WORKING PAPER

INVERSE PROBLEM OF DYNAMICS FOR SYSTEMS DESCRIBED BY PARABOLIC INEQUALITY

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FOREWORD

This paper deals with a specific inverse problem of dynamics for a system described by a parabolic inequality. The aim is to reconstruct the input (the control) of the system on the basis of an on-line measurement corrupted by an error.

The techniques applied to the solution are a combination of those developed in positional control theory and the theory of ill-posed problems. This paper was contributed by the author during his visit to the SDS Program.

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Inverse Problem of Dynamics for Systems Described by Parabolic Inequality

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The considered problem is concerned with the following questions.

Let t be the time variable. Consider an evolutional system Σ on an interval $T = [t_0, \theta]$. We are interested in some unknown characteristic $\xi_1(t)$, $t \in T$ of the system (e.g., ξ_1 may be a collection of some parameters of the system, or of some disturbances acting on the system or of controls etc.). We are to reconstruct $\xi_1(t)$ on the basis of measurements of some other characteristic $\xi_2(t)$, $t \in T$ of the system Σ. The results of measurements $\varsigma(t)$ are not precise, the error being estimated by h.

The smaller h is, the more precise should be the reconstruction (in the appropriate sense). This is the stability property of the reconstruction algorithm D_h.

We consider two types of reconstruction problems. In the problems of the first type (which we call problems of program reconstruction) the measurements $\varsigma(t)$ are known for all $t \in T$ at once. Hence the input of the reconstruction algorithm is the function $\varsigma(t)$, $t_0 \leq t \leq \theta$. The output of D_h is a function $\xi_1^{(h)}(t)$, $t_0 \leq t \leq \theta$ close (in a suitable sense) to the characteristic $\xi_1(t)$, $t_0 \leq t \leq \theta$ for h small enough.

In problems of the second type (we call them problems of dynamical reconstruction) the characteristic ξ_1 is to be restored simultaneously with the process of system motion. Here in every current moment t the input of the algorithm D_h is the previous history $\varsigma_t = \varsigma_t(\cdot) = \{\varsigma(\tau), t_0 \leq \tau < t\}$ of the measurements ς made prior to the moment t. The output of D_h in the moment t is a function

$$\xi_{1t}^{(h)}(\cdot) = \{\xi_1^{(h)}(\tau), t_0 \leq \tau < t\} ,$$

which approximates (in the proper sense) the characteristic

$$\xi_1(\tau),\ t_0 \leq \tau \leq t\ , \text{ for small } h\ .$$

Here D_h is to satisfy the property of physical realizability [2], [3]: if $\varsigma^{(1)}(\tau)$, $t_0 \le \tau \le t_1$ and $\varsigma^{(2)}(\tau)$, $t_0 \le \tau \le t_2$ are such that

$$\varsigma_{t_*}^{(1)} = \varsigma_{t_*}^{(2)},\ t_* \le \min\ \{t_1,t_2\}\ ,$$

then the functions $D_h\varsigma^{(1)}_t(\cdot)$, $D_h\varsigma^{(2)}_t$ are equal on $[t_0,t_*)$.

Below we consider a problem of the second type for a system described by a parabolic inequality. We develop further the method for dealing with such kind of problems proposed in [1-3]. The method is based on some ideas of positional control theory [14-17] and ill-posed problems theory [18].

The present paper is connected with [1-13].

Let $\mathbf{V}$ and $\mathbf{H}$ be real Hilbert spaces, $\mathbf{V}^*$ and $\mathbf{H}^*$ be the spaces dual to $\mathbf{V}$ and $\mathbf{H}$ respectively. We identify $\mathbf{H}$ with $\mathbf{H}^*$. It is supposed that $\mathbf{V} \subset \mathbf{H}$ is dense in $\mathbf{H}$ and is embedded into $\mathbf{H}$ continuously. Denote by $(\cdot,\cdot)_{\mathbf{H}}$ and $|\cdot|_{\mathbf{H}}$ ($(\cdot,\cdot)_{\mathbf{V}}$ and $|\cdot|_{\mathbf{V}}$) the scalar product and the corresponding norm in $\mathbf{H}$ (in $\mathbf{V}$).

Let t be the time variable, $t\in T = [t_0,\theta]$. Consider on T a control system Σ. The state of the system is $y(t)\in\mathbf{V}$. The evolution of the state is given by the following conditions for almost all $t\in T$ the inequality holds ([19,20]):

$$(\dot{y}(t),\ y(t) - \omega)_{\mathbf{H}} + a(y(t),\ y(t)) + \phi(y(t)) - \phi(\omega) \le (Bu(t) + f(t),\ \omega)_{\mathbf{H}} \quad \forall\omega\in\mathbf{V} \tag{1.1}$$

and

$$y(t_0) = y_0\ . \tag{1.2}$$

Here $a(\omega_1,\omega_2)$ is a continuous on $\mathbf{V}$ bilinear symmetrical form satisfying for some $c_1 > 0$ the condition

$$a(\omega,\omega) \ge c_1|\omega|^2_{\mathbf{V}}\ ; \tag{1.3}$$

$\phi:\mathbf{V}\to(-\infty,+\infty]$ is a convex proper lower semicontinuous function (or $\phi:\mathbf{H}\to(-\infty,+\infty]$ is a convex proper lower semicontinuous function satisfying the regularity condition [21,22]; $B : \mathbf{U}\to\mathbf{H}$ is a linear continuous operator, $\mathbf{U}$ is a uniformly convex real Banach space; $f\in L^2(T;\mathbf{H})$; $u(\cdot)$ is a control, i.e. measureable on T function for almost all $t\in T$ having values in bounded closed convex set $P\subset\mathbf{U}$; $y_0\in\{\omega\in\mathbf{V} : \phi(\omega) < +\infty\}$. Under the above assumptions in $\mathbf{W}^{1,2}(T;\mathbf{H}) \cap L^2(T;\mathbf{V})$ there exists a unique function $y(t) = y(t;t_0,y_0,u(\cdot))$, $t\in T$, satisfying (1.1), (1.2) (see [19-22]). We call it a motion of system Σ from the initial state y_0 corresponding to control $u(\cdot)$.

Consider the following problem of dynamical reconstruction. Let $\mathbf{V} = \mathbf{H}_0^1(\Omega)$ (or $\mathbf{V} = \mathbf{H}^1(\Omega)$), $\mathbf{H} = L^2(\Omega)$,$\mathbf{U} = L^2(\Omega)$,B be the identity operator (see notation in [19,20]). Now in (1.1) we take

$$y(t) = y(t,\cdot) = \{y(t,x),\ x\in\Omega\}\ ,$$

$$\dot{y}(t) = \partial y(t,\cdot)/\partial t,\ u(t) = u(t,\cdot)\ .$$

Let the control u be of the form

$$u(t) = u(t,x) = \chi_{G(t)}(x) \times u^0(t,x) \tag{1.4}$$

Here $G(t) \subset \Omega$ is such that the set $\{(t,x) : t\in T,\ x\in G(t)\}$ is Lebesgue measureable; χ_G is the characteristic function of G; the function u^0 satisfies the inequality

$$0 < \beta_1 \le u^0(t,x) \le \beta_2,\ t\in T,\ x\in\Omega\ , \tag{1.5}$$

where β_1, β_2 are positive numbers.

Let the measurement of the system state $y_*(t) = y_*(t,\cdot)$ be possible in every current moment t, the measurement result $\varsigma(t) = \varsigma(t,\cdot)$ satisfying the estimation

$$|\varsigma(t,\cdot) - y_*(t,\cdot)|_{L^2(\Omega)} \le h\ . \tag{1.6}$$

Suppose that the motion being observed is generated by the unique control of the type (1.4), (1.5)

$$u_*(t,x) = \chi_{G_*(t)} u_*^0(t,x),\ t\in T,\ x\in\Omega\ .$$

Consider the problem of dynamical reconstruction with

$$\xi_1(t) = \{u_*(t)\ ;\ S_*(t)\}\ ,$$

$$S_*(t) = \{(\tau,x) : \tau\in[t_0,t),\ x\in G_*(\tau)\}\ ;$$

$$\xi_2(t) = y(t,\cdot)\ .$$

Remark 1.1. Let e.g., (1.1), (1.2) describe the process of diffusion of a substance in a domain Ω and $y(t,\cdot)$ be the concentration of substance in Ω in the moment t. Then we deal with the reconstruction of intensity of the substance sources and their location (see [12]).

We proceed the following way (see [12, 13]). To the system Σ we put into correspondence a control system Σ_1 (the model) which is a copy of Σ.

$$(z(t),\, z(t) - \omega)_{L^2(\Omega)} + a(z(t),\, z(t)) \tag{1.7}$$

$$- \omega) + \phi(z(t)) - \phi(\omega) \le (v(t) + f(t), \omega)_{L^2(\Omega)} \quad \forall w \in V$$

$$z(t_0) = y_0 \ .$$

The control $v(\cdot) \in L^2(T\,;\, L^2(\Omega))$ in the model is chosen for almost all $t \in T$ from convex bounded closed set P which contains all the $L^2(\Omega)$ functions of the form $\chi_B \cdot g(x)$ where $B \subset \Omega$ is a measurable set, $g(\cdot)$ is a measurable function, $g : \Omega \to [\beta_1, \beta_2]$.

Consider a partition τ_i of interval T,

$$t_0 = \tau_0 < \tau_1 < \cdots < \tau_m = \theta \ ;$$

$$m = m(h),\ \delta(h) = \max_i(\tau_{i+1} - \tau_i),\ \delta(h) \le ch,\ c = const > 0 \ .$$

Take

$$v(t) = v^{(h)}(t) = v_i,\ \tau_i \le t < \tau_{i+1}\ , \quad i = 1, \ldots, m$$

where v_i are (the unique) points of minimum of the functional

$$\psi(p) = 2(z(\tau_i\ ;\ t_0,\, y_0,\, v(\cdot)) - \varsigma(\tau_i),\, p)_{L^2(\Omega)} + \alpha(h)|p|^2_{L^2(\Omega)} \ .$$

The function $\alpha(h) > 0$; $\alpha(h) \to 0$, $h/\alpha(h) \to 0$ as $h \to 0$. Form the set

$$S_i^{(h)} = [\tau_i,\, \tau_{i+1}) \times \{x \in \Omega : v_i(x) \ge \mu\}\ , \tag{1.8}$$

where μ is some positive number $\beta_1 \le \mu \le \beta_2$.

Denote

$$S^{(h)} = \bigcup_{i=0}^{m-1} S_i^{(h)}\ ,$$

where $d(S_*(\theta),\, S^{(h)})$ is the Lebesgue measure of the symmetric difference of sets S_*, $S^{(h)}$.

Theorem. If $h \to 0$ then the following is valid

$$|v^{(h)} - u_*|_{L^2(T; L^2(\Omega))} \to 0$$

$$d(S(\theta),\, S^{(h)}) \to 0 \ .$$

Remark 1.2. Similar to [12] one can obtain an estimate of reconstruction accuracy.

2. Consider an example. Let ϕ be a convex continuous function under the assumption of Section 1. Then the system (1.1) is equivalent to the equation

$$\frac{\partial y}{\partial t} = Ay + u + f(t,x),\ t \in T,\ x \in \Omega,\ y|_{\Gamma} = 0 \tag{2.1}$$

Here A is an elliptic coercive operator

$$Ay = \frac{\partial}{\partial x_j}\ (a_{ij}(x)\ \frac{\partial y}{\partial x_i}) - q(x)y,\ a_{ij} = a_{ji}\ , \tag{2.2}$$

$$a_{ij} \in L^{\infty}(\Omega),\ q \in L^{\infty}(\Omega)\ .$$

For (2.1) consider a concrete variant of reconstruction problem [12].

Let Ω be a two-dimensional domain

$$0 < x_1 < \ell_1\ ,\ 0 < x_2 < \ell_2\ ;\ f = 0\ ,\ q = 0$$

and

$$Ay = a^2 \cdot \partial^2 y/\partial x_1^2 + b^2 \cdot \partial^2 y/\partial x_2^2\ .$$

For the sake of simplicity we confine the considerations to the case of reconstruction of location $G(t)$, $t \in T$. Let it be known a priori that the control being restored satisfying the inequality $|u(t,\cdot)|_{L^2(\Omega)} \leq R$.

A closed ball in $L^2(\Omega)$ of radius R is taken as P. Then

$$v_i = [\varsigma(\tau_i) - z(\tau_i\ ;\ t_0,y_0,v(\cdot))]\ /\ \alpha(h)\ \text{ if}$$

$$|\varsigma(\tau_i) - z(\tau_i\ ;\ t_0,y_0,v(\cdot))|_{L^2(\Omega)} \leq R \cdot \alpha(h)\ ,$$

$$v_i = R \cdot [\varsigma(\tau_i) - z(\tau_i\ ;\ t_0,y_0,v(\cdot))]\ /\ |\varsigma(\tau_i) - z(\tau_i\ ;\ t_0,y_0,v(\cdot))|_{L^2(\Omega)}\ ,\ \text{ if}$$

$$|\varsigma(\tau_i) - z(\tau_i\ ;\ t_0,y_0,v(\cdot))|_{L^2(\Omega)} > R \cdot \alpha(h)\ .$$

For the considered variant of the problem the calculations were carried out for the following data

$$a^2 = b^2 = 0.1\ ,\ \ell_1 = \ell_2 = 10\ ,\ t_0 = 0\ ,\ \theta = 1\ ,\ R = 100\ ,$$

$$y_0 = 0\ ,\ \beta_1 = \beta_2 = 10\ ,\ \delta(h) = h\ ,\ \alpha(h) = \sqrt{h}\ ,\ h = 0.1\ .$$

The motions of the dynamical system and the auxiliary model were calculated with the help of an explicit difference scheme with constant time step $\tau = \delta(h)$ and constant spatial steps γ_1 and γ_2 in x_1 and x_2 respectively.

The set $G(t_0)$ is depicted in Fig. 1 and Figs. 2 and 3 show the results of reconstruction of the set

$$G(t) = \{(x_1,x_2) : 0.01 \le x_1 \le 9.99, x_1(t,x_1) \le x_2 \le x_2(t,x_1)\} ,$$

where

$$x_1(t,x_1) = 3.5 + \cos(0.5 \cdot x_1 - 5 \cdot t) + 0.3 \cdot \cos(5 \cdot x_1 + t/h) \cdot \sin(3.2 \cdot x_1 + t/h) ,$$

$$x_2(t,x_1) = 6.5 + \cos(0.5 \cdot x_1 - 5 \cdot t) + 0.3 \cdot \cos(10 \cdot x_1 + t/h) \times \sin(3.2 \cdot x_1 + t/h) ,$$

at the moments $t = 0.5$, $t = 0.9$ respectively for

$$\gamma_1 = \gamma_2 = 10/16 .$$

The unknown set is reconstructed with the help of rectangles with centres in the mesh nodes and sides γ_1 and γ_2 parallel to axes x_1, x_2 respectively.

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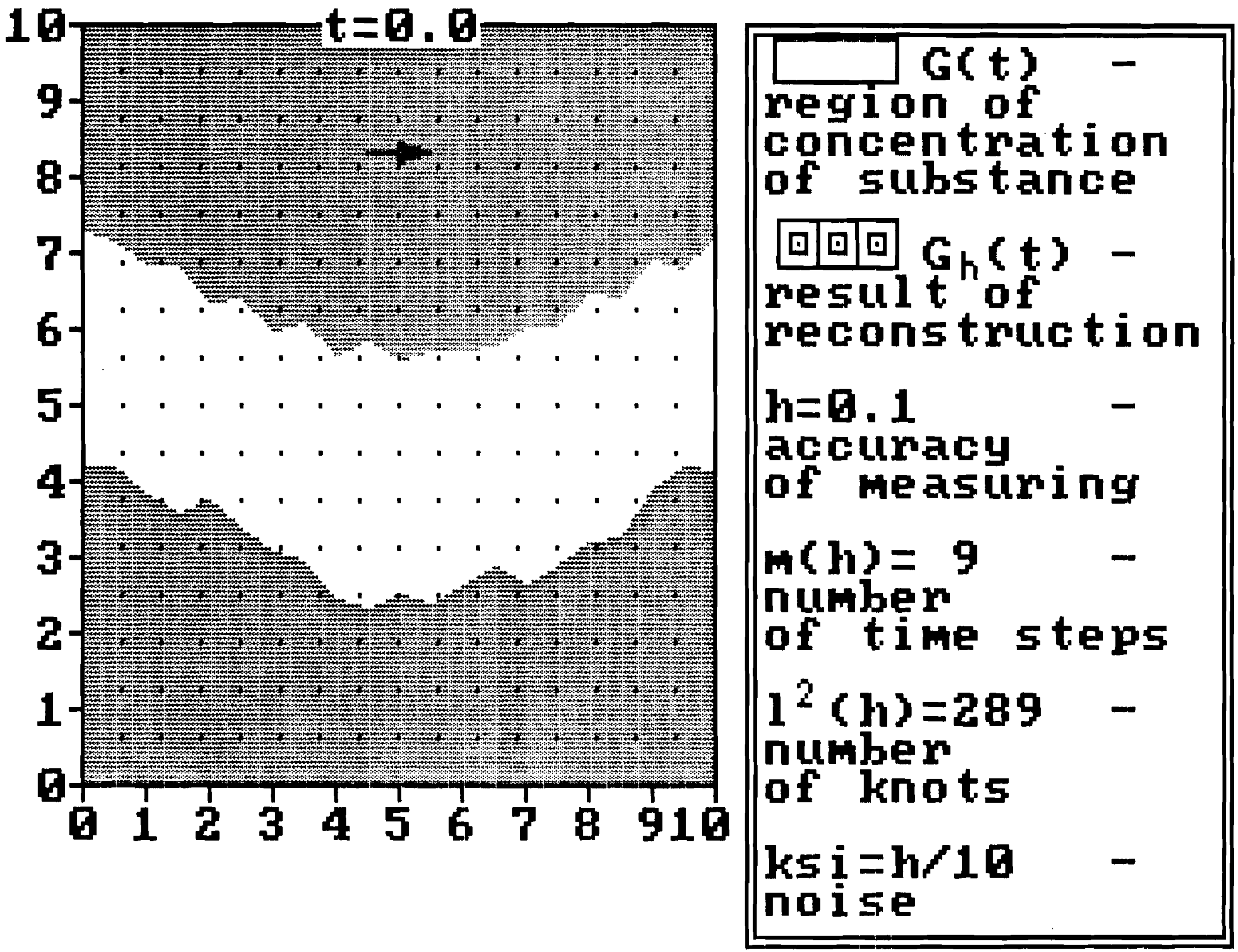
t=0.0
10
9
8
7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9 10
G(t) -
region of
concentration
of substance
G_h(t) -
result of
reconstruction
h=0.1 -
accuracy
of measuring
m(h)= 9 -
number
of time steps
l^2(h)=289 -
number
of knots
ksi=h/10 -
noise

Figure 1

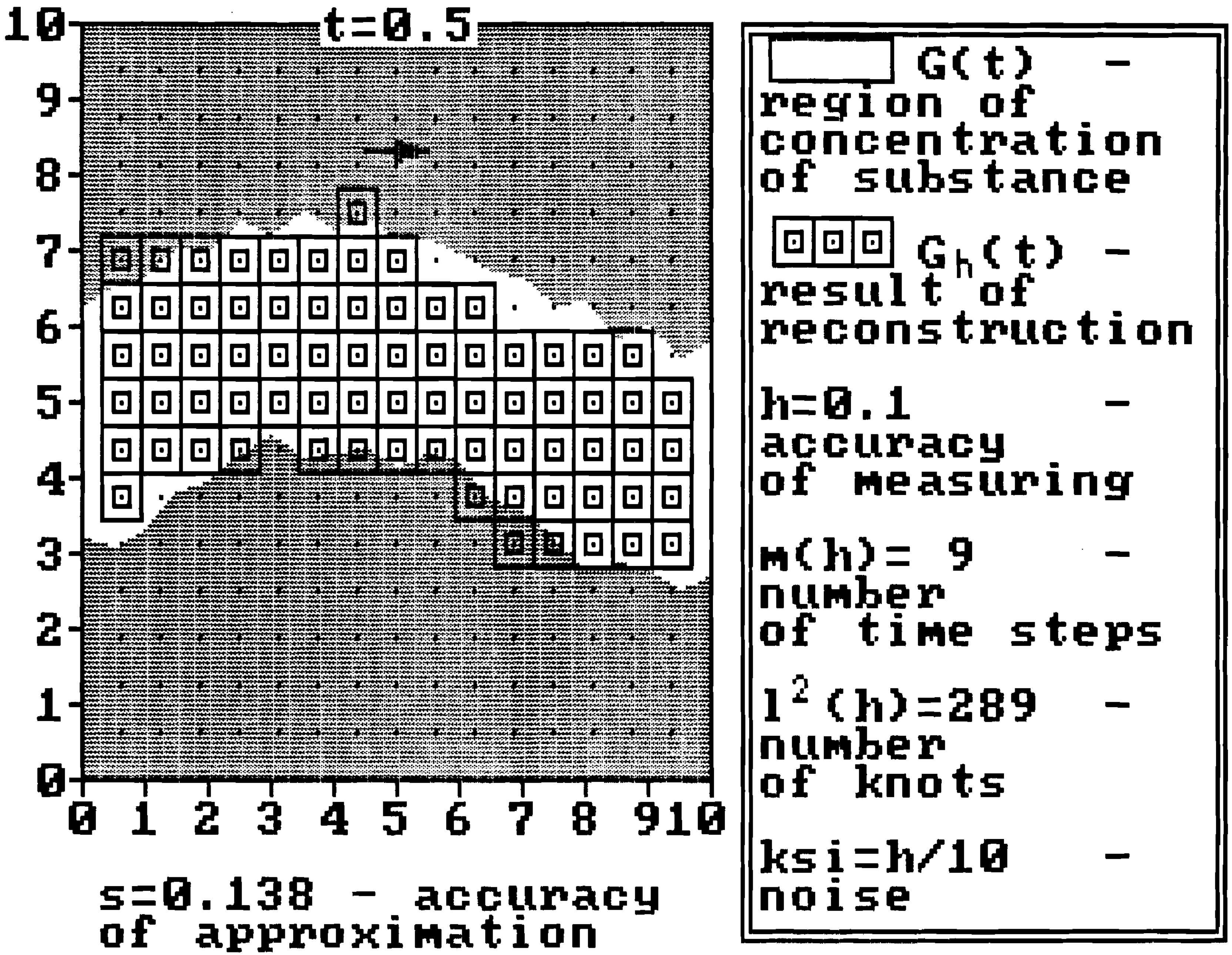
t=0.5
0 1 2 3 4 5 6 7 8 9 10
s=0.138 - accuracy of approximation
G(t) - region of concentration of substance
G_h(t) - result of reconstruction
h=0.1 - accuracy of measuring
m(h)= 9 - number of time steps
l^2(h)=289 - number of knots
ksi=h/10 - noise

Figure 2

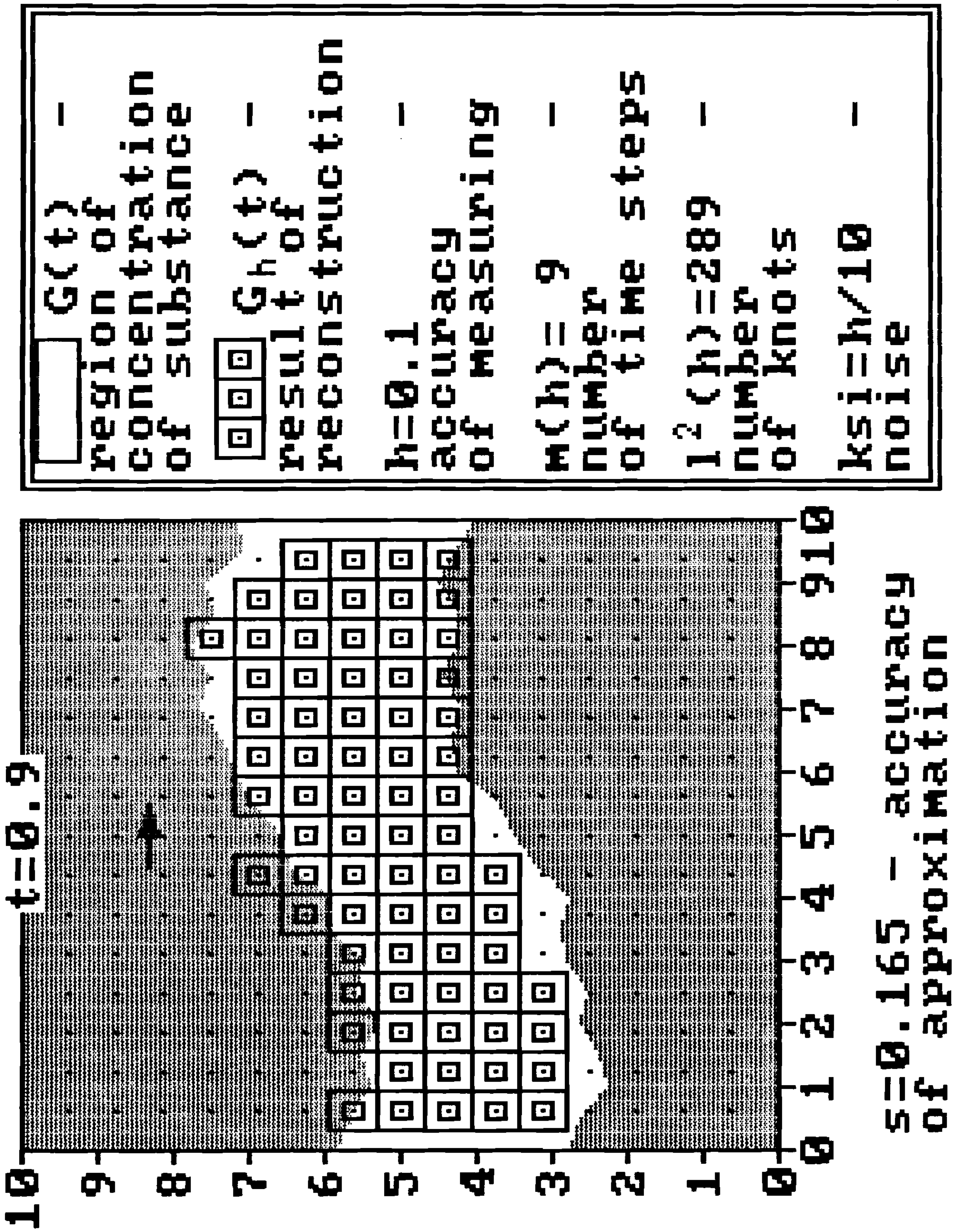
t=0.9
0 1 2 3 4 5 6 7 8 9 10
0 1 2 3 4 5 6 7 8 9 10
s=0.165 – accuracy of approximation
G(t) – region of concentration of substance
G_h(t) – result of reconstruction
h=0.1 – accuracy of measuring
m(h)=9 – number of time steps
l^2(h)=289 – number of knots
ksi=h/10 – noise

Figure 3

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