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FOREWORD

The authors describe a mathematical model associated with the yield of mass production of products in any fabrication process. The problem is called a C_M-embedded problem or is also called the design centering problem in the theory of Integrated Circuits and Systems. Some results associated with conditions of optimality are proposed. To a certain extent they provide the possibility to develop search techniques. The authors mainly treat the case where the Minkowskian norm is just the Euclidian norm.

Alexander Kurzhanski
Chairman
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ABSTRACT

In this paper the C_M-embedded problem which is also called the design centering problem in other papers will be described, and a sufficient optimality condition and some results associated with optimality conditions will be presented. These results hold for general non-convex regions. To a certain extent they provide the possibility to develop search techniques. It should be pointed out that, in this paper, the only case where the Minkowski norm is just the Euclidean norm is treated, whereas we only mention some results related to the general case a little.

Keywords: Design centering, lineality space in a cone, Lipschitzian functions, quasidifferentiable functions, semismooth functions.

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Introduction

During the last decade a mathematical model, the C_M-embedded problem, has started to be used in the optimal design and the analysis of integrated circuits, [1]-[5]. Actually, it could be used to treat problems arising in wider areas and designs in fabrication processes than the area of designs of integrated circuits and systems. This is because in any fabrication process the cost associated with the mass production of products is directly related to the yield that is some percentage of the total number of products manufactured meeting the design specifications. The yield can be regarded as a probability with a joint probability density function γ.

$$\text{Yield} = \int_s \gamma(p \, , c \, , \delta) dp \, ,$$

where p is a statistical parameter vector, c is the mean value of p, δ is the deviation of p, S is a bounded closed and simply connected region, [1], [3], [4]. It is hoped that a maximum yield can be attained. To maximize the yield could be done by choosing the design centre c and the corresponding radius of a contour of γ in a certain sense. This problem is related to the Minkowski norm. It seems that it is necessary to study the DC (Design Centering) problem as a mathematical model in order to solve more practical problems.

Some necessary conditions to this problem, say Kuhn-Tucker and generalized Fritz-John necessary conditions, have been given [1]-[3], [15]. Especially, a sufficient condition for the case where the region is non-convex one whose boundary consists of convex and complementary convex constraints has been proposed in [1], and a proof was given in [14]. But these necessary conditions proposed are very general principles. It would be possible to find further optimality conditions that are suited to solve the C_M-embedded problem with a non-convex region, [19]. The sufficient condition proposed in [1] is only used to handle a kind of problem with convex and complementary convex constraints although it is convenient to be employed. A new sufficient condition and some results given here may be useful to constructing new algorithms and to practical calculations for this problem.

1. C_M-Embedded Problem

The so-called C_M-embedded problem is to find one of the largest convex bodies in a feasible region, which is associated with a Minkowski function which is defined by a predefined convex set containing the origin as its interior point, called unit convex body or unit ball. It can be described below.

The support function $\delta^*(\cdot \mid S)$, of a convex compact set S, which contains the origin, satisfies the following conditions.

$$(a) \quad 0 \le \delta^*(x \mid S) < +\infty \, , \forall \, x \in \mathbf{R}^n \, , \tag{1.1}$$

$$(b) \quad \delta^*(\alpha x \mid S) = \alpha \delta^*(x \mid S) \, , \forall \, x \in \mathbf{R}^n \, \forall \, \alpha \ge 0 \, ,$$

$$(c) \quad \delta^*(x + y \mid S) \le \delta^*(x \mid S) + \delta^*(y \mid S) \, , \forall \, x \, , y \in \mathbf{R}^n$$

In other words, the support function of a convex compact set S , $\delta^*(\cdot \mid S)$, with $0 \in S$ is a non-negative function satisfying sublinear conditions. Given a convex set K in $\mathbf{R}^n$ containing the origin as an interior point, the Minkowski function corresponding to K which is defined by

$$m(x\, ;\, K) := inf\,\{\lambda \mid x\,/\,\lambda \in K\,,\,\lambda > 0\} \tag{1.2}$$

satisfies the conditions in (1.1) [6], [16]. The function $m(x : K)$ also possesses the following properties [6]:

(i) $m(x\, ;\, K)$ is continuous on $x \in \mathbf{R}^n$

(ii) $\bar{K} = \{x \mid m(x\, ;\, K) \le 1\}$, $int\; K = \{x \mid m(x\, ;\, K) < 1\}$.

That is why the Minkowski function corresponding to K is called PDF norm [4] or Minkowski norm [1]. It is always assumed throughout this paper that K is a closed and bounded convex set with the origin as an interior point. We define

$$||\cdot||_{(K)} := m(\cdot\, ;\, K)$$

and $B(0, 1) = K$, where $B(0\,,\,1)$ denotes the Minkowski unit ball. In this paper we only concentrate our attention on such a case where $K = \{x \mid ||\, x\, ||_E \le 1\}$ is considered. Thus $||\cdot||_{(K)} = ||\cdot||_E$, i.e., the Euclidean norm. For the sake of simplicity the subscript on a norm as defined above will be omitted under the case in which the meaning of a norm is clear.

We now visualize the C_M-embedded problem below. The feasible region can be expressed as

$$S := \{x \in \mathbf{R}^n \mid f_i(x) \le f_i^o\,,\, i \in \Omega\}\,,$$

where $\Omega := \{1, 2\,, \ldots,\, m\}$ and $f_i(x) \in C^2$, for every $i \in \Omega$. Assume that S is bounded, $intS \neq \varphi$ and simply connected, and assume further that $\nabla f_i(y) \neq 0$, $y \in bd\; S$, $i \in \Omega$, where we by $bd\, S$ denote the boundary of set S.

$$\max_{x \in S}\; \min_{i \in \Omega}\; \min_{y \in D_i \cap S}\; ||\, x - y\,|| \;\text{ or max min min}||\, x - y\,||^2\,, \tag{1.3}$$

where $D_i :=\{y \in R^n \mid f_i(y) \ge f_i^0 \text{ and } f_j(y) \le f_j^0\,,\, j \in \Omega \setminus \{i\}\}$, $i \in \Omega$, or equivalently,

$$\begin{aligned} &\max_{r\,,\,x}\; r \\ &\quad s.t.\;\; f_i(y) \le f_i^0\,,\, i \in \Omega \\ &\qquad\quad\; y \in B(x\,,\,r) \end{aligned} \tag{1.4}$$

or as follows

$$\begin{aligned} &\max_{r\,,\,x}\; r \\ &\quad s.t.\;\; f_i(x + \omega\, r) \le f_i^0\,,\, i \in \Omega \\ &\qquad\quad\; \omega \in B(0\,,\,1)\,. \end{aligned} \tag{1.5}$$

Note that if $intS \neq \varphi$ then $D_i \neq \varphi\; \forall_i \in \Omega$. The formulae (1.3) - (1.5) were described in [1]-[3]. The equivalence between (1.3) and (1.4) or (1.5) has been established in [2]. Some necessary conditions have been pointed out in [1], [2], [15]. From now on we only consider the case in which $||\cdot||_{(K)} = ||\cdot||_E$. We will study further optimality conditions for some specific situations with the Euclidean norm. These studies are based upon the formula (1.3). Note that any solution to (1.3) always belongs to $intS$. So it would satisfy optimality conditions concerning the unconstrained optimization. In theory, precisely speaking, a closed set close enough to S and included in *intS* should be constructed, but it is not necessary to do it here.

Let

$$Y := \mathop{X}_{i \in \Omega} D_i \subseteq \mathbf{R}^{n \times m}$$

and

$$u = (u^1, \ldots, u^m) \in \mathbf{R}^{n \times m}$$

Define a mapping ψ: $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^{n \times m} \rightarrow \mathbf{R}$ designated by

$$(x, \xi, u) \mapsto \psi(x, \xi, u) = \sum_{i \in \Omega} \omega_i(\xi) \, || \, x - u^i ||^2 .$$

Our discussion is confined to that $\xi \in \mathbb{I}$, $\Omega \subset \mathbb{I} \subset \mathbf{R}$ where $\mathbb{I}$ is a bounded interval, $u \in Y$. The function $\omega_i(\xi)$, $i \in \Omega$, are defined by

$$\omega_i(\xi) = \frac{1}{a_i} \prod_{j \in \Omega \setminus \{i\}} (\xi - j) ,$$

$$a_i = \prod_{j \in \Omega \setminus \{i\}} (i - j) = (-1)^{m-i} (i - 1)! \, (m - i)! ,$$

due to [7]. Clearly, it is easy to be seen that $\psi \in C^1(\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^{n \times m})$ and the restriction of $\psi(x, \xi, u)$ to $S \times \Omega \times Y$, $\psi|_{S \times \Omega \times Y}$, can be replaced by

$$\psi(x, \xi, y) = || \, x - y \, ||^2 , \text{ for } y \in D_\xi , \xi \in \Omega , x \in S ,$$

since $\omega_i(\xi) = 0$ if $i \neq \xi$, $\omega_i(\xi) = 1$ if $i = \xi$ and $\sum_{i \in \Omega} \omega_i(\xi) \, || \, x - u^i \, ||^2 = || \, x - u^\xi \, ||^2$, $\xi \in \Omega$, $U^\xi \in D_\xi$.

Therefore, we can define

$$\begin{aligned} \varphi(x) := & \min_{\xi \in \Omega} \min_{y \in D_\xi} d^2(x, y) \\ = & \min_{\xi \in \Omega} \min_{u \in Y} \psi(x, \xi, u) \\ = & \min_{\xi \in \Omega} \min_{y \in D_\xi} \psi(x, \xi, y) , \end{aligned} \tag{1.6}$$

where $x \in S$. Let $Y(x, \xi)$, $\Omega(x)$ and $Y(x)$ denote respectively

$$Y(x, \xi) := \{ y \in \mathbf{R}^n \mid \psi(x, \xi, y) = \min_{y \in D_\xi} \psi(x, \xi, y) \}, \xi \in \Omega ;$$

$$\Omega(x) := \{ \xi \in \Omega \mid \min_{y \in D_\xi} \psi(x, \xi, y) = \min_{\xi \in \Omega} \min_{y \in D_\xi} \psi(x, \xi, y) \} , x \in S ;$$

$$Y(x) := \bigcup_{\xi \in \Omega(x)} Y(x, \xi) .$$

Now we explore the differentiability of $\varphi(x)$. It is necessary for further studying optimality conditions of the problem (1.3).

Lemma 1.1. The function $\varphi(x)$ is locally Lipschitzian in *int* S, and for any $x \in S$ and $h \in R^n$ the directional derivative of $\varphi(\cdot)$ at x in the direction h exists and is given by

$$\varphi'(x; h) = \lim_{\lambda \downarrow 0} (\varphi(x + h) - \varphi(x))/h \tag{1.7}$$

$$= \min_{\xi \in \Omega(x)} \min_{y \in Y(x, \xi)} < \nabla_x \psi(x, \xi, y), h > .$$

PROOF. The proof follows from [8, Th. 3.2], [5, Prop. 3.1 and Prop. 5.4]. □

A function is said to be quasidifferentiable at a point if there exists a convex compact set such that its directional derivative can be expressed as the support function of that set, [8].

Proposition 1.2. The function $\varphi(x)$ is quasi-differentiable in the sense of [8] in ***int*** S.

PROOF

The directional differentiability of $\varphi(\cdot)$ in ***intS*** issues straightforwardly from Lem. 1.1 and one has

$$\varphi'(x ; h) = \min_{\xi \in \Omega(x)} \min_{y \in Y(x, \xi)} < 2(x - y), h > \quad \forall h \in \mathbf{R}^n .$$

It is clear that $Y(x)$ and $Y(x, \xi)$ are all compact. The formula can be rewritten as

$$\varphi'(x ; h) = \min_{y \in Y(x)}{}^2 < x - y, h > . \tag{1.9}$$

Let $\partial\varphi(x)$ denote

$$co\ 2\ [\ x - Y(x)] . \tag{1.10}$$

Since $\partial\varphi(x)$ is convex compact, we have from [8, Th. 3.4] that

$$\min_{y \in Y(x)} < 2(x - y), h > = \min_{\omega \in \partial\varphi(x)} < \omega, h > . \tag{1.11}$$

Hence $\varphi(\cdot)$ is quasi-differentiable at x . □

The function $\varphi(\cdot)^{1/2}$ is a d.c. (difference of two convex or difference convex) function in ***intS*** when S is determined by convex and complementary convex constraints, [20].

The quasi-differential of φ at x is also called a superdifferential in the sense of [9], i.e.,

$$\bar{\partial}\varphi(x) = \partial\varphi(x) , \underline{\partial}\varphi(x) = \{0\} .$$

The quasi-differential of $\varphi(\cdot)$ at x is $\partial\varphi(x) = co\ 2\ [x - Y(x)]$. This is what we want. It follows from [8] that if x is a solution of the problem (1-3), then it is necessary to satisfy the condition

$$0 \in \partial\varphi(x) = co\ 2\ [x - Y(x)] . \tag{1.12}$$

When $x \in$ ***int*** S the optimality conditions of (1.3) is the same as that of a corresponding unconstrained problem. From this, as for searching techniques the formulation (1.3) can be regarded as an unconstrained problem at least partially. It has been proved in [5, Prop. 5.2] that

$$\partial_c\varphi(x) \subset co\{\nabla_x\psi(x, \xi, y) \mid \xi \in \Omega(x), y \in Y(x, \xi)\} , \tag{1.13}$$

$$x \in int\ S ,$$

where we denote by $\partial_c\varphi(x)$ the generalized gradient of φ at x in the Clarke's sense, [12]. In consequence, the following relation

$$\partial_c\varphi(x) \subset \partial\varphi(x) = co\ 2[x - Y(x)]$$

holds. In addition, (1.12) is equivalent to [2, condition 10]. It is also equivalent to the condition

$$\varphi'(x ; h) = \min_{y \in Y(x)} 2 < x - y, h > = \min_{w \in \partial\varphi(x)} < w, h > \leq 0 \qquad \forall h \in \mathbf{R}^n .$$

Since $0 \notin x - Y(x)$, $x \in int\ S$, (1.12) holds if and only if

$$dimL_{c\partial\varphi(x)} \neq 0$$

or

$$dimL_{c(x - Y(x))} \neq 0 \tag{1.15}$$

where CA means the conical hull of the set A, and L_{CA} means the lineality space of the cone CA, i.e.,

$$L_{CA} := CA \bigcap (-CA) ,$$

due to [10]. The following simple example illustrates the fact that the condition (1.15) is not tight for the problem (1.3).

EXAMPLE. Let $\mathbf{R}^n = \mathbf{R}^2$. The feasible region S is defined by

$$(x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 ,$$

$$(x_1 - 1)^2 + x_2^2 \geq 1/4 ,$$

$$(x_1 - 1)^2 + (x_2 - 2)^2 \geq 1/4 ,$$

where

$$x = (x_1 , x_2)^T \in \mathbf{R}^2 .$$

Take $\bar{x} = (1 , 1)^T$.

Then one has

$$Y(\bar{x}) = \{y_1 , y_2\} = \{(1 , \frac{1}{2})^T , (1 , \frac{3}{2})^T\} .$$

It is easy to see that the formulae (1.12) and (1.15) are satisfied at $\bar{x} = (1 , 1)^T$ and

$$dimL_{c\partial\varphi(\bar{x})} = 1 \neq 0 .$$

But in this case mentioned above, every direction $h \in [(0 , \frac{1}{2})^T]^{\perp}$ is a really expansible direction for this problem at $(1 , 1)^T$.

If $\varphi'(x ; h) > 0$, then the direction h is a really expansible direction because

$$\varphi(x + \alpha h) - \varphi(x) = \alpha\, \varphi'(x ; h) + o(\alpha) > 0$$

for $\alpha \in (0 , \delta)$, δ small enough. In general, if (1.12) does not hold, then the direction is a steepest ascent direction, where $|| Nr\ \partial\varphi(x) ||\ h(x)$ is just the projection of 0 onto $\partial\varphi(x)$.

$$h(x) = Nr\ \partial\varphi(x)\ /\ ||Nr\ \partial\varphi(x)|| \tag{1.16}$$

When the relation (1.12) is satisfied at x, it is impossible to infer in terms of it if the function $\varphi(\cdot)$ has a really expansible direction at x or not. Therefore it is worth to study furthermore and to improve (1.12) at least for some specific cases on the problem (1.3), see [19].

2. Optimality Conditions

In this section we will give some optimality conditions about a (local) solution to the problem

$$\max_{x \in S}\ \min_{\xi \in \Omega}\ \min_{y \in D_\xi}\ ||\ x - y||^2 . \tag{2.1}$$

For any set $Q \subset Y(x)$, we can define a corresponding set

$$G(x\,;\,Q) := \{x - y \in \mathbf{R}^n \mid y \in Q\} = x - Q\,. \tag{2.2}$$

Clearly, $Q_1 \subset Q_2 \subset Y(x)$ implies $CG(x\,;\,Q_1) \subset CG(x\,;\,Q_2) \subset CG(x\,;\,Y(x))$. Furthermore define

$$N^+(x : Q) := \{h \in \mathbf{R}^n \mid < x - y\,, h > \geq 0\,, y \in Q\} = \{h \in \mathbf{R}^n \mid < u\,, h > \geq 0\,, u \in G(x\,;\,Q)\} \tag{2.3}$$

Called the conjugate cone to $\varphi\ G(x\,;\,Q)$, evidently $Q_1 \subset Q_2 \subset Y(x)$ implies $N^+(x\,;\,Q_1) \supset N^+(x\,;\,Q_2) \supset N^+(x\,;\,Y(x))$. From (1.9) and $\partial\varphi(x) = co\ 2\ G(x\,;\,Y(x))$, one has

$$N^+(x\,;\,Y(x)) = N^+(x\,;\,x - \partial\varphi(x)/2) = \{h \in \mathbf{R}^n \mid < u\,, h > \geq 0\,, u \in \partial\varphi(x)\}\,.$$

Proposition 2.1. Suppose that $Q \subset Y(x)$. If

$$Y(x) \bigcap int\ [CG(x\,;\,Q) + x] \neq \emptyset \tag{2.4}$$

(i.e., $CG(x\,;\,Q)$ is blunt), then

$$N^+(x\,;\,Y(x)) = \{0\}\,.$$

PROOF. For the sake of contradiction, suppose that there exists a direction $\bar{h} \neq 0$ such that $\bar{h} \in N^+(x\,;\,Y(x))$. In terms of the hypothesis (2.4), there exists such a y that

$$y \in Y(x) \bigcap int\ [CG(x\,;\,Q) + x]$$

In consequence of $dim\ G(x\,;\,Q) = n$, there exists a set of positive scalars $\{\lambda_i > 0 \mid i = 1\,,\ldots,\,n\}$ and a set of vectors, which constitute a basis in $\mathbf{R}^n$,

$$\{x - y_i \in CG(x\,;\,Q) \mid i = 1\,,\ldots,\,n\} \tag{2.5}$$

such that

$$y - x = \sum_{i=1}^{n} \lambda_i\,(x - y_i)\,.$$

Since $y \in Y(x)$ and $\bar{h} \in N^+(x\,;\,Y(x))$, one has

$$< x - y\,, \bar{h} > = -\sum \lambda_i < x - y_i\,, \bar{h} > \geq 0\,.$$

Every term $< x - y_i\,, \bar{h} >$ in the above expression is nonnegative because of

$$CG(x\,;\,Q) \subset C\partial\varphi(x)\,.$$

Since $\lambda_i > 0\,, i = 1\,,\ldots,\,n$, one has

$$< x - y_i\,, \bar{h} > = 0\,, \qquad i = 1\,,\ldots,\,n\,.$$

Because (2.5) is a basis in $\mathbf{R}^n$, its corresponding matrix is nonsingular, i.e.,

$$\det\,[x - y_1\,,\ldots,\,x - y_n] \neq 0\,.$$

From this it follows that $\bar{h} = 0$. This contradicts $\bar{h} \neq 0$. The proof is completed. □

If $Q \subset \mathbf{R}^n$ is finite, then from the well-known Farkas lemma [10, Th. 2.8.5] and [10, Lem. 2.7.9], it follows straightforwardly that

$$u \in [CG(x \, ; \, Q) + x] \Longleftrightarrow < u - x \, , \, N^+(x \, ; \, Q) > \geq 0$$

where $<u - x \, , \, N^+(x \, ; \, Q) > \geq 0$ means that, for any $v \in N^+(x \, ; \, Q)$,

$$< u - x \, , v > \geq 0$$

always holds. A stronger result will be given below.

Proposition 2.2 Suppose that $N^+ (x \, ; \, Q) \neq \{0\}$ and $int\ CG(x \, ; \, Q) \neq \emptyset$. Then

$$u \in int\ [CG(x \, ; \, Q) + x] \Longleftrightarrow < u - x \, , N^+(x \, ; \, Q) \setminus \{0\} > > O \, .$$

PROOF. To begin with, we prove the necessity of it. $\Longrightarrow$ (only if) : Let $z - x \in N^+(x \, ; \, Q) \backslash \{0\}$. We can form a subspace

$$L\{u - x \, , z - x\} \, ,$$

determined by $u - x$ and $z - x$, with dimension two. Let

$$M(x \, ; \, u - x \, , z - x) : = L\{u - x \, , z - x\} + x$$

denote the linear manifold with dimension two, determined by $u - x$ and $z - x$, through the point x. Since $u \in int[CG(x \, ; \, Q) + x]$, there exists $\lambda > 1$ such that

$$u_G : = \lambda \, u + (1 - \lambda) \, z \in M(x \, ; \, u - x \, , z - x) \bigcap [CG(x \, ; \, Q) + x] \, . \tag{2.6}$$

Clearly $u_G \neq u$ because $\lambda > 1$. Because $z - x \in N^+(x \, ; \, Q)$, one has

$$M(x \, ; \, u - x \, , z - x) \bigcap [N^+(x \, ; \, Q) + x] \neq \emptyset \, .$$

From this there exists a scalar $\mu \geq 1$ such that

$$z_N : = \mu \, z + (1 - \mu) \, u \in M(x \, ; \, u - x \, , z - x) \bigcap [N^+(x \, ; \, Q) + x] \, . \tag{2.7}$$

The points u , u_G , z , z_N are in the same one-dimensional linear manifold. In view of $\lambda > 1$ and $\mu \geq 1$, u and z are included in the interval

$$\{w \mid w = \beta \, z_N + (1 - \beta) u_G \, , \beta \in (0 \, , 1]\} \tag{2.8}$$

Since $u_G - x \in CG(x \, ; \, Q)$ and $z_N - x \in N^+(x \, ; \, Q)$, we have

$$<u_G - x \, , z_N - x > \geq 0 \, .$$

As a result of $\lambda > 1$, $u = \beta \, z_N + (1 - \beta) \, u_G$ for some $\beta \in (0 \, , 1)$. Thus

$$< \frac{u - x}{|| \, u - x \, ||} \, , \frac{z - x}{|| \, z - x \, ||} > \, > \, < \frac{u_G - x}{|| \, u_G - x \, ||} \, , \frac{z_N - x}{|| \, z_N - x \, ||} > \geq 0 \, .$$

So $<u - x \, , z - x> > 0$, i.e., $<u - x \, , N^+(x \, ; \, Q) \setminus \{0\} > > 0$. That is the proof of necessity.

(if) : Prove by contradiction. Suppose that $u \notin int\ (CG(x \, ; \, Q) + x)$. Then there exists a sequence $\{u^i\}_1^\infty$ convergent to u, but not included in $CG(x \, ; \, Q) + x$. The sequence satisfies

$$< u^i - x \, , N^+(x \, ; \, Q) > \not> 0 \qquad \text{for any } i = 1 \, , \ldots , \, .$$

Correspondingly, there exists a sequence $\{z^i\} \subset N^+ (x \, ; \, Q) + x$ such that

$$< u^i - x \, , z^i - x > \leq 0 \, . \tag{2.9}$$

For the sake of simplicity, assume that $\{z^i\}_1^\infty$ is bounded and converges to $z \in N^+(x\ ;\ Q) + x$ different from x. As a result of taking the limit of (2.9), one has

$$< u - x\ ,\ z - x > \leq 0\ .$$

The above inequality contradicts $< u - x\ ,\ N^+(x\ ;\ Q) > > 0$. The proof of sufficiency is completed. □

Note 2.3. When $u \in CG(x\ ;\ Q) + x$ and Q is finite, the well-known Farkas lemma shows that

$$N^+(x\ ;\ Q) + x \subset H^+(x\ ;\ u - x) \triangleq \{w \in \mathbf{R}^n \mid <w - x\ ,\ u - x> \geq 0\}.$$

Prop. 2.2 points out furthermore that, under the same hypotheses as above, if $u \in int\ [CG(x\ ;\ Q) + x]$, then $[N^+(x\ ;\ Q) + x] \setminus \{x\} \subset int\ H^+(x\ ;\ u - x)$ strictly. The result shown in this proposition could be used to construct an ascent algorithm for searching an expansible direction.

In the next theorem, a sufficient condition for a locally optimal solution to the problem in question will be given.

A function $\varphi : \mathbf{R}^n \to \mathbf{R}$ is said to be semismooth at $x \in \mathbf{R}^n$ if it is Lipschitzian in a neighborhood of x and if for each $h \in \mathbf{R}^n$ and for any sequences $\{t_k\} \subset \mathbf{R}_+$, $\{0_k\} \subset \mathbf{R}^n$ and $\{g_k\} \subset \mathbf{R}^n$ such that

$$\{t_k\} \downarrow 0\ ,\ \{0_k\ /\ t_k\} \to 0 \in \mathbf{R}^n\ .$$

$$g_k \in \partial_c\, \varphi(x + t_k h + 0_k)\ ,$$

the sequence $\{<g_k\ ,\ h>\}$ has exactly one accumulation point, [11]. If φ is semismooth at x, then for each $h \in \mathbf{R}^n$, $\varphi'\ (x\ ;\ h)$ exists and equals $\lim_{k \to \infty} <g_k\ ,\ h>$ where $\{g_k\}$ is any sequence just mentioned above, [11, Lem. 2].

Theorem 2.4 If $x \in int\ S$ satisfies the following condition

$$dim\ N^+(x\ ;\ Y(x)) = 0\ , \tag{2.10}$$

then x is a locally optimal solution.

PROOF. The condition (2.10) is satisfied if and only if

$$N^+(x\ ;\ Y(x)) = \{0\}\ ,$$

otherwise $dim\ N^+(x\ ;\ Y(x)) \neq 0$. According to the definition of $N^+(x\ ;\ Y(x))$ for any non-zero $h \in \mathbf{R}^n$, one has

$$\varphi'\ (x\ ;\ h) < 0 \tag{2.11}$$

since if there exists $h \neq 0$ such that

$$\varphi'\ (x\ ;\ h) = \min_{y \in Y(x)} 2 < x - y\ ,\ h > = \min_{u \in \partial\varphi(x)} < u\ ,\ h > \geq 0\ ,$$

then it follows that

$$h \in N^+(x\ ;\ Y(x)) \text{ and } dim\ N^+(x\ ;\ Y(x)) \neq 0\ .$$

For the sake of contradiction, suppose that x is not a locally optimal solution to the problem (2.1). This implies that there exist a sequence $\{\lambda_i\}_1^\infty \downarrow 0$ and a vector sequence $\{h_i\}_1^\infty \subset \mathbf{R}^n$ with $||\ h_i\ || = 1$ such that

$$\varphi(x + \lambda_i\, h_i) > \varphi(x)\ . \tag{2.12}$$

Without loss of generality, assume that $h_i \to \bar{h}$ as $i \to \infty$. The function $\varphi(\cdot)$ is semismooth in the sense of Mifflin in a neighborhood of x, in terms of [5, Props. 5.6 and 5.7] or [11, Ths. 2 and 6] that states that, if Ω is a discrete set, then the function $\varphi(\cdot)$ is semismooth. The function $\varphi(x + \lambda_i h_i)$ can be written as

$$\varphi(x + \lambda_i h_i) = \varphi(x + \lambda_i \bar{h} + \lambda_i(h_i - \bar{h})) .$$

From the mean value theorem [13], one has

$$\varphi(x + \lambda_i h_i) - \varphi(x) = < g_i , \lambda_i h_i > , \tag{2.13}$$

where

$$g_i \in \partial_c \varphi(x + \alpha_i \lambda_i h_i) = \partial_c \varphi(x + \alpha_i \lambda_i \bar{h} + \alpha_i \lambda_i (h_i - \bar{h})) , \alpha_i \in (0 , 1) .$$

The expression can be rewritten as

$$\begin{aligned}\varphi(x + \lambda_i h_i) - \varphi(x) &= \lambda_i <g_i , h_i> \\ &= \lambda_i <g_i , h_i - \bar{h}> + \lambda_i <g_i , h_i> .\end{aligned}$$

In consequence of (2.12), we have

$$<g_i , h_i - \bar{h}> + <g_i , \bar{h}> > 0 . \tag{2.14}$$

Since $h_i \to \bar{h}$ as $i \to \infty$ and the mapping $\partial_c\varphi(\cdot)$ is bounded in a bounded subset $\hat{S} \subset int\ S$ one obtains

$$<g_i , h_i - \bar{h}> \to 0 \quad \text{as } i \to \infty .$$

As far as the second term is concerned, from the fact that

$$g_i \in \partial_c \varphi(x + \alpha_i \lambda_i \bar{h} + \alpha_i \lambda_i (h_i - \bar{h})) \quad \text{for all i} ,$$

$$\alpha_i \lambda_i \to 0 \quad \text{and} \quad (\alpha_i \lambda_i (h_i - \bar{h})) / (\alpha_i \lambda_i) \to 0 \quad \text{as} \quad i \to \infty ,$$

and $\varphi(\cdot)$ is semismooth, it follows that the sequence $\{< g_i , \bar{h} >\}$ converges and

$$\lim_{i \to \infty} < g_i , \bar{h} > = \varphi' (x ; \bar{h}) \tag{2.16}$$

[11, Lem. 2]. Taking the limit to (2.14), one has from (2.14), (2.15) and (2.16) the conclusion that

$$\varphi' (x ; \bar{h}) \geq 0 . \tag{2.17}$$

Clearly $\bar{h} \in N^+(x ; Y(x))$ and $\bar{h} \neq 0$. Comparing (2.11) and (2.17), it leads to a contradiction. The proof is completed. □

Note that when $Y(x)$ is finite, another way to prove this theorem can be given in terms of [17, Prop. 5] where the fact was pointed out that

$$f'(x ; y) = f^0(x ; y) \qquad \text{for all } y \in \mathbf{R}^n .$$

Suppose that $bd\ K$ is smooth (continuously differentiable) where K is a convex closed and bounded set predefined as a unit body in the definition (1.2). Define

$$\bar{\varphi}(x) := \min_{\xi \in \Omega} \min_{y \in D_i} || y - x||_{(K)} . \tag{2.18}$$

we have

$$\bar{\varphi}' (x ; h) = \min_{y \in Y(x)} < - k(y , x) \nabla F(y) , h > , \tag{2.19}$$

where $\nabla_{(y-x)} \| y - x \|_{(K)} = k(y, x) \nabla f(y)$, $y \in Y(x)$ and $k(y, x) > 0$, the meanings of Ω, D_i, $\Omega(X)$, $Y(x, \xi)$ and $Y(x)$ are the same, respectively, as mentioned previously. The formulae (1.9) and (2.19) are the same in substance from the point of view of gradient. Let

$$\partial\bar{\varphi}(x) := co\{w \mid w = -k(y, x) \nabla f(y), y \in Y(x)\} .$$

Then $\bar{\varphi}(x)$ is quasi-differentiable at x and

$$\bar{\varphi}'(x ; h) = \min_{w \in \partial\bar{\varphi}(x)} \langle w, h \rangle . \tag{2.20}$$

$\bar{\varphi}(x)$ is locally Lipschitz, quasi-differentiable in the sense of [8] and also semismooth in the sense of Mifflin [11]. Some results similar to Prop. 2.1 and Th. 2.3 could be established.

The following theorem represents the relation between the number of dimensions of $N^+(x ; Q)$ and $L_{CG(x ; Q)}$. Generally speaking, determining the number of dimensions of the lineality space $L_{CG(x ; Q)}$ is easier than that of $N^+(x ; Q)$ in computation. The result given in the coming theorem could be found in [18], but the proof is integrate and independent.

Theorem 2.7. For any closed set $Q \subset \mathbf{R}^n$, one has the relation

$$dim\, N^+(x ; Q) = n - dim L_{CG(x ; Q)} . \tag{2.21}$$

PROOF. Since $CG(x ; Q)$ is a closed cone, it can be expressed as

$$\begin{aligned} CG(x ; Q) &= [G(x ; Q)^0]^0 \\ &= (-N^+(x ; Q))^0 \\ &= -N^+(x ; Q)^0 \end{aligned} \tag{2.22}$$

or

$$-CG(x ; Q) = N^+(x ; Q)^0 \tag{2.23}$$

where A^0 denotes the polar cone of A. Because $N^+(x ; Q)$ is a cone, $0 \in$ the linear manifold of $N^+(x ; Q)$. Hence $LN^+(x ; Q)$ is the linear manifold of $N^+(x ; Q)$ where LA means that the linear subspace is spanned by A. From this one has

$$L_{CG(x ; Q)} \subset N^+(x ; Q)^\perp = [LN^+(x ; Q)]^\perp . \tag{2.24}$$

It is evident that the following relations hold

$$N^+(x ; Q)^\perp \subset N^+(x ; Q)^0$$

and

$$N^+(x ; Q)^\perp \subset -N^+(x ; Q)^0 .$$

In as much as

$$\begin{aligned} L_{CG(x ; Q)} &= CG(x ; Q) \bigcap (-CG(x ; Q)) \\ &= N^+(x ; Q)^0 \bigcap (-N^+(x ; Q)^0) , \end{aligned}$$

we get

$$N^+(x ; Q)^\perp = [LN^+(x ; Q)]^\perp \subset L_{CG(x ; Q)} . \tag{2.25}$$

Combined (2.24) and (2.25), it follows that

$$[LN^+(x ; Q)]^\perp = N^+(x ; Q)^\perp = L_{CG(x ; Q)} .$$

So (2.21) holds. The proof of this theorem is completed. □

Finally, we point out a remark relative to the decomposability of $CG(x\,;\,Q)$.

Remark 2.8

In [10, Th. 2.10.5], it was pointed out that every cone can be expressed as the direct sum of its lineality space and a pointed cone. So $CG(x\,;\,Q)$ can be decomposed as

$$CG(x\,;\,Q) = L_{CG(x\,;\,Q)} \oplus (L^{\perp}_{CG(x\,;\,Q)} \bigcap CG(x\,;\,Q))\ . \tag{2.26}$$

For any $\bar{g} \in CG(x\,;\,Q)$, there exists a decomposition whose components are $l \in L_{CG(x\,;\,Q)}$ and g belonging to the part of its pointed cone $L^{\perp}_{CG(x\,;\,Q)} \bigcap CG(x\,;\,Q))$. Thus for any $h \in N^{+}(x\,;\,Q)$, one has

$$\begin{aligned} < h\,,\,\bar{g} > &= < h\,,\,l > + < h\,,\,g > \\ &= < h\,,\,g > \\ &= \begin{cases} 0\,, & \text{if } \bar{g} \in L_{CG(x\,;\,Q)} \text{ or } h \in g^{\perp} \\ k > 0\,, & \text{otherwise}\,, \end{cases} \end{aligned}$$

where both h and g belong to $L^{\perp}_{CG(x\,;\,Q)}$, i.e., h and g in the same subspace. And

$$dim\ CG(x\,;\,Q) = dim\ L_{CG(x\,;\,Q)} + dim\ (LN^{+}\ (x\,;\,Q) \bigcap CG(x\,;\,Q))\ .$$

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