CONTROLLABILITY AND OBSERVABILITY OF CONTROL SYSTEMS UNDER UNCERTAINTY

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Foreword

This research report surveys the results of nonlinear systems theory (controllability and observability) obtained at IIASA during the last three summers by Jean-Pierre Aubin, Halina Frankowska, and Cesław Olech.

Classical methods based on differential geometry require some regularity and fail as soon as state-dependent constraints are brought to bear on the controls, or uncertainty and disturbances are involved in the system. Since these important features appear in most realistic control problems, new methods had to be devised, which encompass the classical ones, and allow the presence of *a priori* feedback into the control systems.

This is now possible thanks to two new tools, the development of which IIASA played an important role: differential inclusions and set-valued analysis.

It has been recognized for a long time, particularly in the Polish school around Ważewski and the Soviet school around Filippov to name only two, that classical control problems, as well as the ones mentioned above, could be best treated within the framework of differential inclusions, notwithstanding a natural reluctance to use the unfamiliar set-valued maps (point to set maps) instead of the usual single-valued maps. The lack of an adequate differential calculus for set-valued maps, including an *inverse theorem* which is at the root of most of the important results of analysis and differential geometry, also delayed the use of this approach.

These tools were developed during the last decade for various reasons, and it can safely be said that by now, linear and nonlinear analysis have been adapted to the set-valued case and that many results of differential equations found their counterpart in the theory of differential inclusions.

Some of the most important incentives for developing these techniques were provided by nonsmooth and stochastic optimization dynamical systems under constraints and uncertainty, viability theory and systems theory, all of which form part of the research of the Systems and Decision Sciences Program at IIASA. This report proves this point within the framework of nonlinear systems theory, or, to be more precise, the control of differential inclusions.

Two issues are addressed: controllability and observability. Firstly they are treated in the linear (but set-valued) case, where these concepts are shown to be dual concepts, and where many criteria, including Kalman's, are adapted to this case. They are then treated in the nonlinear case by *linearization*, since the differential calculus of set-valued maps allows this to be done. It is then shown how controllability and observability of the linearized systems apply their local version to the original system.

This survey should convince the reader of the efficiency of the tools provided by differential inclusions and set-valued analysis to solve problems involving constraints and uncertainty, features that are present in most systems that occur in economics, management, biology, and cognitive sciences.

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Contents

Controllability and Observability of Control Systems under Uncertainty

1. Introduction

The purpose of this paper is to review local controllability and/or observability of the system

$$\text{for almost all } t \in [0, T] \quad , \quad x'(t) \in F(t, x(t)) \tag{1.1}$$

whose evolution is described by a differential inclusion.

The overall strategy consists in *linearizing* differential inclusions and deriving these local results from the global controllability and/or observability of the linearized differential inclusion.

Results of this nature are useful when we know how to characterize controllability and/or observability of such linearized differential inclusions: we shall provide necessary and sufficient conditions extending Kalman's celebrated rank condition and show that, in this case, controllability and observability are dual concepts.

There is no longer any need to justify the use of differential inclusions, which provide a unifying framework for dealing with closed loop control systems

$$x' = f(t, x, u) \quad , \quad u \in U(t, x) \quad ,$$

or control systems defined in an implicit way

$$f(t, x, x', u) = 0 \quad , \quad u \in U(t, x) \quad ,$$

or systems under uncertainty, where the set-valued map takes into account disturbances and/or perturbations, or even differential games.

1.1. Linearization through derivatives of set-valued maps

Linearization of the differential inclusion naturally requires a differential calculus of set-valued maps which will be presented in the fourth section.

The idea behind the construction of a differential calculus of set-valued maps is the simple idea of Fermat, and remains the one which most of us have been acquainted with since our teens. It starts with the concept of tangent to the graph of a function: the derivative is the slope of the tangent to the curve. We should say now, that the tangent space to the graph of the curve is the graph of the differential. Based on this, we can adapt the concept of the derivative to the set-valued case.

Consider a set-valued map $F: X \rightrightarrows Y$, which is characterized by its graph (the subset of all pairs (x, y) such that y belongs to $F(x)$).

We first need an appropriate notion of tangent cone to a set in a Banach space at a given point, which coincides with the tangent space when the set is an embedded differentiable manifold and with the tangent cone of convex analysis when the set is convex. Experience shows that four tangent cones seem to be useful:

- Bouligand's contingent cone, introduced in the 1930s.
- Adjacent tangent cone, also known as the *intermediate* cone.
- Clarke's tangent cone, introduced in 1975.
- Bouligand's paratingent cone, introduced in the 1930s.

All four correspond to different regularity requirements. Clarke's tangent cone is always convex. A sufficiently detailed calculus of these cones already exists.

Once a concept of tangent cone is chosen, we can associate with it a notion of the derivative of a set-valued map F at a point (x, y) of its graph: it is a set-valued map $F'(x, y)$, the graph of which is equal to the tangent cone to the graph of F at the point (x, y).

In this way, we associate with the contingent cone, the adjacent cone and the Clarke tangent cones, the following concepts of derivatives:

- Contingent derivative, corresponding to the Gâteaux derivative.
- Adjacent derivative, corresponding to the Fréchet derivative.
- Circatangent derivative, corresponding to the continuous Fréchet derivative.
- Paratingent derivative.

Derivatives of set-valued maps (and also of nonsmooth single-valued maps) are set-valued maps which are positively homogeneous. They are convex (in the sense that their graph is convex) when they depend in a *continuous* way of (x, y). Such maps, whose graphs are closed convex cones, are the set-valued analogs of continuous linear operators, called *closed convex processes.*

They are presented in the second section. Many properties of continuous linear operators can be extended to closed convex processes (including Banach's closed graph and open mapping theorems and the Banach-Steinhauss theorem).

Therefore, the linearized differential inclusion of (1.1) around a given solution $z(\cdot)$ will have the form

$$\text{for almost all } t \in [0,T] \ , \quad w'(t) \in F'(t, z(t), z'(t))\,(w(t)) \ . \tag{1.2}$$

Let S_T denote the solution map (or the funnel) associating with any initial state x_0 the set of solutions to (1.1) starting at x_0.

Can such a linearized differential inclusion (1.2) be regarded as a *variational* inclusion, in the sense that the set of solutions $w(\cdot)$ of (1.2) starting at some u, is related to the derivative of the solution map at $(z_0, z(\cdot))$ in the direction u?

The answer is positive, and is the object of several *variational* theorems presented in the fifth section.

1.2. Local controllability

Let $R(T,\xi) := \{x(T) \,|\, x \in S_T(\xi)\}$ be the reachable set, of (1.1) at time T from the initial state ξ and $M \subset \mathbf{R}^n$, a closed subset, be the target. We shall say that the system is *locally controllable* around M if

$$0 \in \operatorname{Int}(R(T,\xi) - M) \ .$$

This means that a neighborhood U of 0 in $\mathbf{R}^n$ exists, such that for all $u \in U$, there exists a solution $x(\cdot) \in S_T(\xi)$ satisfying $x(T) \in M + u$.

We shall say that the linearized system (1.2), where we take for derivative F' the circatangent derivative, is controllable around $C_M(z(T))$ (the Clarke tangent cone to M at $z(T)$) if

$$R^L(T,0) - C_M(z(T)) = \mathbf{R}^n \ ,$$

where $R^L(T,0)$ denotes the reachable set of (1.2) from 0.

Under adequate assumptions, controllability of the linearized system implies local controllability of the original system. This is derived from a general constrained inverse function theorem. It states that if the derivative $CF(x_0, y_0)$ of a set-valued map F from a Banach space X to a finite dimensional space Y is surjective, then F is invertible around y_0 and its inverse enjoys some kind of Lipschitz property. This result is a simple form of more powerful controllability results obtained by refinements of set-valued analysis, see Frankowska (1987b).

1.3. Local observability

System (1.1) is observed through an observation map H, which is generally a set-valued map from the state space X to some observation space Y, which associates with each solution to the differential inclusion (1.1) an observation $y(\cdot)$ satisfying

$$\forall t \in [0,T] \quad , \quad y(t) \in H(x(t)) \quad . \tag{1.3}$$

Observability concepts deal with the possibility of recovering the initial state $x_0 = x(0)$ of the system when only both the evolution of an observation $t \in [0,T] \longrightarrow y(t)$ during the interval $[0,T]$ and naturally the laws (1.1) and (1.3) are known. Once we obtain the initial state x_0, we may, by studying the differential inclusion, gather information about the solutions starting from x_0, using the results provided by the theory of differential inclusions.

The set-valued character leads to two types of input–output (set-valued) maps:

- Sharp Input–Output map which is the (usual) product

$$\forall x_0 \in X \quad , \quad I_-(x_0) := (H \circ S)(x_0) := \bigcup_{x(\cdot) \in S(x_0)} H(x(\cdot)) \quad .$$

- Hazy Input–Output map which is the square product

$$\forall x_0 \in X \quad , \quad I_+(x_0) := (H \square S)(x_0) := \bigcap_{x(\cdot) \in S(x_0)} H(x(\cdot)) \quad .$$

The ***sharp*** Input–Output map tracks the evolution of *at least* one state which starts from some initial state x_0, whereas the ***hazy*** Input–Output map tracks *all* such solutions.

Recovering the input x_0 from the outputs $I_-(x_0)$ or $I_+(x_0)$ means that the set-valued maps are, in some sense, *injective*.

We shall choose the following strategy for obtaining local observability:

- Provide a general principle for local injectivity of the set-valued maps I_+ and I_-; these properties are derived from the fact that the kernel of an adequate derivative of I_+ or I_- is equal to 0.
- Supply chain rule formulas which allow the computations of the derivatives of the usual product I_- and the square product I_+ from the derivatives of the observation map H and the solution map S.
- Use the various derivatives of the solution map S in terms of the solution maps of the associated variational inclusions provided by the variational theorems.

1.4. Controllability and observability of convex processes

For simplicity, consider now the case when ξ is an equilibrium of a (time-independent) system, i.e., a solution to

$$0 \in F(\xi) \quad , \tag{1.4}$$

where F is assumed to be smooth enough, so that its derivative $A := DF(\xi, 0)$ is a closed convex process.

Therefore, local controllability around ξ, and observability of the system at ξ can be derived from the controllability of the closed convex process

$$x'(t) \in A(x(t)) \quad , \quad x(0) = 0 \tag{1.5}$$

and the observability of this system through the linear operator $H'(\xi)$.

As continuous linear operators, closed convex processes can be transposed. Let A be a convex process; we define its *transpose* A^* by

$$p \in A^*(q) \iff \forall (x, y) \in \text{Graph } A \quad , \quad <p, x> \leq <q, y>$$

We introduce the adjoint differential inclusion

$$\text{for almost all } t \in [0, T] \quad , \quad -q'(t) \in A^*(q(t)) \tag{1.6}$$

and the cones Q_T and Q defined by

- $Q_T := \{\nu \,|\, q(\cdot)$, a solution to (1.6) satisfying $q(T) = \nu\}$;
- $Q := \bigcap_{T>0} Q_T$.

We shall say that the adjoint system is *observable* if $Q = \{0\}$.

We denote by R_T the *reachable set* at *time* T defined by $R_T := \{x(T) \,|\, x(\cdot)$ is a solution to (1.2)$\}$.

We also say that

$$R := \bigcup_{T>0} R_T \quad \text{is the reachable set}$$

and that the differential inclusion (1.5) (or the convex process A) is *controllable* if the reachable set R is equal to the whole space $\mathbf{R}^n$.

The duality method can be stated as follows:

$$R_T^+ \text{ (the positive polar cone of } R_T\text{) is equal to } Q_T \text{ and } R^+ = Q \quad , \tag{1.7}$$

so that A is controllable if and only if A^* is observable. Actually, when the domain of the closed convex process A is the whole space, we can provide eleven necessary and sufficient conditions for the controllability of the convex process A, which will be exposed in the third section. The contents of this survey are as follows.

In the second section the properties of closed convex processes are recalled, which we use for characterizing the controllability and observability properties of linearized differential inclusions in the third section.

The fourth section is devoted to an exposition of tangent cones and derivatives of set-valued maps. We use these concepts to prove the variational theorems in the fifth section and abstract results on local injectivity and surjectivity in the sixth section. The last two sections piece together the above results to prove the local controllability and local observability results which are our objectives.

2. Convex Processes and Their Transposes

A set-valued map from $\mathbf{R}^n$ to $\mathbf{R}^n$ is said to be a *convex process* if its graph is a convex cone. It is *closed* if its graph is closed. It is called *strict* if

$$\text{Dom } A := \{x \in \mathbf{R}^n \mid A(x) \neq \emptyset\} \text{ is the whole space } .$$

Let X be a Hilbert space and $G \subset X$ be a subset. We denote by G^+, the (positive) *polar cone* of G, the closed convex cone defined by

$$G^+ := \{p \in X^* \mid \forall x \in G, \langle p, x \rangle \geq 0\} \ .$$

The separation theorem implies that the *bipolar* G^{++} is the closed convex cone spanned by G. From the above, we can deduce the following:

LEMMA 2.1. (Closed image Lemma) *Let X and Y be two Hilbert spaces, φ be a continuous linear operator from X to Y and L be a closed convex cone of Y. Assume that*

$$\text{Im}\,\varphi - L = Y \ \ (\textit{surjectivity condition}) \ .$$

Then

$$\varphi^{-1}(L)^+ = \varphi^*(L^+) \ .$$

For more details see Aubin & Ekeland (1984). We now recall some properties of convex processes from Rockafellar (1967, 1970, and 1974), Frankowska (1989b), and Aubin and Ekeland (1984, Chapter 3).

DEFINITION 2.2. *Let A be a convex process from $\mathbf{R}^n$ to itself. The transpose A^* of A is the set-valued map from $\mathbf{R}^n$ to itself given by*

$$p \in A^*(q) \iff \forall(x,y) \in \text{Graph}(A) \quad , \quad <p,x> \leq <q,y> \quad .$$

In other words,

$$(q,p) \in \text{Graph}(A^*) \iff (-p,q) \in (\text{Graph } A)^+ \quad .$$

The transpose of A^* is obviously a closed convex process and $A = A^{**}$, if and only if, the convex process A is closed. When A is a linear operator, its transpose as a linear operator coincides with its transpose as a convex process.

If A is a closed convex process, then

$$A(0) = (\text{Dom } A^*)^+ \quad .$$

DEFINITION 2.3. *Let B denote the unit ball. When A is a closed convex process, we define its norm by*

$$\|A\| := \sup_{x \in B \bigcap \text{Dom } A} \; \inf_{y \in A(x)} \|y\| \in (0,+\infty) \quad .$$

PROPOSITION 2.4. *Let A be a strict closed convex process. Then*

(a) $\forall x,y \in \mathbf{R}^n$, $A(x) \subset A(y) + \|A\| \, \|x-y\| \, B$ *(i.e., A is Lipschitzian with a finite Lipschitz constant equal to $\|A\|$).*
(b) $\text{Dom } A^* = A(0)^+$ and A^* *is upper semicontinuous with compact convex images, mapping the unit ball into the ball or radius $\|A\|$.*
(c) *The restriction of A^* to the vector space* $\text{Dom } A^* \cap (-\text{ Dom } A^*)$ *is single-valued and linear [and thus, $A^*(0) = 0$].*

We observe that we always have

$$\sup_{p \in A^*(q_0)} <p,x_0> \leq \inf_{y \in A(x)} <q_0,y> \quad .$$

See Aubin, Frankowska, and Olech (1986b).

LEMMA 2.5. *Let A be a closed convex process. For any $x_0 \in$ Int Dom A, and $q_0 \in$ Dom A^*,*

$$\sup_{p \in A^*(q_0)} <p,x_0> = \inf_{y \in A(x_0)} <q_0,y>$$

[see Rockafeller (1970]. The concepts of invariant subspaces will now be extended to the case of closed convex cones. When K is a subspace and F is a linear operator, we recall that K is invariant by F when $Fx \in K$ for all $x \in K$. When A is a convex process, there are two ways of extending this notion: we shall say that K is invariant by A if, for any $x \in K$, $A(x) \subset K$ and that K is a viability domain for A if, for any $x \in K$, $A(x) \cap K \neq \emptyset$. We also need to extend these notions to the case when K is a closed convex cone. We recall the

DEFINITION 2.6. *If K is a closed convex set and x belongs to K, we say that*

$$T_K(x) := cl \left[\bigcup_{h>0} \frac{1}{h} (K - x) \right]$$

is the tangent cone *to K at x.*

LEMMA 2.7. *When K is a vector subspace, then, for all $x \in K$, $T_K(x) = K$ and when K is a closed convex cone, then*

$$\forall x \in K \quad , \quad T_K(x) = cl(K + \mathbf{R}x) \quad .$$

Now, we can introduce

DEFINITION 2.8. *Let K be a closed convex cone and A be a convex process. We say that K is* invariant *by A if*

$$\forall x \in K \quad , \quad A(x) \subset T_K(x) \quad ,$$

and that K is a viability domain *for A if*

$$\forall x \in K \quad , \quad A(x) \bigcap T_K(x) \neq \emptyset \quad .$$

These are dual notions, as the following proposition shows.

PROPOSITION 2.9. *Let A be a strict closed convex process and K be a closed convex cone containing $A(0)$. Then K is invariant by A if and only if K^+ is a viability domain for A^*.*

Proof. Using Proposition 2.4(b) the condition $A(0) \subset K$ implies that $K^+ \subset A(0)^+ = \text{Dom } A^*$. To say that K is invariant by A amounts to saying that

$$\forall x \in K \quad , \quad \forall q \in T_K(x)^+ \quad , \quad \inf_{y \in A(x)} <q, y> \geq 0 \quad . \tag{2.1}$$

Lemma 2.7 states that $T_K(x) = \overline{\mathbf{R}x + K}$, $T_{K^+}(q) = \overline{\mathbf{R}q + K^+}$. Therefore

$$q \in T_K(x)^+ \iff <q,x> = 0 \quad , \quad q \in K^+ \iff x \in T_{K^+}(q)^+ \quad .$$

On the other hand, Lemma 2.5 implies that $\inf_{y \in A(x)} <q,y> = \sup_{p \in A^*(q)} <p,x>$. Therefore condition (2.1) is equivalent to the condition:

$$\forall q \in K^+ \quad , \quad \forall x \in T_{K^+}(q)^+ \quad , \quad \sup_{p \in A^*(q)} <p,x> \geq 0 \quad . \tag{2.2}$$

According to Proposition 2.4(b), for all $q \in K^+$, the set $A^*(q)$ is compact. The separation theorem implies that $A^*(q)$ has a nonempty intersection with $T_{K^+}(q)$ if and only if for all $x \in \mathbf{R}^n$, $\sup_{p \in A^*(q)} <p,x> \geq \inf_{z \in T_{K^+}(q)} <z,x>$. Since $T_{K^+}(q)$ is a cone, the latter inequality is equivalent to (2.2). This ends the proof. □

The concepts of eigenvalues and eigenvectors of closed convex processes are now introduced.

DEFINITION 2.10. *We shall say that* $\lambda \in \mathbf{R}$ *is an* eigenvalue *of a convex process* A *if* $\mathrm{Im}(A - \lambda I) \neq \mathbf{R}^n$ *and that* $x \in \mathrm{Dom}\, A$ *is an* eigenvector *of* A *if* $x \neq 0$ *if there exists* $\lambda \in \mathbf{R}$ *such that* $\lambda x \in A(x)$.

We observe that half-lines spanned by eigenvectors of A^* are viability domains for A^*.

LEMMA 2.11. *Let* A *be a strict convex process. Then* A^* *has an eigenvector if and only if,* $\mathrm{Im}\,(A - \lambda I) \neq \mathbf{R}^n$ *for some* $\lambda \in \mathbf{R}$.

THEOREM 2.12. *Let* A *be a strict closed convex process. If the largest viability domain* Q *for* A^* *is different from* $\{0\}$ *and contains no line, then* A^* *has at least an eigenvector.*

Example 2.13. Let F be a linear operator from $\mathbf{R}^n$ to itself, L be a closed convex cone of controls, and A be the strict closed convex process defined by $A(x) := Fx + L$.

A cone K is invariant by A if

$$\forall x \in K \quad , \quad Fx + L \subset T_K(x) \quad ,$$

and λ is an eigenvalue of A if

$$\mathrm{Im}\,(F - \lambda I) + L \neq \mathbf{R}^n \quad .$$

The transpose A^* of A is defined by

$$A^*q = \begin{cases} F^*q & \text{if } q \in L^+ \\ \emptyset & \text{if } q \notin L^+ \end{cases} .$$

A cone $P \subset L^+ = \text{Dom } A^*$ is a viability domain for A^* if and only if

$$\forall q \in P \ , \quad F^*q \in T_P(q) \ .$$

An element $q \neq 0$ is an eigenvector of A^* if and only if q is an eigenvector of F^* which belongs to the cone L^+.

Other examples of closed convex processes are provided by *circatangent derivatives* of set-valued maps (see section 4 below). Closed convex processes enjoy most of the properties of continuous linear operators, and in particular, the fundamental Banach theorem.

THEOREM 2.14. (Closed Graph Theorem) *A closed convex process A whose domain is the whole space is Lipschitz, in the sense that*

$$\forall x_1, x_2 \in X \ , \quad A(e_1) \subset A(x_2) + l\|x_1 - x_2\|B \ ,$$

whose open mapping formulation can be stated as follows:

THEOREM 2.15. (Robinson–Ursescu's Open Mapping Theorem) *Assume that a closed convex process $A{:}X \rightrightarrows Y$ is* surjective. *Then there exists a constant $l > 0$ such that,*

$$\forall y \in Y \ , \quad \exists x \in A^{-1}(y) \ \textit{such that} \ \|x\| \leq l\|y\| \ .$$

Banach–Steinhauss's uniform boundedness theorem can be extended to closed convex processes:

THEOREM 2.16. (Uniform Boundedness for Closed Convex Processes) *Let X and Y be reflexive Banach spaces and A_h be a family of closed convex processes from X to Y, i.e.,* pointwise bounded, *in the sense that*

$$\forall x \in X \ , \quad \exists y_h \in A_h(x) \ \textit{such that} \ \sup_h \|y_h\| < +\infty \ .$$

Then this family is uniformly bounded *in the sense that*

$$\sup_h \|A_h\| < +\infty \ .$$

Hence we can speak of *bounded* families of closed convex processes, without specifying whether they are pointwise or uniform. We can then deduce the following:

THEOREM 2.17. *Let us consider a metric space U, reflexive Banach spaces X and Y, and a set-valued map associating with each $u \in U$ a* closed convex process $A(u): X \rightrightarrows Y$. *Let us assume that*

the family of closed convex processes $A(u)$ is bounded.

The following conditions are then equivalent:

- *The set-valued map $u \rightrightarrows$* Graph $(A(u))$ *is lower semicontinuous.*
- *The set-valued map $(u, x) \rightrightarrows A(u)(x)$ is lower semicontinuous.*

See also Robinson (1979), Aubin & Wets (1988), and Aubin (forthcoming).

3. Controllability and Observability of Closed Convex Processes

We begin this section with the duality theorem, which characterizes the polar cones of the reachable sets. Many of the results of this section as well as their proofs can be found in Aubin, Frankowska, and Olech (1986b).

We denote by $W^{1,p}(0, T)$, $p \in [1, \infty]$, the Sobolev space of functions $x \in L^p(0, T; \mathbf{R}^n)$ such that $x'(\cdot)$ belongs to $L^p(0, T; \mathbf{R}^n)$.

Let us consider the Cauchy problem for the differential inclusion

$$\begin{cases} i) & x'(t) \in A(x(t)) \quad \text{for almost all } t \in [0, T] \\ ii) & x(0) = 0 \ . \end{cases} \tag{3.1}$$

We recall that the reachable set R_T is defined by

$$R_T := \{x(T) \mid x \in W^{1,1}(0, T) \ \text{ is a solution to (3.1)}\} \ .$$

We shall characterize its positive polar cone R_T^+. For that purpose, we associate with the differential inclusion (3.1) the adjoint inclusion

$$\begin{cases} i) & -q'(t) \in A^*(q(t)) \quad \text{for almost all } t \in [0, T] \\ ii) & q(T) = \eta \ . \end{cases} \tag{3.2}$$

and we denote by $Q_T \subset \text{Dom}\, A^*$ the set of *final* values η such that the differential inclusion (3.2) has a solution.

$$Q_T := \{\eta \mid \exists q \in W^{1,1}(0, T), \text{a solution to (3.2)}\} \ .$$

THEOREM 3.1. *Let A be a strict closed convex process. Then $R_T^+ = Q_T$.*

Proof.

(a) We denote by S the closed convex cone of solutions to the differential inclusion (3.1) in the Hilbert space

$$X := \{x \in W^{1,2}(0,T) \mid x(0) = 0\} \; .$$

Consider the continuous linear operator

$$\gamma_T : x(\cdot) \in X \longrightarrow x(T) \in \mathbf{R}^n \; .$$

The transpose γ_T^* maps $\mathbf{R}^n$ into the dual X^* of X and for all $\eta \in R_T^+$

$$\forall x \in S \;\; , \quad <\gamma_T^* \eta, x> = <\eta, \gamma_T x> \geq 0 \; . \tag{3.3}$$

It can be checked that S is dense in the $W^{1,1}(0,T)$-solutions to (3.1) in the metric of uniform convergence on $[0,T]$. This and (3.3) yield

$$R_T^+ = \{\eta \mid \gamma_T^* \eta \in S^+\} \; . \tag{3.4}$$

Let us set

$$\begin{cases} i) \;\; Y := L^2(0,T;\mathbf{R}^n) \times L^2(0,T;\mathbf{R}^n) \\ ii) \;\; L := \{(x,y) \in Y \mid y(t) \in A(x(t)) \text{ a.e.}\} \\ iii) \;\; D, \text{ the differential operator defined on } X \text{ by } Dx = x' \; . \end{cases}$$

Then $S = (1 \times D)^{-1}(L)$. The closed image, Lemma 2.1, when applied to the continuous linear operator $\varphi = (1 \times D)$, states that

$$S^+ = (1 \times D)^*(L^+) \tag{3.5}$$

provided that the *surjectivity* assumption,

$$\mathrm{Im}\,(1 \times D) - L = Y \tag{3.6}$$

is satisfied.

(b) This surjectivity assumption can be written

$$\forall (u,v) \in Y \text{ there } \exists \; x \in X \text{ such that } x'(t) \in A(x(t) - u(t)) = v(t) \text{ a.e.} \; .$$

Since the domain of A is the whole space, then A is Lipschitzian.

The set-valued map $F(t,x) := A(x - u(t)) + v(t)$ is then measurable in t, Lipschitzian with respect to x, has closed images and satisfies the following estimate:

$$d(0, F(t,0)) \leq \|A\| \; \|u(t)\| + \|v(t)\| \ .$$

The function $t \to \|A\| \; \|u(t)\| + \|v(t)\|$ being in $L^1(0,T)$, we can apply a Filippov Theorem (Filippov, 1967; see also Clarke, 1983) which states the existence of a solution $x(\cdot)$ to the differential inclusion $x'(t) \in F(t, x(t))$, $x(0) = 0$, satisfying:

$$\|x'(t)\| \leq \|A\| \; e^{\|A\| T} \int_0^T d(0, F(t,0)) \, dt + d(0, F(t,0)) \ .$$

Thus $x \in X$ and the surjectivity assumption (3.6) holds true.

(c) Therefore, by (3.4) and (3.5), we obtain the formula

$$R_T^+ = \{n \mid \gamma_T^* \eta \in (1 \times D)^* \, (L^+)\} \ . \tag{3.7}$$

Let $\eta \in Q_T$ and q be a solution to the adjoint inclusion (3.2). According to Proposition 2.4(b), $q(\cdot) \in W^{1,\infty}(0,T)$ and for all $x \in S$

$$<\eta, x(T)> = <(q', q), (x, x')>_Y \ .$$

This is nonnegative by the definition of A^*. Thus $Q_T \subset R_T^+$. To prove the opposite, let η belong to R_T^+. Using (3.7), there exists $(p,q) \in L^+$ such that

$$<\eta, \gamma_T x> = <p, x>_{L^2} + <q, Dx>_{L^2} \quad \forall x \in X \ . \tag{3.8}$$

By taking x so that $x(T) = 0$, we deduce that $p = Dq$ in the sense of distribution. Since p and q belong to L^2, we infer that q belongs to the Sobolev space $W^{1,2}(0,T)$. Thus $Dq = q'$. Integrating by parts in equation (3.8), and taking into account that $x(0) = 0$, we obtain

$$<\eta, \gamma_T x> = <p - q', x>_{L^2} + <q(T), x(T)> = <q(T), x(T)> \ .$$

The surjectivity of γ_T implies that $\eta = q(T)$. Thus $q(\cdot)$ is a solution to (3.2) and then, η belongs to Q_T. This achieves the proof. □

We now associate with any $\eta \in \text{Dom } A^*$ the *solution set* $S_T(\eta)$ of solutions to the adjoint differential inclusion (3.2) satisfying $q(T) = \eta$ and we denote by Q_T the domain of the *solution map* S_T:

$$Q_T := \eta \in \text{Dom } A^* \mid S_T(\eta) \neq \emptyset \ .$$

We observe that the sequence of the closed domains Q_T decreases:

$$\text{if } T_1 \geq T_2 \ , \quad \text{then } Q_{T_1} \subset Q_{T_2} \ .$$

We now introduce the intersection Q of these cones

$$Q := \bigcap_{T>0} Q_T \ .$$

Since the compact subsets $S^{n-1} \bigcap Q_T$ form a decreasing sequence, we observe that $Q \neq \{0\}$ if and only if all the cones Q_T are different from 0. We shall say that Q is the *largest viability domain*, thanks to the following theorem.

THEOREM 3.2. *Let A be a strict closed convex process. Then the closed convex cone Q is the largest closed convex cone which is a viability domain for A^*.*

Proof. It is not difficult to prove that Q is a closed convex cone containing any viability domain P. It remains to prove that Q is a viability domain, i.e., that

$$\forall q \in Q \ , \quad A^*(q) \bigcap T_Q(q) \neq \emptyset \ .$$

Assume that $Q \neq \{0\}$. Thanks to the necessary condition of the viability theorem (see Haddad, 1981), it is sufficient to prove that for some $T > 0$,

$$\forall \eta \in Q \ , \quad \exists p(\cdot) \in S_T(\eta) \ \text{ which is viable on } Q \ .$$

Since η belongs to Q_{nT} for all $n \geq 2$, there exists a solution $p_n(\cdot) \in S_{nT}\{\eta\}$. By the very definition of Q_t, we know that $p(t) \in Q_t$ for all $t \leq nT$.

Therefore, the translated function $\hat{p}_n(\cdot)$ defined on $[0, T]$ by

$$\hat{p}_n(t) := p_n(t + (n-1)T) \ ,$$

belongs to $S_T(\eta)$ and satisfy for all $t \in [0, T]$, $k \leq n - 1$,

$$\hat{p}_n(t) = p_n(t + (n-1)\,T) \in Q_{t+(n-1)T} \subset Q_{(n-1)T} \subset Q_{kT} \ .$$

But $S_T(\eta)$ is compact in $C(0, T; \mathbf{R}^n)$. Thus there exists a subsequence of $\hat{p}_n(\cdot)$ converging to some $\hat{p}(\cdot) \in S_T(\eta)$ uniformly on $[0, T]$. Since for all $t \in [0, T]$, $k \geq 1$, $\hat{p}(t) \subset Q_{kT}$, we infer that

$$\hat{p}(t) \subset \bigcap_{k \geq 1} Q_{kT} = Q \ . \qquad \square$$

We now translate this result in terms of reachable sets R_T. Since $0 \in A(0)$, the reachable cones $R(T)$ do form an increasing sequence. We define the reachable set of the inclusion (3.1) to be

$$R := \bigcup_{T>0} R(T) \quad .$$

It is a convex cone, which is equal to the whole space if and only if for some $T > 0$, $R(T) = \mathbf{R}^n$.

We say that the closure $\bar{R}$ of R is the *smallest invariant cone* by A. This definition is motivated by the consequences of both Theorem 3.1 and Theorem 3.2.

THEOREM 3.3. *Let A be a strict closed convex process. Then the closed convex cone $\bar{R}$ is the smallest closed convex cone containing $A(0)$ and invariant by A.*
We consider now the largest subspace of Q:

$$Q \cap (-Q) \subset \text{Dom } A^* \cap (-\text{Dom } A^*) \quad .$$

PROPOSITION 3.4. *Let A be a strict closed convex process. The subspace $Q \cap (-Q)$ is the largest subspace invariant by A^* and its orthogonal space $R - R$ is invariant by A in the sense that:*

$$\forall x \in R - R \quad , \quad A(x) \subset R - R \quad .$$

We consider now the cones $A(0)$, $A^2(0) := A(A(0)), \ldots, A^k(0) = A(A^{k-1}(0))$, etc. Since 0 belongs to $A(0)$, these convex cones form an increasing sequence. We introduce the cone

$$N := cl\left[\bigcup_{k \geq 1} A^k(0)\right]$$

and the vector subspace

$$M \text{ spanned by } N \quad .$$

THEOREM 3.5. *Let A be a strict closed convex process. Then*

- $A(N) \subset N$.
- $\bar{R} \subset N \subset M \subset R - R$.
- $Q \cap (-Q) \subset \bigcap_{k \geq 1} A^k(0)^{\perp} \subset \bigcap_{k \geq 1} A^k(0)^{+} \subset Q$.

Remark. When the reachable set R is a vector space, the subsets R, N, M and $R - R$ coincide. This happens, for instance, when A is symmetric (in the sense that $A(-x) = -A(x)$), i.e., when the graph of A is a vector subspace.

According to the duality Theorem 3.1, the following dual version of this theorem holds true.

THEOREM 3.6. *Let A be a strict closed convex process. Assume that the reachable set R is different from $\mathbf{R}^n$ and spans the whole space. Then A has at least one eigenvalue.*

We shall deduce from the preceding results several characterizations of the controllability of closed convex processes.

DEFINITION 3.7. *We say that (3.1) is* controllable at time T (respectively, controllable) *if* $R_T = \mathbf{R}^n$ (respectively, $R = \mathbf{R}^n$). *We shall say that the adjoint inclusion (3.2) is observable at time T* (respectively, observable) *if* $Q_T = \{0\}$ (respectively, $Q = \{0\}$).

We also observe the following property.

LEMMA 3.8. *Let A be a strict closed convex process. The following three properties are equivalent.*

$$\begin{cases} a) & \exists m \geq 1 \ \textit{ such that } \ A^m(0) - A^m(0) = \mathbf{R}^n \\ b) & \exists m \geq 1 \ \textit{ such that } \ A^m(0)^{\perp} = \{0\} \\ c) & \exists m \geq 1 \ \textit{ such that } \ \mathrm{Int}\, A^m(0) \neq \emptyset \ . \end{cases} \tag{3.9}$$

It is convenient here to introduce the *rank condition.*

Rank Condition. We say that a convex process A satisfies the rank condition if one of the equivalent properties of (3.9) holds true.

LEMMA 3.10. *Consider the strict closed convex process $A(x) = Fx + L$, where $F \in \mathbf{R}^{n \times n}$ is a matrix and L is a vector subspace of $\mathbf{R}^n$. Then A satisfies the rank condition, if and only if, $A^n(0) - A^n(0) = \mathbf{R}^n$.*

We begin by stating characteristic properties of observability of the adjoint system (3.2) and then use the duality results to infer the equivalent characteristic properties of system (3.1).

THEOREM 3.11. *Let A be a strict closed convex process. The following properties are equivalent:*

(a*) *The adjoint inclusion (3.2) is observable.*
(b*) *The adjoint inclusion (3.2) is observable at time $T > 0$ for some T.*
(c*) $\{0\}$ *is the largest closed convex cone which is a viability domain for A.*

(d*) *A^* has neither a proper invariant subspace nor eigenvectors.*
(e*) *The rank condition is satisfied and A^* has no eigenvectors.*

THEOREM 3.12. *Let A be a strict closed convex process. All the properties of Theorem 3.11 are equivalent to the following properties:*

(a) *The differential inclusion (3.1) is controllable.*
(b) *The differential inclusion (3.1) is controllable at some time $T > 0$.*
(c) *$\mathbf{R}^n$ is the smallest closed convex cone containing $A(0)$ which is invariant by A.*
(d) *A has neither proper invariant subspace nor eigenvalues.*
(e) *The rank condition is satisfied and A has no eigenvalues.*
(f) *For some $m \geq 1$, $A^m(0) = (-A)^m(0) = \mathbf{R}^n$.*

In the case when the set-valued map A is defined by $A(x) := Fx + L$, we derive known results from Kalman when L is a vector space of control, from Brammer (1972), Korobov (1980), and Saperstone and Yorke (1971), when L is an arbitrary set of controls containing 0.

4. Tangent Cones and Derivatives of Set-Valued Maps

We devote this section to the definitions of some (and maybe too many) of the tangent cones which have been used in applications, and in particular, for defining derivatives of set-valued maps. Unfortunately, for arbitrary subsets, we are forced to introduce and study several concepts of tangent cones which correspond to different regularity requirements.

However, the idea remains the same, i.e., implement one of the possible mathematical descriptions of the concept of tangency, without requiring *a priori* a vector space of tangent vectors, as in differential geometry.

DEFINITION 4.1. (Tangent cones) *Let $K \subset X$ be a subset of a Banach space X and $x \in \overline{K}$ belong to the closure of K. We denote by*

$$S_K(x) := \bigcup_{h>0} \frac{K - x}{h}$$

the cone spanned by $K - x$.

We now introduce four tangent cones:

(1) *The* contingent cone $T_K(x)$, *defined by*

$$T_K(x) := \left\{ v \mid \liminf_{h \to 0+} d_K(x + hv)/h = 0 \right\} .$$

(*From the Latin* contingere, *to touch on all sides; introduced by G. Bouligand in 1931.*)

(2) *The* adjacent cone $T^b_K(x)$, *defined by*

$$T^b_K(x) := \left\{ v \mid \lim_{h \to 0+} d_K(x + hv)/h = 0 \right\} .$$

[*From the Latin* adjacere, *to lie near, recently referred to as the intermediate cone by* Frankowska (1987b) *and the* derivable cone *by* Rockafellar (1987a and 1987b).]

(3) *The* Clarke tangent cone $C_K(x)$, *defined by*

$$C_K(x) := \left\{ v \mid \lim_{h \longrightarrow 0+,\, K \ni x' \longrightarrow x} d_K(x' + hv)/h = 0 \right\} .$$

[*From* Clarke (1983); *we shall use the adjective* circatangent *when referring to properties derived from this tangent cone, for instance, circatangent derivatives.*]

(4) *If $L \subset K$ is a subset of K, the* paratingent cone $P^L_K(x)$ *to K relative to L at $x \in L$ defined by*

$$P^L_K(x) := \left\{ v \mid \limsup_{h \to 0+, L \ni x' \to x} d_K(x' + hv)/h = 0 \right\} .$$

(*Introduced by Bouligand in 1931.*)

We see at once that these tangent cones are closed, that these tangent cones to K and the closure $\overline{K}$ of K do coincide, that

$$C_K(x) \subset T^b_K(x) \subset T_K(x) \subset S_K(x) \quad ,$$

and that

$$\text{if } x \in \operatorname{Int}(K) \quad , \quad \text{then } C_K(x) = X \quad .$$

The Clarke tangent cone $C_K(x)$ is a closed convex cone satisfying the following properties

- $C_K(x) + T_K(x) \subset T_K(x)$.
- $C_K(x) + T^b_K(x) \subset T^b_K(x)$.

DEFINITION 4.2. *We say that a subset $K \subset X$ is* sleek *at $x \in K$ if the set-valued map $K \ni x' \rightrightarrows T_K(x')$ is lower semicontinuous at x and* sleek *if and only if it is* sleek *at every point x of K.*

We shall say that K is derivable *at $x \in K$, if and only if, $T_K^b(x) = T_K(x)$ and* derivable *if and only if it is derivable at every $x \in K$. The following property is very useful.*

THEOREM 4.3. (Tangent Cones of Sleek Subsets) *Let K be a weakly closed subset of a reflexive Banach space. If K is sleek at $x \in K$, then the contingent and Clarke tangent cones do coincide, and consequently, are convex* (Aubin and Clarke, 1977).

Example. (Tangent Cones to Convex Sets) Let us assume that K is convex. Then the contingent cone $T_K(x)$ to K at x is convex and

$$C_K(x) = T_K^b(x) = T_K(x) = \overline{S_K(x)} \ .$$

Furthermore, if the dimension of X is finite, then any closed convex subset is sleek. The same is true for smooth manifolds (see Aubin and Ekeland, 1984).

Remark. We are prompted to introduce this ménagerie of tangent cones because each of them corresponds to a classical regularity requirement. It will be shown later that the contingent cone is related to Gâteaux derivatives, the adjacent cone is related to the Fréchet derivative and the Clarke tangent cone to the continuous Fréchet derivative.

The contingent cone plays a crucial role in characterizing the subsets $K \subset \mathbf{R}^n$ that enjoy the *viability property*: for every $x_0 \in K$, there exists a solution to the differential inclusion $x' \in F(x)$ which is viable in the sense that $x(t) \in K$ for all $t \geq 0$.

When F is upper semi-continuous with closed convex images and linear growth, Haddad's viability theorem (Haddad, 1981) an extension of the 1943 Nagumo theorem, states that K enjoys the viability property if and only if

$$\forall x \in K \ , \quad F(x) \bigcap T_K(x) \neq \emptyset \ .$$

Adjacent tangent cones play an important role in Lebesgue and Sobolev spaces.

The charm of the Clarke tangent cone (and thus of sleek subsets) is the convexity, that allows for dual formulations and statements by polarity and transposition. But the price that has to be paid in terms of loss of information for playing with duality simply to conserve some familiar dual formulation is, indeed, too high in many situations. This is one of the reasons why we shall not use normal cones and generalized gradients here.

From each concept of tangent cone to a subset, we now derive an associated concept of *graphical derivative* of a set-valued map F from a topological vector space X to another vector space Y. The idea is very simple, and goes back a long way in the history of differential calculus, when Pierre de Fermat, in the first half of the seventeenth century, introduced the concept of a tangent to the graph of a function:

The tangent space to the graph of a function f at a point (x, y) of its graph is the line of slope $f'(x)$, i.e., the graph of the linear function $u \to f'(x)\,u$.

It is possible to implement this idea for any set-valued map F since we have introduced (unfortunately several) ways to implement the concept of tangency for any subset of a topological vector space. Therefore, within the framework of a given problem, we can choose an adequate concept of tangent cone, and thus, regard this tangent cone to the graph of the set-valued map F at some point (x, y) of its graph, as the graph of the associated *graphical* derivative of F at the point (x, y).

Since the tangent cones are at least ... cones, all these derivatives are at least *positively homogeneous* set-valued maps (also called *processes*). However, they are closed convex processes, i.e., set-valued analogs of continuous linear operators, when the tangent cones happen to be closed and convex (which is the case when we use the Clarke tangent cone).

Hence, we begin with some definitions and notations.

DEFINITION 4.4. *Let $F{:}X \rightrightarrows Y$ be a set-valued map from a Banach vector space X to another vector space Y. We introduce four graphical derivatives:*

(1) *The* contingent derivative $DF(x, y)$, *defined by*

$$\text{Graph }(DF(x,y)) := T_{\text{Graph }(F)}(x,y) \ .$$

(2) *The* adjacent derivative $D^{bF(x,}y)$, *defined by*

$$\text{Graph }(D^bF(x,y)) := T^b_{\text{Graph }(F)}(x,y) \ .$$

(3) *The* circatangent derivative $CF(x, y)$, *defined by*

$$\text{Graph }(CF(x,y)) := C_{\text{Graph }(F)}(x,y) \ .$$

(4) *The* paratingent derivative $PF(x, y)$, *defined by*

$$\text{Graph }(PF(x,y)) := P_{\text{Graph }(F)}(x,y) \ .$$

We shall say that F is sleek *at $(x, y) \in$ Graph (F) if and only if*

$$(x', y') \rightrightarrows \text{Graph }(DF)\,(x', y') \quad \textit{is lower semicontinuous at} \quad (x, y)$$

and it is sleek *if it is* sleek *at every point of its graph.*

We shall say that F is derivable at $(x, y) \in$ Graph (F) *if and only if the contingent and adjacent derivatives coincide:*

$$DF(x,y) := D^b F(x,y)$$

and that it is derivable *if it is derivable at every point of its graph.*

But what about Newton and Leibnitz who introduced the derivatives as limits to differential quotients? Our first task is to characterize the various *graphical definitions* as adequate limits of differential quotients. Unfortunately, the formulas often become quite ugly, and nobody would have invented them in this form had they not been derived from the graphical approach.

However, all these limits are pointwise limits, which classify all generalized derivatives in a class different from the class of distributional derivatives introduced by L. Schwartz and S. Sobolev in the fifties, for solving partial differential equations. (Their objective was to keep the linearity of the differential operators by allowing the convergence of the differential quotients in weaker and weaker topologies; the price to be paid is that derivatives may no longer be functions, but distributions.)

For instance, the *contingent derivative* $DF(x,y)$ of F at (x,y) is the set-valued map from X to Y defined by

$$v \in DF(x,y)(u) \iff \liminf_{h\to 0+, u'\to u} d\left(v, \frac{F(x+hu')-y}{h}\right) = 0 \ ,$$

and the *paratingent derivative* $PF(x,y)$ of F at (x,y) is the set-valued map from X to Y defined by

$$v \in PF(x,y)(u) \iff \liminf_{h\to 0+, (x',y') \underset{F}{\longrightarrow} (x,y), u'\to u} d\left(v, \frac{F(x'+hu')-y'}{h}\right) = 0 \ ,$$

where $\underset{F}{\longrightarrow}$ denotes the convergence in Graph (F).

When F is lipschitzian around $x \in$ Int (Dom (F)), the above formulas become

$$\begin{cases} i) \ \ v \in DF(x,y)(u) \iff \liminf_{h\to 0+} d\left(v, \dfrac{F(x+hu)-y}{h}\right) = 0 \\ ii) \ \ v \in PF(x,)(u) \iff \liminf_{h\to 0+, (x',y') \underset{F}{\longrightarrow} (x,y)} d\left(v, \dfrac{F(x'+hu)-y'}{h}\right) = 0 \ . \end{cases}$$

Moreover, if k denotes the Lipschitz constant of F at x, then for every $y \in F(x)$ the derivative $DF(x,y)$ has nonempty images and is k-lipschitzian.

Despite the fact that both adjacent and circatangent derivatives can be defined as limits of difference quotients for any set-valued map F, the formulas are simpler when we deal with lipschitzian set-valued maps. Since we use them only in this context in this paper, we provide their formulas in this limited case.

Assume that F is lipschitzian around an element $x \in \mathrm{Int}\,(\mathrm{Dom}\,(F))$, then the *adjacent derivative* $D^bF(x,y)$ and the *circatangent derivative* $CF(x,y)$ are the set-valued maps from X to Y respectively defined by

$$v \in D^bF(x,y)(u) \iff \lim_{h\to 0+} \left(v, \frac{F(x+hu)-y}{h} \right) = 0 \ ,$$

and

$$v \in CF(x,y)(u) \iff \lim_{h\to 0+,(x',y') \underset{F}{\to} (x,y)} d\left(v, \frac{F(x'+hu)-y'}{h} \right) = 0 \ .$$

A brief explanation is necessary. First, all these derivatives are positively homogeneous and their graphs are closed. We observe the obvious inclusions,

$$CF(x,y)(u) \subset D^bF(x,y)(u) \subset DF(x,y)(u) \subset PF(x,y)(u) \ ,$$

and that the definitions of contingent and adjacent derivatives on the one hand, the paratingent and circatangent derivatives, on the other hand, are symmetric. When $F := f$ is single-valued, we set

$$Df(x) := Df(x,f(x)) \ , \quad D^bf(x) := D^bf(x,f(x)) \ , \quad Cf(x) := Cf(x,f(x)) \ .$$

It can easily be seen that:

$$\begin{cases} Df(x)(u) = f'(x)\,u & \text{if } f \text{ is Gâteaux differentiable and Lipschitz} \\ D^bf(x)(u) = f'(x)\,u & \text{if } f \text{ is Fréchet differentiable} \\ Cf(x)(u) = f'(x)\,u & \text{if } f \text{ is continuously differentiable} \ . \end{cases}$$

This also allows us to define and use derivatives of restrictions $F := f|_K$ of single-valued maps f to subsets $K \subset X$, which are defined by

$$f|_K(x) := \begin{cases} f(x) & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \ . \end{cases}$$

If f is continuously differentiable around a point $x \in K$, then the *derivative of the restriction is the restriction of the derivative to the corresponding tangent cone.*

The most familiar instance of set-valued maps is the inverse of a noninjective single-valued map. In this case, the *derivative of the inverse of a set-valued map F is the inverse of the derivative:*

$$D(F)^{-1}(y,x) = DF(x,y)^{-1} \ , \quad D^b(F)^{-1}(y,x) = D^bF(x,y)^{-1} \ ,$$
$$C(F)^{-1}(y,x) = CF(x,y)^{-1} \ .$$

The circatangent derivatives are closed convex processes, because their graphs are closed convex cones, i.e., they are set-valued analogs of the continuous linear operators.

Remark: Kernel of the Derivative – The kernels of the various derivatives characterize the associated tangent cones to the inverse image.

PROPOSITION 4.5. *Let $F: X \rightrightarrows Y$ be a set-valued map and (x,y) belong to its graph. Then*

$$\begin{cases} i) & T_{F^{-1}(y)}(x) \subset \ker DF(x,y) := DF(x,y)^{-1}(0) \\ ii) & T^b_{F^{-1}(y)}(x) \subset \ker D^bF(x,y) \ . \end{cases}$$

If F^{-1} is pseudo-lipschitzian around (y,x), in the sense that there exists $l > 0$ such that for any $(\hat{x},\hat{y}) \in$ Graph(F) in a neighborhood of (x,y), $d(\hat{x}, F^{-1}(y)) \leq l\|y - \hat{y}\|$ we have

$$\begin{cases} i) & \ker DF(x,y) = T_{F^{-1}(y)}(x) \\ ii) & \ker D^bF(x,y) = T^b_{F^{-1}(y)}(x) \\ iii) & \ker CF(x,y) \subset C_{F^{-1}(y)}(x) \ . \end{cases}$$

We now provide chain rule formulas for computing the composition product of a set-valued map $G: X \rightrightarrows Y$ and a set-valued map $H: Y \rightrightarrows Z$.

One can conceive of two dual ways for defining composition products of set-valued maps (that coincide when G is single-valued):

DEFINITION 4.6. *Let X, Y, Z be Banach spaces and $G: X \rightrightarrows Y$, $H: Y \rightrightarrows Z$ be set-valued maps:*

- *The usual composition product (called simply the* product*) $H \circ G: X \rightrightarrows Z$ of H and G at x is defined by*

$$(H \circ G)(x) := \bigcup_{y \in G(x)} H(y) \ .$$

- *The* square product *$H \,\square\, G: X \rightrightarrows Z$ of H and G at x is defined by*

$$(H \,\square\, G)(x) := \bigcap_{y \in G(x)} H(y) \ .$$

We recall that there are two ways of defining the inverse image by a set-valued map G of a subset M:

$$\begin{cases} a) \ \ G^{-}(M) := \{x \,|\, G(x) \cap M \neq \emptyset\} & \text{(inverse image of M)} \\ b) \ \ G^{+}(M) := \{x \,|\, G(x) \subset M\} & \text{(cone of M)} \end{cases}$$

We deduce the following formulas

$$\begin{cases} i) \ \ \text{Graph } (F \circ G) = (G \times 1)^{-}\text{Graph } (H) = (1 \times H)\,\text{Graph}(G) \\ ii) \ \ \text{Graph } (F \,\square\, G) = (G \times 1)^{+}\text{Graph}(H) \ . \end{cases}$$

as well as the formulas which state that the inverse of a product is the product of the inverses (in reverse order):

$$\begin{cases} i) \ \ (H \circ G)^{-1}(y) = G^{-}(H^{-1}(y)) \\ ii) \ \ (H \,\square\, G)^{-1}(y) = G^{+}(H^{-1}(y)) \ . \end{cases}$$

We begin with the simple result:

THEOREM 4.7. *Let us consider a set-valued map $G{:}X \rightrightarrows Y$ and a set-valued map $H{:}Y \rightrightarrows Z$.*

Let us assume that H is lipschitzian around y where y belongs to $G(x)$. Then, for any $z \in H(y)$, we have

$$D^{b}H(y,z) \circ DG(x,y) \subset D(H \circ G)\,(x,z) \ .$$

Let us assume that G is lipschitzian around x. Then, for all $y \in G(x)$ and $z \in (H \,\square\, G)\,(x)$, we have

$$D(H \,\square\, G)\,(x,z) \subset DH(y,z) \,\square\, D^{b}G(x,y) \ .$$

In particular, if $G := g$ is single-valued, differentiable, and lipschitzian around x, we obtain

$$D(Hg)\,(x,z)\,(u) \subset DH(g(x),z)\,(g'(x)\,u) \ ,$$

and the equality holds true when H is lipschitzian around $g(x)$.

More powerful results can now be stated which can be derived from the inverse function theorem found in the next section.

THEOREM 4.8. *Let us consider a set-valued map $G{:}X \rightrightarrows Y$ and a set-valued map $H{:}Y \rightrightarrows Z$. We suppose that*

$$\mathrm{Im}\,(CG(x_0,y_0)) - \mathrm{Dom}\,(CH(y_0,z_0)) = Y \ .$$

If the dimension of Y is finite, then

$$\begin{cases} i) & D^bH(y_0,z_0) \circ DG(x_0,y_0) \subset D(H \circ G)\,(x_0,z_0) \\ ii) & D^bH(y_0,z_0) \circ D^bG(x_0,y_0) = D^b(H \circ G)\,(x_0,z_0) \\ iii) & CH(y_0,z_0) \circ CG(x_0,y_0) \subset C(H \circ G)\,(x_0,z_0) \ . \end{cases}$$

The next proposition provides chain rule formulas for square products.

PROPOSITION 4.9. *Let us consider a set-valued map G from a Banach space X to a Banach space Y and a single-valued map H from Y to a Banach space Z. Assume that G is lipschitzian around x^*. If H is differentiable around some $y^* \in G(x^*)$, then:*

- *The contingent derivative of $H \square G$ is contained in the square product of the derivative of H and the adjacent derivative of G: for all $u \in \mathrm{Dom}(D^bG(x^*,y^*))$ we have*

$$D(H \square G)\,(x^*,H(y^*))\,(u) \subset H'(y^*) \square D^bG(x^*,y^*)\,(u) \ .$$

- *If H is continuously differentiable around y^* then the paratingent derivative of $H \square G$ is contained in the square product of the derivative of H and the circatangent derivative of G : $\forall u \in \mathrm{Dom}\,(CG(x^*,y^*))$ we have*

$$P(H \square G)\,(x^*,H(y^*))\,(u) \subset H'(y^*) \square CG(x^*,y^*)\,(u) \ .$$

We can extend this theorem to the case where P is set-valued. For this purpose we have to define the lop-sided paratingent derivatives $P_xF(x,y)$ and $P_yF(x,y)$ in the following way:

$$\mathrm{Graph}\,(P_1F(x,y)) := P^{\mathrm{Dom}\,(F)}_{\mathrm{Graph}\,(F)}\,(x,y) \ \&\ \mathrm{Graph}\,(P_2F(x,y))$$

$$:= P^{\mathrm{Im}\,(F)}_{\mathrm{Graph}\,(F)}\,(x,y) \ .$$

THEOREM 4.10. *Assume that G is lipschitzian around x. Then*

- *Y is a finite dimensional vector-space and $G(x)$ is bounded, then*

$$D(H \circ G)\,(x,z) \subset \bigcup_{y \in G(x)} P_1H(y,z) \circ P_2G(x,y) \ .$$

- *and*

$$P(H \,\square\, G)(x,z) \subset \bigcap_{y \in G(x)} PH(y,z) \,\square\, CG(x,y) \quad .$$

Proofs of the above results can be found in Aubin and Frankowska (1989).

5. Variational Inclusions

We now provide estimates of the contingent, adjacent and circatangent derivatives of the solution map S associated to the differential inclusion

$$x'(t) \in F(t, x(t)) \quad . \tag{5.1}$$

We shall express these estimates in terms of the solution maps of suitable linearizations of the differential inclusion (5.1) of the form

$$w'(t) \in F'(t, x(t), x'(t))(w(t)) \quad ,$$

where, for almost all t, $F'(t,x,y)(u)$ denotes one of the (contingent, adjacent or circatangent) derivatives of the set-valued map $F(t,\cdot,\cdot)$ at a point (x,y) of its graph (in this section the set-valued map F is regarded as a family of set-valued maps $x \rightrightarrows F(t,x)$ and *the derivatives are taken with respect to the state variable only*).

These linearized differential inclusions can be called the *variational inclusions*, since they extend, in various ways, the classical variational equations of ordinary differential equations.

Let $\bar{x}$ be a solution of the differential inclusion (5.1). We assume that F satisfies the following assumptions:

$$\left\{ \begin{array}{ll} i) & \forall x \in X \;,\; \text{the set-valued map } F(\cdot, x) \text{ is measurable} \\ ii) & \forall t \in [0,T] \;,\; \forall x \in X \;,\; F(t,x) \text{ is a closed set} \\ iii) & \exists \beta > 0 \;,\; k(\cdot) \in L^1(0,T) \text{ such that for almost all } t \in (0,T) \\ & \text{the map } F(t,\cdot) \text{ is } k(t) - \text{Lipschitz on } \bar{x}(t) + \beta B \quad . \end{array} \right. \tag{5.2}$$

Consider the *adjacent variational inclusion*, which is *linearized* along the trajectory $\bar{x}$ inclusion

$$\left\{ \begin{array}{l} w'(t) \in D^b F(t, \bar{x}(t), \bar{x}'(t))(w(t)) \text{ a.e. in } [0,T] \\ w(0) = u \quad . \end{array} \right. \tag{5.3}$$

where $u \in X$. In Theorems (5.1) and (5.2) below we consider the solution map S as the set-valued map from $\mathbf{R}^n$ to the Sobolev space $W^{1,1}(0, T; \mathbf{R}^n)$. First we provide a short proof of a result from Frankowska (1987b).

THEOREM 5.1. (Adjacent variational inclusion) *If the assumptions (5.2) hold true then for all $u \in X$, every solution $w \in W^{1,1}(0, T; X)$ to the linearized inclusion (5.3) satisfies $w \in D^b S(\bar{x}(0), \bar{x})(u)$. In other words,*

$$\{w(\cdot) \mid w'(t) \in D^b F(t, \bar{x}(t), \bar{x}'(t))(w(t)), w(0) = u\} \subset D^b S(\bar{x}(0), \bar{x})(u) \ .$$

Proof. Filippov's theorem [see, for example, Aubin and Cellina (1984), Theorem 2.4.1, p. 120] implies that the map $u \to S(u)$ is lipschitzian on a neighborhood of $\bar{x}(0)$. Let $h_n > 0$, $n = 1,2,\ldots$ be a sequence converging to 0. Then, by the very definition of the adjacent derivative, for almost all $t \in [0, T]$,

$$\lim_{n \to \infty} d\left[w'(t), \frac{F(t, \bar{x}(t) + h_n w(t)) - \bar{x}'(t)}{h_n}\right] = 0 \ . \tag{5.4}$$

Moreover, since $\bar{x}'(t) \in F(t, \bar{x}(t))$ a.e. in $[0, T]$, using (5.2), for all sufficiently large n and almost all $t \in [0, T]$.

$$d(\bar{x}'(t) + h_n w'(t), F(t, \bar{x}(t) + h_n w(t))) \le h_n(\|w'(t)\| + k(t)\|w(t)\|) \ .$$

Thus, (5.4) and the Lebesgue dominated convergence theorem yield

$$\int_0^T d(\bar{x}'(t) + h_n w'(t), F(t, \bar{x}(t) + h_n w(t)))\, dt = o(h_n) \ . \tag{5.5}$$

where $\lim_{n \to \infty} o(h_n)/h_n = 0$. According to the Filippov Theorem and (5.5), there exist $M \ge 0$ and solutions $y_n \in S(\bar{x}(0) + h_n u)$ satisfying

$$\|y_n' - \bar{x}' - h_n w'\|_{L^1(0,T;X)} \le M o(h_n) \ .$$

Since $(y_n(0) - \bar{x}(0))/h_n = u = w(0)$ this implies that

$$\lim_{n \to \infty} \frac{y_n - \bar{x}}{h_n} = w \text{ in } C(0, T; x) \ ; \ \lim_{n \to \infty} \frac{y' - \bar{x}'}{h_n} = w' \text{ in } L^{1(0,} T; X) \ .$$

Hence

$$\lim_{n \to \infty} d\left[w, \frac{S[\bar{x}(0) + h_n u] - \bar{x}}{h_n}\right] = 0 \ .$$

Since u and w are arbitrary the proof is complete.

Consider next the *circatangent variational inclusion*, which is the linearization involving circatangent derivatives:

$$\begin{cases} w'(t) \in CF(t, \bar{x}(t), \bar{x}'(t))\,(w(t)) \text{ a.e. in } [0, T] \\ w(0) = u \ . \end{cases} \tag{5.6}$$

where $u \in X$.

THEOREM 5.2. (Circatangent variational inclusion) *Assume that conditions (5.2) hold true. Then for all $u \in X$, every solution $w \in W^{1,1}(0, T; X)$ to the linearized inclusion (5.6) satisfies $w \in CS(\bar{x}(0), \bar{x})\,(u)$. In other words,*

$$\{w(\cdot)\,|\,w'(t) \in CF(t, \bar{x}(t), \bar{x}'(t))\,(w(t)),\ w(0) = u\} \subset CS(\bar{x}(0), \bar{x})\,(u) \ .$$

Proof. According to Filippov's theorem, the map $u \to S(u)$ is lipschitzian on a neighborhood of $\bar{x}(0)$. Consider a sequence x_n of trajectories of (5.1) converging to $\bar{x}$ at $W^{1,1}(0, T; X)$ and let $h_n \to 0+$. Then there exists a subsequence $x_j = x_{n_j}$ such that

$$\lim_{j\to\infty} x_j'(t) = x_0't) \text{ a.e. in } [0, T] \ . \tag{5.7}$$

Set $\lambda_j = h_{n_j}$. Then, by definition of the circatangent derivative and according to (5.7), for almost all $t \in [0, T]$

$$\lim_{j\to\infty} d\left[w'(t), \frac{F(t, x_j(t) + \lambda_j w(t)) - x_j'(t)}{\lambda_j} \right] = 0 \tag{5.8}$$

Moreover, using the fact that $x_j'(t) \in F(t, x_j(t))$ a.e. in $[0, T]$, we obtain, for almost all $t \in [0, T]$

$$d\Big[x_j'(t) + \lambda_j w'(t), F(t, x_j(t) + \lambda_j w(t)) \Big] \le \lambda_j(\|w'(t)\| + k(t)\|w(t)\|) \ .$$

This, (5.8), and the Lebesgue dominated convergence theorem yield

$$\int_0^T d\Big[x_j'(t) + \lambda_j w'(t), F(t, x_j(t) + \lambda_j w(t)) \Big]\, dt = o(\lambda_j) \ , \tag{5.9}$$

where $\lim_{j\to\infty} o(\lambda_j)/\lambda_j = 0$. According to the Filippov Theorem and (5.9), there exist $M \ge 0$ and solutions $y_j \in S(x_j(0) + \lambda_j u)$ satisfying

$$\|y_j' - x_j' - \lambda_j w'\|_{L^1(0,T;X)} \le Mo(h_j) \ .$$

Since $(y_j(0) - x_j(0))/\lambda_j = u = w(0)$, this implies that

$$\lim_{j\to\infty} \frac{y_j - x_j}{h_{n_j}} = w \text{ in } C(0,T;X); \ \lim_{j\to\infty} \frac{y_j' - x_j'}{h_{n_j}} = w' \text{ in } L^1(0,T;X) \ .$$

Hence

$$\lim_{j\to\infty} d\left[w, \frac{S(x_j(0) + h_{n_j}u) - x_j}{h_{n_j}} \right] = 0 \ . \tag{5.10}$$

Therefore we have proved that for every sequence of solutions x_n to (5.1) converging to $\bar{x}$ and every sequence $h_n \to 0+$, there exists a subsequence $x_j = x_{n_j}$ which satisfies (5.9). This yields, for every sequence of solutions x_n converging at $\bar{x}$ and $h_n \to 0+$

$$\lim_{n\to\infty} d\left[w, \frac{S(x_n(0) + h_n u) - x_n}{h_n} \right] = 0 \ .$$

Since u and w are arbitrary the proof is complete.

We consider now the contingent variational inclusion

$$\begin{cases} w'(t) \in \overline{co}DF(t,\bar{x}(t),\bar{x}'(t))(w(t)) \text{ a.e. in } [0,T] \\ w(0) = u \ . \end{cases} \tag{5.11}$$

THEOREM 5.3. (Contingent variational inclusion) *Let us consider the solution map S as a set-valued map from $\mathbf{R}^n$ to $W^{1,\infty}(0,T;\mathbf{R}^n)$ supplied with the weak-$*$ topology and let $\bar{x}(\cdot)$ be a solution of the differential inclusion (5.11) starting at x_0. Then the contingent derivative $DS(x_0,\bar{x}(\cdot))$ of the solution map is contained in the solution map of the contingent variational inclusion (5.11), in the sense that*

$$DS(x_0,\bar{x}(\cdot))(u) \subset \{w(\cdot) \mid w'(t) \in \overline{co}DF(t,\bar{x}(t),\bar{x}'(t))(w(t)), w(0) = u\} \tag{5.12}$$

Proof. Fix a direction $u \in \mathbf{R}^n$ and let $w(\cdot)$ belong to $DS(x_0,\bar{x}(\cdot))(u)$. By definition of the contingent derivative, there exist sequences of elements $h_n \to 0+$, $u_n \to u$ and $w_n(\cdot) \to w(\cdot)$ in the weak-$*$ topology of $W^{1,\infty}(0,T;\mathbf{R}^n)$ and $c > 0$ satisfying

$$\begin{cases} i) \ \ \|w_n'(t)\| \le c \text{ a.e. in } [0,T] \\ ii) \ \ \bar{x}'(t) + h_n w_n'(t) \in F(t,\bar{x}(t) + h_n w_n(t)) \text{ a.e. in } [0,T] \\ iii) \ \ w_n(0) = u_n \ . \end{cases} \tag{5.13}$$

Hence

$$\begin{cases} i) & w_n(\cdot) \text{ converges pointwise to } w(\cdot) \\ ii) & w_n'(\cdot) \text{ converges weakly in } L^1(0,T;\mathbf{R}^n) \text{ to } w'(\cdot) \ . \end{cases} \tag{5.14}$$

By using Mazur's Theorem and (5.14) ii), a sequence of convex combinations

$$v_m(t) := \sum_{p=m}^{\infty} a_m^p w_p'(t)$$

converges strongly to $w'(\cdot)$ in $L^1(0,T;X)$. Therefore a subsequence (again denoted) $v_m(\cdot)$ converges to $w'(\cdot)$ almost everywhere. According to (5.13) i) and ii), for all p, and almost all $t \in [0,T]$

$$w_p'(t) \in \left[\frac{1}{h_p} F(t, \bar{x}(t) + h_p w_p(t)) - \bar{x}'(t) \right] \cap cB \ .$$

Let $t \in [0,T]$ be a point where $v_m(t)$ converges to $w'(t)$ and $x'(t) \in F(t,x(t))$. Fix an integer $n \geq 1$ and $\epsilon > 0$. Based on (5.14) i), there exists m such that $h_p \leq 1/n$ and $\|w_p(t) - w(t)\| \leq 1/n$ for all $p \geq m$.

Then, by setting

$$\Phi(y,h) := \frac{1}{h} \left[F(t, \bar{x}(t) + hy) - \bar{x}'(t) \right] \cap cB \ .$$

we obtain

$$v_m(t) \in K_n := \overline{co} \left(\bigcup_{h \in]0,\frac{1}{n}], y \in w(t) + \frac{1}{n}B} \Phi(y,h) \right) ,$$

and therefore, by letting m go to ∞,

$$w'(t) \in \overline{co} \left(\bigcup_{h \in]0,\alpha], y \in w(t) + \frac{1}{n}B} \Phi(y,h) \right) .$$

Since this is true for any n, we deduce that $w'(t)$ belongs to the convex upper limit:

$$w'(t) \in \bigcap_{n \geq 1} \overline{co} \left(\bigcup_{h \in]0,\frac{1}{n}], y \in w(t) + \frac{1}{n}B} \Phi(y,h) \right) .$$

Since the subsets $\Phi(y,h)$ are contained in the ball of radius c, we infer that $w'(t)$ belongs to the closed convex hull of the Kuratowski upper limit:

$$w'(t) \in \overline{co} \bigcap_{\epsilon>0, n\geq 1} \left[\bigcup_{h\in]0,\frac{1}{n}[, y\in w(t)+\frac{1}{n}B} \Phi(y,h) + \epsilon B \right] .$$

We observe now that

$$\bigcap_{\epsilon>0, n\geq 1} \left[\bigcup_{h\in]0,\frac{1}{n}[, y\in w(t)+\frac{1}{n}B} \Phi(y,h) + \epsilon B \right] \subset DF(t,\bar{x}(t),\bar{x}'(t))(w(t))$$

to conclude that $w(\cdot)$ is a solution to the differential inclusion

$$\begin{cases} w'(t) \in \overline{co}\ DF(t,\bar{x}(t),\bar{x}'(t))(w(t)) \text{ a.e. in } [0,T] \\ w(0) = u \ . \end{cases}$$

Since $w \in DS(x_0, \bar{x}(\cdot))(u)$ is arbitrary, we have proved (5.12).

6. Local Injectivity and Surjectivity of Set-Valued Maps

Let $\mathcal{F}$ be a set-valued map from a Banach space X to a Banach space Y. We study its local invertibility (injectivity and surjectivity) at point (x^*, y^*) of its graph. We shall derive local injectivity of a set-valued map $\mathcal{F}: X \rightrightarrows Y$ from a general principle based on the differential calculus of set-valued maps. For that purpose, we use its contingent and paratingent derivatives $D\mathcal{F}(x^*,y^*)$ and $P\mathcal{F}(x^*,y^*)$, which are closed processes from X to Y.

Since 0 member $D\mathcal{F}(x^*,y^*)(0)$, we observe that the *linearized system* $D\mathcal{F}(x^*,y^*)$ enjoys inverse univocity, which means that the inverse image $D\mathcal{F}(x^*,y^*)^{-1}(0)$ contains only one element, i.e., that its kernel is naturally defined by

$$\ker D\mathcal{F}(x^*,y^*) := D\mathcal{F}(x^*,y^*)^{-1}(0) \ ,$$

and is reduced to zero.

THEOREM 6.1. *Let $\mathcal{F}$ be a set-valued map from a finite dimensional vector-space X to a Banach space Y and (x^*,y^*) belong to its graph.*

- *If the kernel of the contingent derivatives $D\mathcal{F}(x^*,y^*)$ of $\mathcal{F}$ at (x^*,y^*) is equal to $\{0\}$, then there exists a neighborhood $N(x^*)$ such that*

$$\{x \text{ such that } y^* \in \mathcal{F}(x)\} \bigcap N(x^*) = \{x^*\} \ .$$

- *Let us assume that there exists $\gamma > 0$ such that $\mathcal{F}(x^* + \gamma B)$ is relatively compact and that $\mathcal{F}$ has a closed graph. If for all y member $\mathcal{F}(x^*)$ the kernels of the paratingent derivatives $P\mathcal{F}(x^*, y)$ of $\mathcal{F}$ at (x^*, y) are equal to $\{0\}$, then $\mathcal{F}$ is locally injective around x^*.*

Proof. We provide the proof for the second statement only. Proof of the first statement can be found in Aubin and Frankowska (1987a).

Assume that $\mathcal{F}$ is not locally injective. Then there exists a sequence of elements $x_n^1, x_n^2 \in N(x^*)$, $x_n^1 \neq x_n^2$, converging to x^* and y_n satisfying

$$\forall n \geq 0 \ , \quad y_n \in \mathcal{F}(x_n^1) \bigcap \mathcal{F}(x_n^2) \ .$$

Let us set $h_n := \|x_n^1 - x_n^2\|$, which converges to 0, and $u_n := (x_n^1 - x_n^2)/h_n$. The elements u_n belong to the unit sphere, which is compact. Hence a subsequence again denoted by u_n does converge to some u different from 0. Then for all large n

$$y_n \in \mathcal{F}(x_n^1) \cap \mathcal{F}(x_n^2) := \mathcal{F}(x_n^2 + h_n u_n) \cap \mathcal{F}(x_n^2) \subset \mathcal{F}(x^* + \gamma B)$$

so that we deduce that a subsequence again denoted by y_n converges to some $y \in \mathcal{F}(x^*)$ (because Graph $(\mathcal{F})$ is closed). Since the above equation implies that

$$\forall n \geq 0 \ , \quad y_n + h_n 0 \in \mathcal{F}(x_n^2 + h_n u_n) \ ,$$

we deduce that

$$0 \in P\mathcal{F}(x^*, y)\,(u) \ .$$

Hence we have proved the existence of a nonzero element of the kernel of $P\mathcal{F}(x^*, y)$ which is a contradiction.

For local surjectivity, we shall obtain some regularity property of $\mathcal{F}^{-1}$ around $y^* \in \mathcal{F}(x^*)$. For that purpose we need the following

DEFINITION 6.2. *A set-valued map G from Y to Z is* pseudo-Lipschitz *around* $(y^*, z^*) \in$ Graph (G) *if there exist neighborhoods V of y^* and W of z^* and a constant l such that*

$$\begin{cases} i) \ \forall y \in V \ , \ G(y) \neq \emptyset \\ ii) \ \forall y_1, y_2 \in V \ , \ G(y_1) \bigcap W \subset G(y_2) + l\|y_1 - y_2\| B \ . \end{cases}$$

THEOREM 6.3. *Let $\mathcal{F}$ be a set-valued map from a Banach space X to a finite dimensional space Y and (x^*, y^*) belong to the graph of $\mathcal{F}$. If the circatangent derivative $C\mathcal{F}(x^*, y^*)$ is surjective, then $\mathcal{F}^{-1}$ is pseudo-Lipschitz around $(y^*, x^*) \in$* Graph $(\mathcal{F}^{-1})$.

See Aubin and Frankowska (1987a) for the proof of the above result. As a corollary we obtain the following inverse function theorem for single-valued maps under certain constraints.

COROLLARY 6.4. *Let X be a Banach space, Y be a finite dimensional space, $K \subset X$ be a closed subset of X and x_0 belong to K. Let A be a differentiable map from a neighborhood of K to Y. We assume that A' is continuous at x_0 and that*

$$A'(x_0)\, C_K(x_0) = Y \ .$$

Then $A(x_0)$ belongs to the interior of $A(K)$ and there exist constants ρ and l such that,

$$\begin{cases} \text{for all } y_1, y_2 \in A(x_0) + \rho B \text{ and any solution } x_1 \in K \text{ to the equation } A(x_1) = y_1 \\ \text{satisfying } \|x_0 - x_1\| \leq l\rho, \text{there exists a solution } x_2 \in K \text{ to the equation } A(x_2) = y_2 \\ \text{satisfying } \|x_1 - x_2\| \leq l\|y_1 - y_2\| \ . \end{cases}$$

For further extensions on inverse function theorems for maps from a complete metric space to a Banach space and higher order results, see Frankowska (1986a; 1987c; 1987e; 1989d; and forthcoming).

7. Local Observability of Differential Inclusions

Let us consider a set-valued input–output system of the following form built through a differential inclusion

$$\text{for almost all } t \in [0, T] \quad , \quad x'(t) \in F(t, x(t)) \quad , \tag{7.1}$$

whose dynamics are described by a set-valued map F from $[0, T] \times X$ to X, where X is a finite dimensional vector-space (the *state space*) and $0 < T \leq \infty$. It governs the (uncertain) evolution of the state $x(\cdot)$ of the system. The *inputs* are the *initial states* x_0 and the *outputs* are the *observations* $y(\cdot) \in H(x(\cdot))$ of the evolution of the state of the system through a single-valued (or set-valued) map H from X to an *observation space* Y.

Let $S := S_F$ from X to $C(0, T; X)$ denote the solution map associating with every initial state $x_0 \in X$ the (possibly empty) set $S(x_0)$ of solutions to the differential inclusion (7.1) starting at x_0 at the initial time $t = 0$.

In other words, we have introduced an Input–Output system where the

- *Inputs*, are the *initial states* x_0.
- *Outputs*, are the *observations* $y(\cdot) \in H(x(\cdot))$ of the evolution of the state of the system through H

$$\begin{array}{ccccc}
\textbf{Inputs} & \overset{S}{\rightrightarrows} & \textbf{States} & \overset{H}{\rightrightarrows} & \textbf{Outputs} \\
\downarrow & & \downarrow & & \downarrow \\
X \ni x_0 & \rightrightarrows & x(\cdot) \in S(x_0) & \rightrightarrows & y(\cdot) \in H(x(\cdot)) \\
\uparrow & & \uparrow & & \uparrow \\
\text{Initial States} & & \begin{cases} x'(t) \in F(t, x(t)) \\ x(0) = x_0 \end{cases} & & \text{Observations}
\end{array} \; .$$

It remains to define an Input–Output map. But, because of the set-valued character (the presence of uncertainty), one can conceive two dual ways of defining composition products of the set-valued maps S from X to the space $C(0, T; X)$ and H from $C(0, T; X)$ to $C(0, T; Y)$. So, for systems under uncertainty, we have to deal with *two Input–Output maps* from X to $C(0, T; Y)$:

- The *sharp Input–Output map*, which is the (usual) product

$$\forall x_0 \in X \quad , \quad I_-(x_0) := (H \circ S)(x_0) := \bigcup_{x(\cdot) \in S(x_0)} H(x(\cdot)) \quad .$$

- The *hazy Input–Output map*, which is the *square product*

$$\forall x_0 \in X \quad , \quad I_+(x_0) := (H \,\square\, S)(x_0) := \bigcap_{x(\cdot) \in S(x_0)} H(x(\cdot)) \quad .$$

The *sharp* Input–Output map tracks *at least* the evolution of a state starting at some initial state x_0, whereas the *hazy* Input–Output map tracks *all* such solutions.

Opinions may differ as to which would be the *right* Input–Output map, since it depends upon the context in which a given problem is stated. Therefore we will study the observability properties of *both* the sharp and hazy Input–Output maps.

When the observation map is single-valued, the use of a nontrivial hazy Input–Output map requires that all solutions $x(\cdot) \in S(x_0)$ yield the same observation $y(\cdot) = H(x(\cdot))$. Hence we have to ascertain when this possibility occurs, by projecting the differential inclusion (7.1) onto a differential equation which *tracks* all the solutions to the differential inclusion.

We shall tackle this issue by *projecting* the differential inclusion given in the state space X, onto a differential inclusion in the observation space Y, in such a way that solutions to the projected differential inclusion are observations of solutions of the original differential inclusion.

We project the differential inclusion (7.1) to a differential inclusion (or a differential equation) on the observation space Y described by a set-valued map G (or a single-valued map g):

$$y'(t) \in G(t, y(t)) \text{ (or } y'(t) = g(t, y(t))) \quad , \quad y(0) = y_0 \tag{7.2}$$

which allows us to partially or completely track solutions $x(\cdot)$ to the differential inclusion (7.2) in the following sense:

$$\begin{cases} a) & \forall (x_0, y_0) \in \text{Graph } (H), \text{ there exist solutions } x(\cdot) \text{ and } y(\cdot) \\ & \text{to (7.1) and (7.2) such that } \forall t \in [0,T],\ y(t) \in H(x(t)) \\ b) & \forall (x_0, y_0) \in \text{Graph } (H), \text{ all solutions } x(\cdot) \text{ and } y(\cdot) \\ & \text{to (7.1) and (7.2) satisfy } \forall t \in [0,T],\ y(t) \in H(x(t)) \end{cases} \tag{7.3}$$

The second property means that the differential inclusion (7.2) is *blind* to the solutions to the differential inclusion (7.1). When it is satisfied, we see that for all $x_0 \in H^{-1}(y_0)$, all the solutions to the differential inclusion (7.1) do satisfy

$$\forall t \in [0,T] \quad , \quad y(t) \in H(x(t)) \quad .$$

In the next Proposition, we denote by $DH(x,y)$ the contingent derivative of H at (x,y).

PROPOSITION 7.1. *Let us consider a closed set-valued map H from X to Y.*

(1) *Let us assume that F and G are nontrivial upper semicontinuous set-valued maps with nonempty compact convex images and with linear growth. We assume:*

$$\forall (x,y) \in \text{Graph } (H) \quad , \quad G(t,y) \bigcap (DH(x,y) \circ F)(t,x) \neq \emptyset \quad . \tag{7.4}$$

Then property (7.3) a) holds true.

(2) *Let us assume that $F \times G$ is lipschitzian on a neighborhood of the graph of H and has a linear growth. We assume:*

$$\forall (x,y) \in \text{Graph } (H) \quad , \quad G(t,y) \subset (DH(x,y) \,\square\, F)(t,x) \quad . \tag{7.5}$$

Then property (7.3) b) is satisfied.
[See Aubin and Frankowska (1989) for the proof.]

In particular, we have obtained a sufficient condition for the hazy Input–Output set-valued map I_+ to be nontrivial.

First, it will be convenient to introduce the following definition:

DEFINITION 7.2. *Let us consider $F:[0,T] \times X \rightrightarrows X$ and $H:[0,T] \times X \rightrightarrows Y$. We say that a set-valued map $G:[0,T] \times Y \rightrightarrows Y$ is a* lipschitzian square projection *of a set-valued map $F:[0,T] \times X \rightrightarrows X$ by H if and only if*

$$\begin{cases} i) & F \times G \text{ is lipschitzian around } [0,T] \times \text{Graph } (H) \\ ii) & \forall (x,y) \in \text{Graph } (H) \ , \ G(t,y) \subset (DH(x,y) \,\square\, F)(t,x) \quad . \end{cases}$$

Therefore, now that we are able to use nontrivial hazy Input–Output maps, we shall use the following results from Proposition 7.1:

PROPOSITION 7.3. *Let us assume that* $F:[0,T] \times X \rightrightarrows X$ *and* $H:X \rightrightarrows Y$ *are given. If a lipschitzian square projection of* F *by* H *exists, then the hazy Input–Output map* $I_+ := H \square S$ *has nonempty values for any initial value* $y_0 \in H(x_0)$.

We observe that when the set-valued maps F and G are time-independent, Proposition 7.1 can be reformulated in terms of commutativity of schemes for square products.

Let Φ denote the solution map associating with any y_0, a solution to the differential inclusion (equation) (7.2) starting at y_0 (when G is single-valued, such a solution is unique). Then we can deduce that property (7.3) b) is equivalent to

$$\forall y_0 \in \mathrm{Im}(H) \ , \quad \Phi(y_0) \subset ((H \square S) \square H^{-1})(y_0) \ .$$

Condition (7.5) becomes: for all $y \in \mathrm{Im}(H)$,

$$G(y) \subset \bigcap_{x \in H^{-1}(y)} \ \bigcap_{v \in F(x)} DH(x,y)(v) := (DH(x,y) \square F) \square H^{-1}(y) \ .$$

In other words, the second part of Proposition 7.1 implies that if the scheme

$$\begin{array}{ccc} & & \\ & X \overset{F}{\rightrightarrows} X & \\ H^{-1} & \uparrow \qquad \downarrow & DH(x,y) \\ & Y \overset{G}{\rightrightarrows} Y & \end{array}$$

is *commutative for the square products*, then the derived scheme

$$\begin{array}{ccc} X & \overset{S}{\rightrightarrows} & C(0,T;X) \\ H \downarrow\uparrow H^{-1} & & \downarrow H \\ Y & \overset{\Phi}{\rightrightarrows} & C(0,T;Y) \end{array}$$

is also commutative for the square products. Using these definitions we are able to adapt some of the observability concepts to the set-valued case.

DEFINITION 7.4. *Assume that the sharp and hazy Input–Output maps are defined on nonempty open subsets. Let* $y^* \in H(S(x_0))$ *be an observation associated with an initial state* x_0.

We say that the system is sharply observable *at* (*respectively* locally sharply observable at) x_0 *if and only if the sharp Input–Output map* I_- *enjoys the global inverse univocity* (*respectively* local). Hazily observable *and locally hazily observable systems are defined in the same way when the sharp Input–Output map is replaced by the hazy Input–Output map* I_+.

The system is said to be hazily (locally) observable around (x_0, y^*) *if the hazy Input–Output map* I_+ *is (locally) injective.*

Remark 7.5. Several observations are in order. We observe that the system is sharply locally observable *at* x_0, if and only if, there exists a neighborhood $N(x_0)$ of x_0 such that

$$\text{if } x(\cdot) \in S(N(x_0)) \text{ is such that } y^*(\cdot) \in H(x(\cdot)) \quad , \quad \text{then } x(0) = x_0 \quad ,$$

i.e., sharp observability means that an observation $y^*(\cdot)$ which characterizes the input x_0.

The system is hazily locally observable *at* x_0 if and only if there exists a neighborhood $N(x_0)$ of x_0 such that, for every $x_1 \in N(x_0^*)$,

$$\text{if } \forall x(\cdot) \in S(x_1) \quad , \quad y^*(\cdot) \in H(x(\cdot)) \quad , \quad \text{then } x_1 = x_0 \quad .$$

It is also clear that sharp local (respectively global) observability implies hazy local (respectively global) observability.

If we consider two systems $\mathcal{F}_1$ and $\mathcal{F}_2$ such that

$$\forall x \in X \quad , \quad \mathcal{F}_1(x) \subset \mathcal{F}_2(x) \quad ,$$

then:

- If $\mathcal{F}_2$ is sharply locally (respectively globally) observable, so is $\mathcal{F}_1$.
- If $\mathcal{F}_1$ is hazily locally (respectively globally) observable, so is $\mathcal{F}_2$.

In this section we piece together the general principle of local inverse univocity and local injectivity (Theorem 6.1), the chain rule formulas and the estimates of the derivatives of the solution map in terms of solution maps of the variational equations (Theorems 5.1, 5.2 and 5.3) to prove the statements on local hazy and sharp observability we have made.

We assume from now on, that H is differentiable and F has a linear growth. We also impose some regularity assumptions on the set-valued map F. In the next theorem it is assumed that F is derivable in the sense that its contingent and adjacent derivatives do coincide.

THEOREM 7.6. *Let us assume that F is derivable, satisfies assumptions (5.2), and that it has a lipschitzian square projection G by H. Let $\bar{x}(\cdot) \in S(x_0)$. If the contingent variational inclusion*

$$\text{for almost all } t \in [0, T] \quad , \quad w'(t) \in DF(t, \bar{x}(t), \bar{x}'(t))(w(t)) \tag{7.6}$$

is globally hazily observable through $H'(\bar{x}(\cdot))$ at 0, then the system (7.1) is locally hazily observable through H at x_0.

In the next theorem we assume that F is sleek, so that its contingent and circatangent derivatives do coincide.

THEOREM 7.7. *Let us assume that F is sleek, has convex images, satisfies assumptions (5.2), and that it has a lipschitzian square projection G by H. If for all $\bar{x}(\cdot) \in S(x_0)$ the contingent variational inclusion (7.6) is globally hazily observable through $H'(\bar{x}(\cdot))$ at 0, then the system (7.1) is hazily observable through H around x_0.*

We consider now the sharp Input–Output map.

THEOREM 7.8. *Let us assume that the graphs of the set-valued maps $F(t, \cdot): X \rightrightarrows X$ are closed and convex. Let H be a linear operator from X to another finite dimensional vector-space Y. Let $\bar{x}(\cdot)$ be a solution to the differential inclusion (7.1). If the contingent variational inclusion (7.6) is globally sharply observable through H around 0, then the system (7.1) is globally sharply observable through H around x_0.*

Whenever we know that the chain rule holds true, we can state the following proposition: a consequence of the general principle (Theorem 6.1) and of Theorem 5.3 on the estimate of the contingent derivative of the solution map.

PROPOSITION 7.9. *Let us assume that the solution map of the differential inclusion (7.1) and the differentiable observation map H do satisfy the chain rule*

$$DI_{-}(x_0, y_0)(u) = (H'(\bar{x}) \circ S(x_0, \bar{x}(\cdot)))(u)$$

If the contingent variational inclusion

$$\text{for almost all } t \in [0, T] \quad , \quad w'(t) \in \overline{co}DF(t, \bar{x}(t), \bar{x}'(t))(w(t))$$

is globally sharply observable through $H'(\bar{x}(\cdot))$ around 0, then the system (7.1) is locally sharply observable through H around x_0.

However, we can bypass the chain rule formula and attempt to obtain directly other criteria of local sharp observability in the nonconvex case.

THEOREM 7.10. *Assume that F has closed convex images, is continuous, Lipschitz in the second variable with a constant independent of t, and that the growth of F is linear with respect to the state. Let H be a twice continuously* *differentiable function from X to another finite dimensional vector-space Y. Consider an observation $y^* \in I_-(x_0)$ and assume that for every solution $\bar{x}(\cdot)$ to the differential inclusion (7.1) satisfying $y^*(\cdot) = H(\bar{x}(\cdot))$ and for all $t \in [0, T]$ we have*

$$\ker H'(\bar{x}(t)) \subset (F(t, \bar{x}(t)) - F(t, \bar{x}(t)))^{\perp} \ .$$

If for all $\bar{x}$ as above, the contingent variational inclusion

$$\text{for almost all } t \in [0, T] \quad , \quad w'(t) \in \overline{co}\ DF(t, \bar{x}(t), \bar{x}'(t))\,(w(t))$$

is globally sharply observable through $H'(\bar{x}(t))$ around 0, then the system (7.1) is locally sharply observable through H at (x_0, y^*).

Example: Observability around an Equilibrium. Let us consider the case of a time-independent system (F, H): this means that the set-valued map $F{:}X \rightrightarrows X$ and the observation map $H{:}X \rightrightarrows Y$ do not depend upon time.

We shall observe this system around an equilibrium $\bar{x}$ of F, i.e., a solution to the equation

$$0 \in F(\bar{x}) \ .$$

For simplicity, we shall assume that the set-valued map F is *sleek* at the equilibrium. Hence all the derivatives of F at $(\bar{x},0)$ do coincide with the contingent derivative $DF(\bar{x},0)$, which is a closed convex process from X to itself.

The theorems on local observability reduce the local observability around the equilibrium $\bar{x}$, to the study of the observability properties of the variational inclusion

$$w'(t) \in DF(\bar{x},0)\,(w(t)) \tag{7.7}$$

through the observation map $H'(\bar{x})$ around the solution 0 of this variational inclusion.

We mention below a characterization of sharp observability of the variational inclusion in terms of *viability domains* of the restriction of the derivative $DF(\bar{x},0)$ to the kernel of $H'(\bar{x})$.

PROPOSITION 7.11. *Let us assume that F is sleek at its equilibrium $\bar{x}$ and that H is differentiable at $\bar{x}$. Then the variational inclusion (7.7) is sharply observable at 0 if and only if the* largest closed viability domain *of the restriction to kernel $H'(\bar{x})$ of the contingent derivative $DF(\bar{x},0)$ is equal to zero.*

On the other hand, the variational inclusion is hazily observable if and only if the largest closed invariance domain of the restriction to kernel $H'(\bar{x})$ of the derivative $DF(\bar{x},0)$ is equal to zero.

Therefore we derive from the duality results of the first section, that the sharp observability of the variational inclusion at 0 is equivalent to the controllability of the adjoint system

$$-p'(t) \in DF(\bar{x},0)^*(p(t)) + H'(\bar{x})^* u(t) \quad , \quad u(t) \in Y^* \ .$$

PROPOSITION 7.12. *Using the assumptions of Proposition 7.11, we assume that* $DF(\bar{x},0)\,(0) = 0$ *and that*

$$\ker H'(\bar{x}) + \mathrm{Dom}\,(DF(\bar{x},0)) = X \ .$$

Then the concepts of sharp and hazy observability of the variational inclusion coincide, and are equivalent to the controllability of the adjoint system.

8. Applications to Local Controllability

Let us consider a bounded set-valued map F from a closed subset $K \subset \mathbf{R}^n$ to $\mathbf{R}^n$ with closed graph and convex values, satisfying

$$\forall x \in K \ , \quad F(x) \cap T_K(x) \neq \emptyset \ .$$

Using Haddad's Theorem, we know that for all $\xi \in K$, the subset $S_T(\xi)$ of viable solutions (a trajectory $t \to x(t)$ is viable if, for all $t \in [0,T]$, $x(t) \in K$) to the differential inclusion

$$x'(t) \in F(x(t)) \quad , \quad x(0) = \xi \tag{8.1}$$

is nonempty and closed in $C(0,T;\mathbf{R}^n)$ for all $\xi \in K$.

Let $R(T,\xi) := \{x(T) \,|\, x \in S_T(\xi)\}$ be the reachable set and $M \subset R^n$ the target, be a closed subset. We shall say that the system is locally controllable around M if

$$0 \in \mathrm{Int}\,(R(T,\xi) - M) \ .$$

This means that for a neighborhood U of 0 in $\mathbf{R}^n$ and for all $u \in U$, there exists a solution $x(\cdot) \in S_T(\xi)$ such that $x(T) \in M + u$. We denote by $\mathcal{K} \subset S_T(\xi)$ the subset of solutions $x \in S_T(\xi)$ such that $x(T) \in M$.

Let $z(\cdot) \in \mathcal{K}$ be such a solution. We linearize the differential inclusion (8.1) around $z(\cdot)$ using the circatangent derivative:

$$\begin{cases} w'(t) \in CF(z(t),z'(t))\,(w(t)) \\ w(0) = 0 \ . \end{cases} \tag{8.2}$$

and we denote by $R^L(T,0)$ its reachable set from zero at time T.

When ξ is an equilibrium and $z(\cdot) \equiv \xi$, the differential inclusion (8.2) becomes

$$\begin{cases} w'(t) \in CF(\xi,0)\,(w(t)) \\ w(0) = 0 \ . \end{cases}$$

where $CF(\xi,0)$ is a closed convex process. Its controllability can then be derived from Theorems 3.11 and 3.12.

THEOREM 8.1. *Using the assumptions of Theorem 5.2, if*

$$R^L(T,0) - C_M(z(T)) = \mathbf{R}^n \ , \tag{8.3}$$

[*i.e., if the linearized system is controllable around the Clarke tangent cone to M at $z(T)$*], *then the original system is locally controllable around M and there exists a neighborhood U of x_0 and a constant $l > 0$ such that, for any solution $x \in S_T(\xi)$ in U,*

$$d(x(\cdot), K) \le l d_M(x(T)) \ .$$

Proof. We apply Theorem 6.3 to the continuous linear map A from $C(0, T; \mathbf{R}^n)$ $\times$ $\mathbf{R}^n$ to $\mathbf{R}^n$ defined by $A(x,y) := x(T) - y$, to the subset $S_T(\xi) \times M$, at $(z, z(T)) \in S_T(\xi) \times M$. We observe that $A(z, z(T)) = 0$ and that condition (8.3) can be written

$$AC_{S_T(\xi)}(x_0) - C_M(x_0(T)) = \mathbf{R}^n \ .$$

Hence 0 belongs to the interior of $A(S_T(\xi) \times M) = R(T,\xi) - M$ and there exist constants $r > 0$ and $l > 0$, such that $u \to A^{-1}(u) \cap (S_T(\xi) \times M)$ is pseudo-Lipschitz around $(0, z, z(T))$. Let us now consider a ball U of center x_0 and radius r. Let us take a solution $x \in S_T(\xi) \cap U$ to the inclusion (8.1) so that $d_M(x(T)) \le \|x(T) - z(T)\| \le r$. Let y belong to $\pi_M(z(T))$. Then $\|A(x,y)\| = d_M(x(T))$ and we deduce from the fact that $u \to A^{-1}(u) \bigcap (S_T(\xi) \times M)$ is pseudo-Lipschitz that there exists $\tilde{x}$ such that $A(\tilde{x}, \tilde{x}(T)) = 0$ (i.e., an element $\tilde{x} \in K$), such that $d(x,K) \le \|x - \tilde{x}\| \le l\|0 - A(x,y)\| = l d_M(x(T))$.

Remark. When $M = (\xi)$, the considered notion of controllability around ξ coincides with the one that is often used in the literature. In this case a stronger result was proved in Frankowska (1987a). Under the assumption that ξ is an equilibrium a *larger* linerarization was considered namely

$$\begin{cases} w'(t) \in CF(\xi,0)\,w(t) + T_{coF(\xi)}(0) \\ w(0) = 0 \ . \end{cases} \tag{8.4}$$

Observe that the map $x \to \overline{CF(\xi,0)x + T_{coF(\xi)}(0)}$ is a convex process with closed images. Moreover if Dom $CF(\xi,0) = \mathbf{R}^n$ then it is also closed. Hence we may apply the results from Section 3 to study controllability of (8.4).

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