect to p. Therefore the boundedness above of $f'(x, p)$ with respect to p in the neighborhood $U(0)$ follows from the last inequality.

LEMMA 4 *A dierential $\partial_p f'(x, 0)$ of the convex function $f'(x, p)$ with respect to argument p at the point 0 coincides with $\partial f(x)$.*

PROOF Dierential $\partial_p f'(x, 0)$ is dened as followed

$$\partial_p f'(x, 0) = \{z \in H : \langle z, p\rangle \le f'(x, p) \quad \text{for all } p \in H\} \ .$$

The inequality (5) implies

$$f(y) - f(x) \ge \frac{f(x+\alpha(y-x)) - f(x)}{\alpha} + (1-\alpha) r(x, y), \ \alpha > 0 \ .$$

Passing to limit $\alpha \downarrow 0$

$$f(y) - f(x) \ge f'(x, y-x) + r(x, y) \ .$$

Let z belong to $\partial_p f'(x, 0)$, then from the last inequality we have

$$f(y) - f(x) \ge \langle z, y-x\rangle + r(x, y) \ ,$$

consequently $\partial_p f'(x, 0) \subset \partial f(x)$.

Let us prove the reverse conclusion. If $z \in \partial f(x)$, then

$$f(x + \lambda p) - f(x) \geq \lambda <z, p> + r(x + \lambda p, x) .$$

Hence

$$\frac{f(x + \lambda p) - f(x)}{\lambda} \geq <z, p> + \frac{r(x + \lambda p, x)}{\lambda}$$

and $f'(x, p) \geq <x, p>$, consequently $\partial f(x) \subset \partial_p f'(x, o)$. According to Minkovski duality, since $f'(x, p)$ is a convex positive homogeneous function with respect to p and continuous at the point 0 then the set $\partial_p f'(x, 0)$ is non-empty bounded convex closed and

$$f(x, p) = \max\{<z, p> : z \in \partial_p f(x, 0)\} .$$

Consequently the set $\partial f(x)$ is non-empty bounded convex close and relation (8) is true. The theorem has been proved.

We shall give equivalent denition of the weakly convex functions.

DEFINITION 2 *Let X be a convex subset of a Hilbert space H. We say that a function continuous on X is weakly convex on X, if for any $x \in X$ the set $G(x)$ consisting of the vectors g, such that*

$$f(y) - f(x) \geq <g, y - x> + \zeta(x, y) \quad \text{for all } y \in X \tag{10}$$

is empty, and remainder term $\zeta(x, y)$ in each compact subset $K \subset X$ is uniformly small relatively to $\|x - y\|$, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ that

$$\frac{|r(x, y)|}{\|x - y\|} < \varepsilon$$

for $\|x - y\| < \delta$; $x, y \in K$.

E. Nurminski [15] has introduced this denition in case $X = H = R^n$ where R^n is an Euclidean n-dimenional space.

THEOREM 2 *Let X be a convex open subset of a Hilbert space H, then the denitions 1 and 2 are equivalent in the following sense:*

a) *if a function is weakly convex in the sense of the denition 1, then it is weakly convex in the sense of the denition 2 and $\zeta(x, y) = r(x, y)$;*

b) *if a function is weakly convex in the sense of the denition 2, then it is weakly convex in the sense of the denition 1 and*

$$r(x, y) = \inf_{\substack{\alpha_1 + \alpha_2 = 1 \\ \alpha_1, \alpha_2 \geq 0}} \left\{ \frac{\zeta(x, x + \alpha_2(y - x))}{\alpha_2} + \frac{\zeta(y, y + \alpha_1(x - y))}{\alpha_1} \right\} .$$

PROOF It is not dicult to see that (6) implies the uniform convergence on a compact subset K. Hence the statement a) of the theorem follows from the denition of a weakly convex function and theorem 1.

Let us prove the statement b). In the view of (10), we have

$$\alpha_1 f(x) \geq \alpha_1 f(\alpha_1 x + \alpha_2 y) + \alpha_1\alpha_2 < g(\alpha_1 x + \alpha_2 y), x - y >$$
$$+ \alpha_1 \zeta(x, \alpha_1 x + \alpha_2 y) ,$$
$$\alpha_2 f(y) \geq \alpha_2 f(\alpha_1 x + \alpha_2 y) + \alpha_1\alpha_2 < g(\alpha_1 x + \alpha_2 y), y - x >$$
$$+ \alpha_2 \zeta(y, \alpha_1 x + \alpha_2 y)$$

for $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$.

From last two inequalities we obtain

$$\alpha_1 f(x) + \alpha_2 f(y) \geq f(\alpha_1 x + \alpha_2 y) + \alpha_1 \zeta(x, \alpha_1 x + \alpha_2 y) +$$
$$\alpha_2 \zeta(y, \alpha_1 x + \alpha_2 y) \geq f(\alpha_1 x + \alpha_2 y) +$$
$$\alpha_1\alpha_2 \inf_{\substack{\alpha_1 + \alpha_2 = 1 \\ \alpha_1, \alpha_2 \geq 0}} \left[\frac{\zeta(x, x + \alpha_2(y - x)}{\alpha_2} + \frac{\zeta(y, y + \alpha_1(x - y))}{\alpha_1} \right] .$$

As $\zeta(x, y)/ \|x - y\| \to 0$ for $x \to z, y \to z$, then $\inf_{\substack{\alpha_1 + \alpha_2 = 1 \\ \alpha_1, \alpha_2 \geq 0}} (\zeta(x, x + \alpha_2(y - x))/ \alpha_2\|y - x\| + \zeta(y, y + \alpha_1(x - y))/ \alpha_1\|y - x\|) \to 0$ for $x \to z$, $y \to z$.

The theorem has been proved.

We shall note some cases, where the remainder terms in the denitions 1, 2 coincide.

COROLLARY 1 *Let X be a convex open subset of a Hilbert space H. If a weakly convex function $f(x)$ satises*

$$f(y) - f(x) \geq < g, y - x > + \mu\|x - y\|^2, \mu \in R$$

for all $x, y \in X$, then

$$\alpha_1 f(y) + \alpha_2 f(x) \geq f(\alpha_1 x + \alpha_2 y) + \alpha_1\alpha_2\mu\|x - y\|^2$$

for $\alpha_1 + \alpha_2 = 1$; $\alpha_1, \alpha_2 \geq 0$; $x, y \in X$.

Conversely the second inequality implies the rst one for any $g \in \partial f(x)$.

This corollary is well known, if $\mu > 0$, in this case the function $f(x)$ is strongly convex.

Let us consider the case when function $f(x)$ is twice continuously dierentiable at each $x \in X$, where X is an open subset of a Hilbert space H, i.e. for all $x, y \in X$

$$f(y) = f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2} \langle A(x)(y - x), y - x \rangle$$
$$+ o(\|y - x\|^2) \; , \tag{11}$$

$\nabla f(x)$ is a gradient at a point x; $A(x) : X \longrightarrow H$ is a linear operator generating the symmetric bilinear function

$$\langle A(x)h, z \rangle; \; h, z \in H; \; \frac{o(\|h\|^2)}{\|h\|^2} \longrightarrow 0 \quad \text{for} \quad \|h\| \longrightarrow 0 \; .$$

The function is called twice continuously dierentiable on $X \subset H$, if it is dierentiable at each point $x \in X$ and

$$\|A(x + h) - A(x)\| \longrightarrow 0 \quad \text{if} \quad \|h\| \longrightarrow 0 \quad \text{for all} \quad x, x + h \in X \; . \tag{12}$$

COROLLARY 2 *If a function $f(x)$ is twice continuously dierentiable on an open subset X of a Hilbert space H, i.e. the conditions (11), (12) are true, then*

$$\alpha_1 f(y) + \alpha_2 f(x) \geq f(\alpha_1 x + \alpha_2 y) + \alpha_1 \alpha_2 [\frac{1}{2} \langle A(x)(x - y), x - y \rangle$$
$$+ o(\|x - y\|^2)] \quad \text{for all} \quad x, y \in X \; , \tag{13}$$

where

$$\frac{o(\|x - y\|^2)}{\|x - y\|^2} \longrightarrow 0 \quad \text{for} \quad \|x - y\| \longrightarrow 0; \; \alpha_1 + \alpha_2 = 1; \; \alpha_1, \alpha_2 \geq 0 \; .$$

Conversely, the inequality (13) implies (11).

PROOF The converse statement follows from the a) of the theorem 2. Let us prove the direct statment using the b) of the theorem 2. In this case we can denote

$$\zeta(x \; y) = \frac{1}{2} \langle A(x)(y - x), y - x \rangle + o(\|y - x\|^2) \; ,$$

consequently

$$r(x, y) = \inf_{\substack{\alpha_1+\alpha_2=1 \\ \alpha_1, \alpha_2 \geq 0}} \left[\frac{\zeta(x, x + \alpha_2(y - x))}{\alpha_2} + \frac{\zeta(y, y + \alpha_1(x - y))}{\alpha_1} \right]$$

$$= \inf_{\substack{\alpha_1+\alpha_2=1 \\ \alpha_1, \alpha_2 \geq 0}} \left[\frac{1}{2} \alpha_2 < A(x)(x - y), x - y > + \frac{1}{2} \alpha_1 < A(y)(y - x), \right.$$

$$\left. y - x > + o(\|x - y\|^2) \right] = \frac{1}{2} < A(x)(y - x), y - x > + o(\|x - y\|^2) .$$

Thus, the inequality (13) is true.

3. ON THE ANTI-MONOTONICITY OF THE DIFFERENTIAL MAPS FOR THE WEAKLY CONVEX FUNCTIONS

Let $\Psi : X \times X \longrightarrow R$ be a function dened on a product $X \times X$, where X is a convex open subset of a Hilbert space H. The function $\Psi(x, y)$ is weakly convex on X with respect to the rst argument, i.e.

$$\alpha_1 \Psi(x, z) + \alpha_2 \Psi(y, z) \geq \Psi(\alpha_1 x + \alpha_2 y, z) + \alpha_1 \alpha_2 r_z(x, y) \tag{14}$$

for all $x, y, z \in X$; $\alpha_1 + \alpha_2 = 1$; $\alpha_1, \alpha_2 \geq 0$ and

$$\frac{r_z(x, y)}{\|x - y\|} \longrightarrow 0 \quad \text{if } \|x - y\| \longrightarrow 0 \quad \text{for all } z \in X .$$

We suppose that the function $\Psi(x, y)$ is weakly concave with respect to the second argument on X, i.e.

$$\alpha_1 \Psi(z, y) + \alpha_2 \Psi(z, y) \leq \Psi(z, \alpha_1 x + \alpha_2 y) + \alpha_1 \alpha_2 \mu_z(x, y) \tag{15}$$

for all $x, y, z \in X$; $\alpha_1 + \alpha_2 = 1$; $\alpha_1, \alpha_2 \geq 0$ and also

$$\frac{\mu_z(x, y)}{\|x - y\|} \longrightarrow 0 \quad \text{if } \|x - y\| \longrightarrow 0 \quad \text{for all } z \in X .$$

We say the function $\Psi(x, y)$ is weakly convex-concave, if it satises (14), (15). Let

$$\Psi(x, x) \equiv 0 \quad \text{for all } x \in X . \tag{16}$$

Denote $G(x) = \partial_y \Psi(x, y)|_{y=x}$, i.e. $G(x)$ is a dierential of the function $\Psi(x, x)$ with respect to the second argument at a point (x, x).

We shall formulate the sucient conditions of the anti-monotonicity of the multivalued map $G(x)$, i.e. for all $g(x) \in G(x)$, $g(y) \in G(y)$

$$<g(x) - g(y), y - x> \geq 0 \quad \text{for all } x, y \in X .$$

THEOREM 3 *Let X be an open convex subset of a Hilbert space H, a function $\Psi: X \times X \longrightarrow R$ be weakly convex-concave, the remainder $r_z(x, y)$ be continuous with respect to z, the function Ψ satisfy condition (16). Then for all $x, y \in X$; $g(x) \in G(x)$, $g(y) \in G(y)$*

$$<g(x) - g(y), y - x> \geq r_y(x, y) - \mu_x(y, x) \quad \text{for all } x, y \in X . \tag{17}$$

PROOF We can assume $y = 0$. It also can be assumed that $g(0) = 0$ as an anti-monotonicity of the map $G(x)$ does not depend upon a linear term of the function $\Psi(x\ y)$ with respect to the second argument. It is necessary to prove that

$$<g(x), -x> \geq r_0(x, 0) - \mu_x(0, x) .$$

In the view of (14) we get

$$\alpha \Psi(x, \alpha x) + (1 - \alpha)\Psi(0, \alpha x) \geq \Psi(\alpha x, \alpha x) + \alpha(1 - \alpha) r_{\alpha x}(x, 0) .$$

Taking into account the properties of the weakly concave functions from the last inequality obtain

$$\begin{aligned} \alpha[\Psi(x, \alpha x) - \Psi(x, x)] &\geq (1 - \alpha)[\Psi(0, 0) - \Psi(0, \alpha x)] + \alpha(1 - \alpha) r_{\alpha x}(x, 0) \\ &\geq (1 - \alpha)[<g(0), -\alpha x> - \mu_0(\alpha x, 0)] + \alpha(1 - \alpha) r_{\alpha x}(x, 0) \\ &= (\alpha - 1)\mu_0(\alpha x, 0) + \alpha(1 - \alpha) r_{\alpha x}(x, 0) . \end{aligned}$$

Since $\mu_0(\alpha x, 0)/\alpha \longrightarrow 0$ if $\alpha \downarrow 0$, then the last inequality implies

$$\Psi(x, 0) - \Psi(x, x) = \lim_{\alpha \downarrow 0} [\Psi(x, \alpha x) - \Psi(x, x)] \geq$$

$$\lim_{\alpha \downarrow 0} \left[(\alpha - 1)\frac{\mu_0(\alpha x, 0)}{\alpha} + (1 - \alpha) r_{\alpha x}(x, 0) \right] \geq r_0(x, 0) .$$

Consequently, taking into account the weakly concavity of the function $\Psi(x, y)$ with respect to the second argument, we obtain

$$<g(x), -x> \geq \Psi(x, 0) - \Psi(x, x) - \mu_x(0, x) \geq r_0(x, 0) - \mu_x(0, x) .$$

Theorem has been proved.

REMARK We can see from the proof of the theorem that condition (14) can be replaced by

$$\Psi(x, y) - \Psi(x, x) \le \langle g(x), y - x\rangle + \mu_x(y, x) \quad \text{for all } x, y \in X .$$

COROLLARY 1 *Let all conditions of the theorem 3 be fullled and* $r_y(x, y) - \mu_x(y, x) \ge 0$ *for all* $x, y \in X$, *then the map* $G(x)$ *is anti-monotone on* X.

COROLLARY 2 *Let* X *be an open subset of a Hilbert space* H, *a function* $\Phi(x, -y)$ *be continuous and convex-concave on* $X \times X$, *the function* $\Phi(x, x)$ *be concave on* X, *then the multivalued map* $G(x) = \partial_y \Phi(x, y)|_{y=x}$ *is antimonotine on* X.

PROOF It is easy to get this statement if assume

$$\Psi(x, y) = \Phi(x, y) - \Phi(x, x) .$$

COROLLARY 3 *Let all conditions of the theorem 3 be fullled and*

$$r_y(x, y) - \mu_x(y, x) \ge \nu\|x - y\|^2 + o(\|x - y\|^2), \ \nu > 0 \tag{18}$$

for all $x, y \in X$, *where*

$$\frac{o(\|x - y\|^2)}{\|x - y\|^2} \to 0 \quad \text{if } x \to z, \ y \to z$$

uniformly with respect to $z \in X$, *then*

$$\langle g(x) - g(y), y - x\rangle \ge \nu\|x - y\|^2$$

for all $x, y \in X$ *and for all* $g(x) \in G(x)$, $g(y) \in G(y)$.

PROOF Let $x, y \in X$; $x \ne y$. In the view of (17), (18) we get

$$\langle g(x) - g(y), y - x\rangle = \lim_{n\to\infty} \left\langle \sum_{i=1}^{n} \left[g\left(y + \frac{i}{n}(x - y)\right) - g\left(y + \frac{i-1}{n}(x - y)\right)\right], y - x \right\rangle = \lim_{n\to\infty} n \sum_{i=1}^{n} \left\langle g\left(y + \frac{i}{n}(x - y)\right) - g\left(y + \frac{i-1}{n}(x - y)\right), \right.$$

$$\left. \left(y + \frac{i-1}{n}(x - y) - (y + \frac{i}{n}(x - y)\right)\right\rangle$$

$$\ge \lim_{n\to\infty} n \sum_{i=1}^{n} \left[\nu\left\|\frac{y - x}{n}\right\|^2 + o\left(\left\|\frac{y - x}{n}\right\|^2\right)\right] = \lim_{n\to\infty} \nu\|x - y\|^2 \times$$

$$\left[1 + \frac{\sum_{i=1}^{n} o(\|\frac{y-x}{n}\|^2)}{\nu \sum_{i=1}^{n} \|\frac{y-x}{n}\|^2}\right] = \nu \|x - y\|^2 .$$

Corollary has been proved.

Let us suppose that function $\Psi(x, y)$ is twice dierentiable on each argument. Denote by $A(x, y) = \Psi_{xx}(x, y)$ the second derivative with respect to the rst argument (see (11)), and i the same way $B(x, y) = \Psi_{yy}(x, y)$ with respect to the second argument and $g(x) = \nabla_y \Psi(x, y)|_{y=x}$.

THEOREM 4 *Let X be an open convex subset of a Hilbert space H, function $\Psi: X \times X \to R$ be twice dierentiable with respect to each argument, and*

$$\|A(x, y) - A(z, z)\| \to 0, \quad \text{if } x \to z, y \to z \quad \text{uniformly for } z \in X ;$$

$$\|B(x, y) - B(z, z)\| \to 0, \quad \text{if } x \to z, y \to z \quad \text{uniformly'for } z \in X ;$$

the operator $Q(x, x) = A(x, x) - B(x, x)$ satisfy

$$<Q(x, x)h, h> \geq \nu \|h\|^2, \nu > 0 \tag{19}$$

for all $x \in X$, $h \in H$ and ν do not depend on x, h. Then

$$<g(x) - g(y), y - x> \geq \frac{1}{2} \nu \|x - y\|^2 \quad \text{for all } x, y \in X . \tag{20}$$

PROOF The statement of the theorem follows from the corollary 2 of the theorem 2, and the corollary 3 of the theorem 3. According to the corollary 2 of the theorem2

$$r_y(x, y) = \frac{1}{2} <A(x, x)(x - y), x - y> + o(\|x - y\|^2) ,$$

$$\mu_x(y, x) = \frac{1}{2} <B(x, x)(x - y), x - y> + o(\|x - y\|^2) .$$

In the view of (19)

$$r_y(x, y) - \mu_x(y, x) = \frac{1}{2} <A(x, x) - B(x, x))(x - y), x - y> +$$

$$o(\|x - y\|^2) \geq \frac{1}{2} \nu \|x - y\|^2 + o(\|x - y\|^2), \nu > 0 .$$

Thus, the conditions of the corollary 3 of the theorem 3 are satised.

Let us consider the example which illustrates the last theorem.

EXAMPLE 4 *Model for international oil trade* [1]. There is a market of a single homogeneous product, which consists of some sellers (exporters) and a single buyer (importer). Let $i = 1, \dots, n$ be the exporters, $f^i(z)$ be the marginal cost of which any exporter i produces the amount z of the product for marketing and $r(z)$ be the price at which the importer would agree to buy the amount z of the product. If x_i denotes the amount of the product sold by exporter i, then the revenue $\varphi_i(x)$ of the exporter i, can be expressed as follows:

$$\varphi_i(x) = r(x_1 + \cdots + x_n) - \int_0^{x_i} f^i(z)dz \ .$$

Note that to the sense of the problem $x_i \geq 0$, $f^i(z) \geq 0$, $r(z) \geq 0$, $i = 1, \dots, n$. We assume also that exporter i, $i = 1, \dots, n$ is able to sell no more then μ_i amount of the product. If we suppose that each seller is going to choose the amount x_i in order to maximize his revenue in any market's situation characterized by a vector $x = (x_1, \dots, x_n)$, then the problem will be as follows: to nd an equilibrium's situation $x^* = (x_1^*, \dots, x_n^*)$ such that

$$\varphi_i(x^*) = \max_{0 \leq y_i \leq \mu_i} \varphi_i(x_1^*, \dots, x_{i-1}^*, y_i, x_{i+1}^*, \dots, x_n^*), \ i = 1, \dots, n \ .$$

The admissible set $X = \{x \in R^n : 0 \leq x_i \leq \mu_i, \ i = 1, \dots, n\}$ is convex and compact. The function $\Psi(x, y)$ (see example 1) denes as

$$\Psi(x, y) = \sum_{i=1}^{n} r(x_1 + \cdots + x_{i-1} + y_i + x_{i+1} + \cdots + x_n) y_i$$

$$- \sum_{i=1}^{n} \int_0^{y_i} f^i(z)dz - \sum_{i=1}^{n} r(x_1 + \cdots + x_n)x_i + \sum_{i=1}^{n} \int_0^{x_i} f^i(z)dz \ .$$

We assume that the functions $f^i(z), i = 1, \dots, n$ are continuously dierentiable and the function $r(z)$ is twice continuously dierentiable on some open subset $\{z \in R : z = \sum_{i=1}^{n} x_i, \ x \in U \subset R^n\}$, where U is open subset such that $X \subset U \subset R^n$. Denote

$$A(x, x) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \ B(x, x) = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot \\ b_{n1} & \cdots & b_{nn} \end{bmatrix},$$

$$Q(x,x)=\begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \cdot & \cdot & \cdot \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}, \quad z=\sum_{k=1}^{n} x_i \ .$$

It is not dicult to nd

$$a_{ii}=-r_{zz}(z)x_i-2r_z(z)+f^i_{x_i}(x_i),\ i=1,\dots n\ ;$$

$$a_{ij}=-r_{zz}(z)(x_i+x_j)-2r_z(z),\ i\neq j;\ i,j=1,\ldots,n\ ;$$

$$b_{ii}=r_{zz}(z)x_i+2r_z(z)-f^i_{x_i}(x_i),\ i=1,\ldots,n\ ;$$

$$b_{ij}=0,\ i\neq j;\ i,j=1,\ldots,n\ ;$$

$$q_{ii}=-2r_{zz}(z)x_i-4r_z(z)+2f^i_{x_i}(x_i),\ i=1,\ldots,n\ ;$$

$$q_{ij}=-r_{zz}(z)(x_i+x_j)-2r_z(z),\ i\neq j;\ i,j=1,\ldots,n\ ;$$

$$Q(x,x)=-r_{zz}(z)\begin{bmatrix} x_1x_1 & \ldots & x_1 \\ x_2x_2 & \cdots & x_2 \\ \cdot & \cdot & \cdot \\ x_nx_n & \cdots & x_n \end{bmatrix}-r_{zz}(z)\begin{bmatrix} x_1x_2 & \ldots & x_n \\ x_1x_2 & \cdots & x_n \\ \cdot & \cdot & \cdot \\ x_1x_2 & \cdots & x_n \end{bmatrix}$$

$$-2r_z(z)\begin{bmatrix} 11 & \cdots & 1 \\ 11 & \cdots & 1 \\ \cdot & \cdot & \cdot \\ 11 & \cdots & 1 \end{bmatrix}+2\begin{bmatrix} f^1_{x_1}(x_1)-r_z(z) & & 0 \\ & \ddots & \\ 0 & & f^n_{x_n}(x_n)-r_z(z) \end{bmatrix}.$$

Let us investigate under what conditions the matrix $Q(x,x)$ satises (19). Let $e=(1,\ldots,1)$ be the n-dimensional vector, and

$$\rho(x)=\min_{1\le 1\le n}(f^i_{x_i}(x_i)-r_z(z))\ .$$

Assume that

$$r_z(z)\le 0,\ r_{zz}(z)\ge 0 \quad \text{for } x\in X\ . \tag{21}$$

We can write the following inequality

$$<Q(x,x)h,h>\ \ge -2r_{zz}(z)<x,h><e,h>-2r_z(z)<e,h>^2+$$

$$2\rho(x)\|h\|^2\ge -2r_{zz}(z)\|x\|\sqrt{n}\,\|h\|^2+2\rho(x)\|h\|^2\ge$$

$$\left[-2\sqrt{n\sum_{i=1}^{n}\mu_i^2}\,r_{zz}(z)+2\rho(x)\right]\|h\|^2\ .$$

Hence, $\left[-2\sqrt{n\sum_{i=1}^{n}\mu_i^2 r_{zz}(z)} + 2\rho(x)\right].+ \nu > 0$ implies (19). The last inequality is correct if

$$-\sqrt{n\sum_{i=1}^{n}\mu_i^2 r_{zz}(z)} + f_{x_i}^i(x_i) - r_z(z) \geq \frac{1}{2}\nu > 0 \qquad (22)$$

for all $x \in X$, $1 \leq i \leq n$. Consequently (21), (22) imply (20) and the map $g(x) = (\nabla_{x_1}\varphi_1, \ldots, \nabla_{x_n}\varphi_n)$ is anti-monotone.

It should be noted that in the view of the (21), (22) the function $\Psi(x, y)$ is strongly concave with respect to y because the matrix $B(x, y)$ is negatively dened and the Nash point equilibrium exists [20].

4. THE MONOTONICITY OF DIFFERENTIAL MAP FOR QUASI CONVEX-CONCAVE FUNCTIONS

Let X be a closed convex subset of a Hilbert space H, U be an open subset of H and $X \subset U$. A function $\Psi : X \times X \to R$ is quasiconvex with respect to the rst argument, i.e.

$$\max[\Psi(z, y), \Psi(x, y)] \geq \Psi(\alpha_1 x + \alpha_2 z, y) \qquad (23)$$

for all $\alpha_1, \alpha_2 \geq 0$; $\alpha_1 + \alpha_2 = 1$ and for all $x, y, z \in X$ and quasiconcave with respect to the second argument, i.e.

$$\min[\Psi(y, z), \Psi(y, x)] \leq \Psi(y, \alpha_1 x + \alpha_2 z) \qquad (24)$$

for all $\alpha_1, ,\alpha_2 \geq 0$; $\alpha_1 + \alpha_2 = 1$ and for all $x, z, y \in X$.

For the further development we assume that the function $\Psi(x, y)$ satises regularity condition with respect to the second argument, i.e. for any $x \in X$ and any y such that $y \neq \arg\max_{z \in X} \Psi(x, z)$

$$int\{u \in U : \Psi(x, u) = \Psi(x, y)\} = 0 \ ,$$

where $int A$ denotes the interior of a set A and $\Psi(x, x) = 0$ for all $x \in X$. The families of weakly convex and quasiconvex functions intersect but are not embedded into each other.

A cone $Df(x) = \{g \in H : \langle g, y - x\rangle \leq 0$ for all $y \in \mu(x)\}$ is called a dierential of a quasiconvex function $f(x)$ at a point x on a set $U \subset H$, where $M(x) = \{y : f(y) \leq f(x),\ y \in U\}$. A dierential of the quasiconcave function $\varphi(x)$ is dened as a dierential of the quasiconvex function $-\varphi(x)$ taken with sign minus.

We say a function $\Psi(x, y)$ is quasi convex-concave if it is quasiconvex with respect to the rst argument and is quasiconcave with respect to the second one.

Denote $G(x) = D_y \Psi(x, y)|_{y=x}$ where D_y is a dierential of a function $\Psi(x, y)$ with respect to y in the sense described above.

Let a point $x^* \in X$ be a solution of the equation

$$\max_{y \in X} \Psi(x^*, y) = 0 \quad . \tag{25}$$

We will prove that (25) implies

$$<g(x), x^* - x> \geq 0 \tag{26}$$

for all $x \in X$ and for all $g(x) \in G(x)$. Consequently for nding a point x^* results of the paper [14] can be used, for example.

THEOREM 5 *Let U be an open subset of a Hilbert space H, X be a closed convex subset of H and $X \subset U$, a function $\Psi : X \times X \longrightarrow R$ be continuous on $X \times X$ and quasi convex-concave on $X \times X$, at least one of the inequalities (23), (24) be strict (for $\alpha_1 \neq 0$, $\alpha_1 \neq 1$), $\Psi(x, x) = 0$ for all $x \in X$. If a point $x^* \in X$ satises (25) then the variational inequality (26) holds for all $x \in X$ and for all $g(x) \in G(x)$.*

PROOF We can assume $x^* = 0$. Consequently it is necessary to prove that

$$<g(x), -x> \geq 0 \quad \text{for all } x \in X,\ g(x) \in G(x) \quad .$$

Assume at rst that the inequality (23) is strict, then

$$\max[\Psi(x, \alpha x), \Psi(0, \alpha x)] > \Psi(\alpha x, \alpha x) = 0,\ 0 < \alpha < 1 \quad . \tag{27}$$

The equality (25) implies

$$\Psi(0, \alpha x) \leq \Psi(0,0) = 0 \quad \text{for } 0 \leq \alpha \leq 1,\ x \in X \quad .$$

We get $\Psi(x, \alpha x) > 0$ for $0 < \alpha < 1$ taking into account (27). Passing to limit when $\alpha \downarrow 0$ obtain $\Psi(x, 0) \geq 0$ and $\Psi(x, 0) - \Psi(x, x) \geq 0$. In accordance to the denition of the dierential of a quasiconcave function we obtain statement of the theorem from the last inequality.

Let us consider the case with the strict inequality (24). In this case

$$\Psi(0, \alpha x) < \Psi(0, 0) = 0 \quad \text{for } 0 < \alpha < 1, x \in X, x \neq 0 .$$

The inequality (23) implies

$$\max[\Psi(x, \alpha x), \Psi(0, \alpha x)] \geq \Psi(\alpha x, \alpha x) = 0 \quad \text{for } 0 \leq \alpha \leq 1 .$$

Taking into account the last two inequalities we get $\Psi(x, \alpha x) \geq 0$ for $0 < \alpha < 1$. Further consideration coincides with the previous case. The theorem is proved.

Let us consider the case when a function $\Psi(x, y)$ is dierentiable with respect to the argument y. Denote

$$q(x) = \nabla_y \Psi(x, y)|_{y=x} .$$

COROLLARY *Let all conditions of the theorem 5 be satised and the function $\Psi(x, y)$ is dierentiable with respect to the second argument. Then the equality (25) implies.*

$$<q(x), x^* - x> \geq 0 \quad \text{for all } x \in X \tag{28}$$

PROOF The theorem statement follows from the inclusion $q(x) \in G(x)$, where $G(x)$ is a dierential of quasiconcave function $\Psi(x, x)$ with respect to the second argument.

Let us consider an example illustrating this corollary.

EXAMPLE *Wald's production model* [2]. Let n products are produced and r resources used in an economy, $a_1, \ldots, a_r$ be the amounts of these resources. The values $a_{ij} (i = 1, \ldots, n; j = 1, \ldots, r)$ denote the input of j-th resource necessary to produce the unit of i-th product. The prices of the products depend on the amounts of the produced products. Let $f_i(x) = f_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, n)$ be the price of a unit of i-the product if the products are produced in the amounts $x_1, \ldots, x_i, \ldots x_n$; X be a feasible set

$$X = \left\{ x \mid \sum_{i=1}^{n} a_{ij} x_i \leq a_j, \ j = 1, \ldots, r; \ x_i \geq 0, \ i = 1, \ldots, n \right\}$$

Under some conditions on the functions $f_i(x)$, $(i = 1, \ldots, n)$ the existence of a non-negative production vector $x^* = (x_1^*, \ldots, x_n^*)$ and non-negative resource price vector such that

$$\left.\begin{array}{l} \sum_{i=1}^{n} a_{ij}x_i^* \le a_j (j=1,\ldots,r);\ \sum_{j=1}^{r} a_{ij}U_j^* \ge f_i(x_1^*,\ldots,x_n^*) \ , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (i=1,\ldots,n)\ ; \\ \left[a_j - \sum_{i=1}^{n} a_{ij}x_i^*\right]U_j^* = 0 (j=1,\ldots,r);\ \left[\sum_{j=1}^{r} a_{ij}U_j^* - f_i(x^*)\right]x_i^* = 0 \ , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (i=1,\ldots,n)\ . \end{array}\right\} \quad (29)$$

was proved in [2], [17]. For all non-negative and continuous on the set $x \ge 0$ functions the existence of a non-negative production vector $x^* \in X$ being the solution of the following linear programming problem

$$\max_{x \in X} \sum_{i=1} f_i(x^*)x_i = \sum_{i=1}^{n} f_i(x^*)x_i^* \qquad (30)$$

was proved in [21]. The existence of vectors x^*, U^* satisfying (29) follows from the last equality and the duality theorem of the linear programming.

Denote

$$\Psi(x, y) = \sum_{i=1}^{n} f_i(x)y_i - \sum_{i=1}^{n} f_i(x)x_i \ .$$

Since the function $\Psi(x, y)$ is linear with respect to y then it is concave in y. If the point x^* is a solution of the problem (30), then x^* is also the solution of the problem (25). Hence in order to satisfy inequality (28) it is sucient that the function $\Psi(x, y)$ be strictly quasiconvex with respect to x on X, i.e.

$$\max[\Psi(z, y), \Psi(x, y)] > \Psi(\alpha_1 x + \alpha_2 z, y)$$

for all $\alpha_1, \alpha_2 \ge 0$, $\alpha_1 \ne 1$, $\alpha_1 + \alpha_2 = 1$ and for all $x, z, y \in X$, $z \ne x$.

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