WORKING PAPER

OPTIMAL INPUTS FOR GUARANTEED IDENTIFICATION

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FOREWORD

While treating the problem of identifying a vector parameter under unknown but bounded errors this paper deals with the selection of an optimal input for the identification process. This would ensure a smallest possible diameter for the worst-case set of states consistent with the measurements and the restrictions on the unknowns.

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Optimal Inputs for Guaranteed Identification

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This paper deals with the problem of identifying a finite dimensional vector parameter on the basis of observations that are generated by an infinite dimensional input and corrupted by an unknown but bounded "noise". The specific problem solved here is one of selecting an optimal input that would ensure the smallest worst-case error for the identification procedure. This is taken as the diameter of the smallest ball that would contain the set of states consistent with the measurement and the given constraints on the unknowns. The paper continues the investigation of [1-8].

1. Assume the following notations: H stands for a Hilbert space, $\mathbf{R}^n$ for the n-dimensional Euclidean space, the respective inner products for those spaces being $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ and the norms being $\|\cdot\|$ and $|\cdot|$.

The problem under discussion is as follows. Consider a system

$$y = \sum_{i=1}^{m} z_i a_i + \zeta \tag{1}$$

where

$$y, a_i, \zeta \in H, z_i \in \mathbf{R}, (i = 1, \dots, m)$$

With y, a_i given, one is to identify the unknown vector $z = (z_1, \dots, z_m)$ under the restriction $\|\zeta\| \leq 1$.

Here y is *the available measurement*, a_i are the given *inputs*, ζ is the *unknown but bounded disturbance.* We further assume the elements a_i to be linearly independent.

Also denote H^m to be the Hilbert space of columns so that $x \in H^m$ if

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, x_i \in H,$$

If C is a matrix of dimension $k \times m$ with elements $C_{ij} \in H$, then Cx is a column with k elements

$$\left\{ \sum_{j=i}^{m} C_{ij} x_j \right\}, i = 1, \dots, k,$$

so that

$$Cx \in H^k$$

Let the asterisk indicate the transpose for a vector or a matrix. Then for $\psi_i \in \mathbf{R}, a_i \in H$ we will have $\psi^* a = \sum_{i=1}^{m} \psi_i a_i$,

The operations on matrices whose elements belong to H are performed according to the standard rules of "ordinary" matrix calculations except that the products of respective elements are taken as scalar products in H, e.g.

$$a^* a = \sum_{i=1}^{m} \langle a_i, a_i \rangle = \sum_{i=1}^{m} \|a_i\|^2$$

$$aa^* = \begin{pmatrix} \langle a_1, a_1 \rangle & ,\ldots, & \langle a_1, a_n \rangle \\ \ldots & \ldots & \ldots \\ \langle a_m, a_1 \rangle & ,\ldots, & \langle a_m, a_m \rangle \end{pmatrix}$$

Finally assume

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} \in \mathbf{R}^m$$

Formula (1) may now be rewritten as

$$y = z^* a + \zeta \tag{2}$$

2. Given y, a, let us find the set of states of system (2) consistent with the constraint $||\zeta|| \leq 1$:

$$z(y) = \{z : \exists \zeta \in H, ||\zeta|| \leq 1, y = z^* a + \zeta\}$$

From (2) it follows that

$$||\zeta||^2 = yy - 2a^* z + z^* aa^* z \leq 1$$

or, taking,

$$p = Aq, p = ay, A = aa^*$$

that

$$(z - q)^* A(z - q) \leq 1 - h^2(y), \tag{3}$$

where $h^2(y) = yy - q^* p$ and obviously $0 \leq h^2(y) \leq 1$.

Inequality (3) describes an *ellipsoid* $E(q, A)$ in $\mathbf{R}^{\mathbf{n}}$ whose matrix A and center q depends upon the measurement y. The *diameter* of this ellipsoid is defined as twice the radius $r(y)$ of the smallest ball that includes it.

According to a well-known property of the eigenvalues of a positive definite quadratic form, we have, [9]

$$r(y) = (1 - h^2(y))\lambda^{-1}(a)$$

where $\lambda(a)$ is the *smallest eigenvalue* of the form

$$x' aa^* x$$

It is clear that the diameter $d(y) = 2r(y)$ will be the largest iff $h^2(y) = 0$ which happens if and only if $y = 0$ (the "worst case" realization).

Our objective will now be to select the input a in such a way that the "worst case" diameter $d(0)$ would be as small as possible.

Hence we are to minimize $\lambda^{-1}(a)$ - the inverse of the smallest eigenvalue $\lambda(a)$ of the matrix $A = aa^*$ of the ellipsoid

$$E(0, A) = \mathcal{E}(A) = \{x : x'Ax \leq 1\}$$

(the location of the center does not matter and may be taken to be the origin).

As $A = aa^*$ is invertible, the minimization of $\lambda^{-1}(a)$ is equivalent to the maximization of $\lambda(a)$. The procedure makes sense (the solution remains in H^m) once the admissible values of a are bounded by a certain set $\mathcal{M}$.

The problem to be discussed is therefore as follows: *specify an element* $a \in \mathcal{M}$ *such that* $\lambda(a)$ *would attain its maximal value.*

Remark 2.1. The center $q = A^{-1}ay$ could be presented as by where $b = A^{-1}a$ is a vector biorthogonal to a, i.e.

$$ab^* = ba^* = A^{-1}aa^* = I_m,$$

where I_m is an m-dimensional unit matrix.

3. According to the theory of necessary conditions of optimality let us first investigate the local behaviour of $\lambda(a)$ by calculating the directional derivative

$$\lambda'(a, \bar{a}) \equiv \lim_{\gamma \downarrow 0} \frac{\lambda(a + \gamma\bar{a}) - \lambda(a)}{\lambda}, \bar{a} \in H^m$$

Due to the extremal properties of the eigenvalues of A we have

$$\lambda(a) = \min\{(\psi, A\psi) \mid \ \mid \psi \mid = 1\} =$$
$$= \min\{\langle\psi^* a, \psi^* a\rangle \mid \ \mid \psi \mid = 1\} \tag{4}$$

Denote

$$\Psi(a) = \{\psi \in \mathbf{R}^m : (\psi, A\psi) = \lambda(a), \mid \psi \mid = 1\}$$

Clearly $\Psi(a)$ is the set of normalized eigenvectors that correspond to the minimal eigenvalue $\lambda(a)$ of A.

Since

$$\frac{d}{d\gamma}\langle\psi^*(a + \gamma\bar{a}), \psi^*(a + \gamma\bar{a})\rangle \mid_{\gamma=0} =$$

$$= 2\langle\psi^*\bar{a}, \psi^* a\rangle = 2a^*\psi \cdot \psi^*\bar{a}$$

it follows from [10] that

$$\lambda'(a, \bar{a}) = \min\{2a^*\psi \cdot \psi^*\bar{a} \mid \psi \in \Psi(a)\} \tag{5}$$

Denote

$$\partial\lambda(a) = co\{2a^*\psi\psi^* : \psi \in \Psi(a)\} =$$

$$= 2a^* co\{\psi\psi^* : \psi \in \Psi(a)\} \tag{6}$$

Relations (5), (6) yield

Theorem 1. The following formula is true

$$\lambda'(a, \bar{a}) = \min\{w\bar{a} \mid w \in \partial\lambda(a)\} \tag{7}$$

Let us discuss the latter relation in more detail.

According to the terminology of convex analysis [11, 12] the set $\partial\lambda(a)$ is defined as the ***subdifferential*** (of function $\lambda(a)$ at point a) and its elements as the respective ***subgradients.*** The finite dimensionality of $\partial\lambda(a)$ also implies that $\partial\lambda(a)$ is a convex compact set.

Following [11, 12] it is possible to indicate that if an $m \times m$-dimensional matrix

$$\Gamma \in co\{\psi\psi^* : \psi \in \Psi(a)\}$$

then there exists an integer $k \leq m^2 + 1$ such that

$$\Gamma = \sum_{j=1}^{k} \gamma_j \psi_j \psi_j^*, \psi_j \in \Psi(a); j = 1, \ldots, k, \tag{8}$$

$$\sum_{j=1}^{k} \gamma_j = 1; \gamma_j \geq 0, j = 1, \ldots, k$$

Therefore all the elements of $\partial\lambda(a)$ turn to have the form $2a^*\Gamma$ where Γ is given through the relation (8).

4. Let us now proceed with the necessary conditions of optimality for the basic problem which is to maximize $\lambda(a)$ under the restriction $a \in \mathcal{M}$. For doing this we will need the notion of *tangent cone*, [12].

Recall that a *tangent cone* $K(a)$ to set $\mathcal{M}$ at point a is a convex cone such that $\bar{a} \in K_{\mathcal{M}}(a)$ yields the existence of a function $\psi(\sigma) : [0, 1] \to H^m$ that ensures for a sufficiently small $\varepsilon > 0$ the inclusion

$$a + \sigma\bar{a} + \psi(\sigma) \in \mathcal{M}; \sigma < \varepsilon;$$

and

$$\lim_{\sigma \to 0} \psi(\sigma)\sigma^{-1} = 0$$

With M convex

$$K_{\mathcal{M}}(a) = \{\bar{a} \in H^m : \bar{a} = \gamma(w - a), \gamma > 0, w \in \mathcal{M}\}$$

Denote $K^*_{\mathcal{M}}(a)$ to be the *adjoint cone* for $K_{\mathcal{M}}(a)$ so that

$$K^*_{\mathcal{M}}(a) = \{w^* \in H^{m^*} : w^*\bar{a} \geq 0, \forall \bar{a} \in K_{\mathcal{M}}(a)\},$$

$$w^* = (w_1, \ldots, w_m), w_i \in H,$$

Theorem 2. Once the element a delivers the maximum of function $\lambda(a)$ on the set $\mathcal{M}$ there exists an array of values $\gamma_j > 0, j \leq k \leq m^2 + 1$ and normalized eigenvectors ψ_j of the matrix $A = aa^* :|\psi_j| = 1, A\psi_j = \lambda(a)\psi_j$, that

$$-a^*\Gamma \in K^*_{\mathcal{M}}(a) \quad , \quad \Gamma = \sum_{j=1}^{k} \gamma_j \psi_j \psi_j^* \tag{9}$$

Proof. According to the theory of necessary conditions of optimality at point a, one must have, [10, 12]

$$(-\partial\lambda(a)) \bigcap K^*_{\mathcal{M}}(a) \neq \emptyset.$$

But the elements of $\partial\lambda(a)$ are of the form $2a^*\Gamma$, the structure of Γ being defined by (8). As $K^*_{\mathcal{M}}(a)$ is a cone, its elements could be multiplied by any positive constant with the resulting element still in $K^*_{\mathcal{M}}(a)$. The multiplier 2 and the normalizing relation for the sum of γ_j's being equal to unity may therefore be substituted by the requirement that $\gamma_j \geq 0$ for all $j = 1, \ldots, k$.

Consider some specific properties of the matrix

$$\Gamma = \sum_{j=1}^{k} \gamma_j \psi_j \psi_j^*$$

that may facilitate the further analysis:

(a) The matrix Γ is symmetric and positive definite. Indeed, once $W \in \mathbf{R}^m$ we have

$$(W, \Gamma W) = \sum_{j=1}^{k} \gamma_j W^* \psi_j \psi_j^* W = \sum_{j=1}^{k} \gamma_j (W^* \psi_j)^2 \geq 0,$$

(b) For each column $\Gamma_i, i = 1, \ldots, m$, of the matrix $\Gamma(\Gamma = \Gamma_1, \ldots, \Gamma_m)$ we have $A\Gamma_1 = \lambda(a)\Gamma_1$. By direct calculation

$$A\Gamma = \sum_{j=1}^{k} \gamma_j A\psi_j \psi_j^* = \lambda(a) \sum_{j=1}^{k} \gamma_j \psi_j \psi_j^* = \lambda(a)\Gamma$$

and further, due to the rules of matrix multiplication
$A\Gamma = (A\Gamma_1, \ldots, A\Gamma_m) = \lambda(a)(\Gamma_1, \ldots, \Gamma_m)$
which proves the assertion.

(c) If there exists an eigenvector ψ such that

$$A\psi = \lambda\psi, \lambda > \lambda(a),$$

then $\Gamma\psi = 0$ (the matrix Γ is degenerate). Under the conditions of the above $(\psi_j, \psi) = 0$. Therefore

$$\Gamma\psi = \sum_{j=1}^{k} \gamma_j \psi_j (\psi_j^*, \psi) = 0$$

Let us now specify some particular cases.

5. Suppose

$$\mathcal{M} = \{a \in H^m : f(a) \leq 0\}$$

with $f(a) \equiv f(a_1, \ldots, a_m)$ assumed to be a smooth function with a nondegenerate gradient

$$f'(a) = (f_1'(a), \ldots, f_m'(a))$$

($f_i'(a)$ stands for the partial derivative of f in a_i).
As it is well known [12] in this case

$$K_m^*(a) = \{-\sigma f'(a) : \sigma \geq 0, \sigma f(a) = 0\}$$

On the other hand the matrix Γ is nonzero as for example

$$(\psi_1, \Gamma\psi_1) = \sum_{j=1}^{k} \gamma_j(\psi_1^*\psi_j)^2 \geq \gamma_1 \mid \psi_1 \mid^2 = \gamma_1 > 0$$

Therefore at least one of its vector columns is nonzero, for example, $\Gamma_1 \neq 0$. The necessary condition for the case under consideration yields

$$a^*\Gamma = \sigma f'(a), \sigma \geq 0 \tag{10}$$

If $\sigma = 0$, then $a^*\Gamma = 0$, hence $a^*\Gamma_1 = 0, \Gamma_1 \neq 0$, i.e. the a_i are linearly dependent which contradicts with the condition that $\lambda(a) > 0$.

We have just proved

Corollary 1. If $\mathcal{M} = \{a \in H^m : f(a) \leq 0\}$ then the maximizing point for $\lambda(a), a \in \mathcal{M}$, satisfies the relations

$$a^*\Gamma = \sigma f'(a), \sigma > 0, f(a) = 0$$

$$\Gamma = \sum_{j=1}^{k} \gamma_j\psi_j\psi_j^*,\ \gamma_j > 0\ ,\ aa^*\psi_j = \lambda(a)\psi_j,\ \mid \psi_j \mid = 1$$

Particularly if

$$f(a) = \sum_{i=1}^{m} ||a_i||^2 - 1 = a^*a - 1$$

then

$$f'(a) = 2(a_1, \ldots, a_m) = 2a^*$$

and the necessary condition yields

$$a^*\Gamma = 2\sigma a^*, \sigma > 0, a^*a = 1,$$

If matrix Γ would be degenerate we would have $\Gamma\psi = 0$ for a certain $\psi \in \mathbf{R}^m, \mid \psi \mid = 1$. Therefore

$$a^*\Gamma\psi = 2\sigma a^*\psi = 0$$

i.e. $a^*\psi = 0, a_1, \dots, a_m$ would be linearly dependent and $\lambda(a) = 0$ which contradicts with the maximality of $\lambda(a) > 0$. The matrix Γ is therefore nondegenerate.

From the representation (8) of matrix Γ it follows that it may be nondegenerate only if among the vectors $\psi_j, j = 1, \dots, k$, there exists a subset of m linearly independent vectors. In this case all of the latter eigenvectors of A would correspond to $\lambda(a)$. This is possible only if

$$A = a \cdot a^* = \lambda(a) I_m,$$

i.e.

$$(a_i, a_j) = 0, i \neq j, \|a_i\|^2 = \lambda(a).$$

Hence the solution to the basic problem results in an array of orthogonal vectors a_i with equal norms.

Since

$$\sum_{i=1}^{m} \|a_i\|^2 = m\lambda(a) = 1.$$

we have

$$\lambda(a) = m^{-1}$$

6. Consider a specific problem of controlling the observation process when

$$a \in H^m, H = L^2[0, T]$$

The set $\mathcal{M}$ is the set of solutions to the m-dimensional differential system

$$\dot{a} = Ca + Bu, \quad t \in [0, T], \quad a[0] = a_0, \tag{11}$$

with control $u(t)$ selected from a convex set U of functions that ensure the existence of solutions to (11).

On the interval [0,T] we are therefore considering the measured signal

$$y(t) = a^*(t)z + \zeta(t), \ \zeta(\cdot) \in L^2[0, T],$$

$$\int_0^T \zeta^2(t)dt \leq 1$$

The optimal control problem now consists in the selection of a control $u(\cdot) \in U$ that would maximize the minimal eigenvalue of the matrix A with elements

$$\int_0^T a_i(t)\, a_j(t)\, dt$$

Once $u_0(t)$ is the optimal control and $a^0(t)$ the respective solution to system (11), the adjoint cone would be determined as

$$K_M^*(\cdot) = \{\psi^*(\cdot) : \int_0^T \psi^*(t)(a(t) - a^0(t))dt \geq 0\}, \tag{12}$$

where the inequality should be fulfilled for all the solutions $a(t)$ to equation (11) generated by all the controls $u(\cdot) \in U$.

Moreover

$$\psi(t) = \begin{pmatrix} \psi_1(t) \\ \vdots \\ \psi_m(t) \end{pmatrix}, \psi_i(\cdot) \in L^2[0,T];\ i = 1, \ldots, m,$$

Since

$$a(t) = (\exp\ Ct)a(0) + \int_0^t (\exp\ C(t-\tau))\ Bu(\tau)d\tau,$$

this may be substituted into inequality which yields (12). After an obvious calculation this yields

$$\int_0^T (\int_\tau^T \psi^*(\sigma)(\exp\ C(t-\tau))dt)\ B(u(\tau) - u_o(\tau))d\tau \geq 0, \tag{13}$$

$$u(\tau) \in U$$

Denoting

$$\psi^*(\tau) = -\int_\tau^T \psi^*(t)(\exp\ C(t-\tau))dt \tag{14}$$

we come to

Theorem 3. The inclusion $\psi^*(\cdot) \in K_M^*(a^0(\cdot))$ *if and only if the inequality*

$$\int_0^T \psi^*(\tau)\ B(u(\tau) - u_0(\tau))d\tau \leq 0 \tag{15}$$

for any $u(\cdot) \in U$.

Passing to the necessary conditions of optimality we have to check the condition of theorem 2 which is

$$-a^{0*}(t)\Gamma = \psi^*(t), \psi^*(\cdot) \in K_M^*(a^0(\cdot)).$$

Combining this with (14) we come to the relation

$$\psi^*(\tau) = \int_\tau^T a^{0*}(t)\ \Gamma\ (\exp\ C(t-\tau))dt \tag{16}$$

which should be coupled with inequality (15).

The principal result now sounds as follows.

Theorem 4. In order that the control $u \in U$ *and the respective trajectory* $a_0(t), t \in [0,T]$ *would determine the maximum for the minimal eigenvalue of the matrix*

$$A = \{\int_0^T a_i^0(t)\ a_j^0(t)\ dt\}$$

it is necessary that one could indicate such numbers $\gamma_j > 0$ *and such eigenvectors* ψ_j *of the matrix* $A, (i = 1, \ldots, k)$ *that the following relations would be true:*

1. $\dot{a}^0(t) = Ca^0(t) + Bu_0(t),\ t \in [0,T]$,

2. $\dot{\psi}^*(\tau) = -a^*(\tau)\Gamma - \psi^*(\tau)C,\ \tau \in [0,T], \psi^*(T) = 0$

3. $\Gamma = \sum_{j=1}^{k} \gamma_j \psi_j$

4. $\int_0^T \psi^*(t)Bu(t)dt \leq \int_0^T \psi^*(t)Bu_0(t)dt;\ \ u(\cdot) \in U$

The proof follows from above having in view that relation (2) is obtained by a direct differentiation of (16) in τ.

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