WORKING PAPER

EVOLUTION OF PRICES UNDER THE INERTIA PRINCIPLE

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

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Jean-Pierre Aubin
CEREMADE, Université de Paris-Dauphine
&
IIASA, International Institute for Applied Systems Analysis

FOREWORD

The decentralized evolution of allocations of resources among consumers described by their change functions is characterized by a regulation rule associating with each allocation the subset of prices regulating them. Sufficient budgetary conditions analog to the Walras law for the viability of such evolutions are then provided. Next, the issue of finding feedback controls is tackled: conditions under which slow evolutions are given.

More to the point, dynamical feedback controls obeying the *inertia principle are provided: prices are changed only when the viability of the evolution mechanism is at stakes.* We then derive the differential equations governing the *heavy* evolution of prices: for a given bound on inflation rates, the price evolves with minimal velocity.

Alexander B. Kurzhanski
Chairman
System and Decision Science Program

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Introduction

Let M denote the set of scarce resources and K the set of allocations $x = (x_1, \dots, x_n)$ of scarce commodities $y = \sum_{i=1}^{n} x_i \in M$ to n consumers. We interpret the basic law of economics:

It is impossible to consume more physical goods than available

by taking the allocation set K as a viability set, i.e., by requiring that the consumptions of the n consumers are allocations of available commodities.

Static models assign one or several elements $\bar{x}$ in the allocation set K. But it may be time to answer the wish J. von Neumann and O. Morgenstern expressed in 1944 at the end of the first chapter of their monograph "Theory of Games and Economic Behavior":

"Our theory is thoroughly static. A dynamic theory would unquestionably be more complete and therefore, preferable..."
"Our static theory specifies equilibria ... A dynamic theory, when one is found — will probably describe the changes in terms of simpler concepts."

We study here some mechanisms which govern the evolution of allocations of scarce resources[1].

In these dynamical models, the laws which govern the evolution of allocations are most often represented by differential equations (or differential inclusions) with or without memory. Static models are particular cases yielding "constant functions" $x(\cdot) \equiv \bar{x}$, which are also called "equilibria[2]".

For that purpose, the behavior of each consumer is described by a demand function d_i allowing the consumer to choose a commodity $x_i = d_i(p)$

[1] By the way, in dynamical models, we can assume that the subset $K(t)$ of allocations depends upon t, or even, upon the history of the evolution.

[2] The concept of equilibrium often covers two different meanings in economics. The first one is derived from mechanics, where an equilibrium is a constant function, or a "rest point". The second meaning is covered here by what we call the viability constraints, such as the sum of consumptions must be less than or equal to the sum of supplies, etc.

knowing only the price p. The problem is then to find a price $\bar{p}$ (the Walrasian equilibrium price) such that $(d_1(\bar{p}), \ldots, d_n(\bar{p}))$ is an allocation. This is a decentralized model because consumers do not need to know neither the choices of other consumers nor the set M of allocations. The basic Arrow-Debreu Theorem states in this case that such an equilibrium exits whenever a budgetary rule, the Walras law — it is forbidden to spend more monetary units than earned — is obeyed by consumer's demand functions.

Furthermore, such a price $\bar{p}$ is an equilibrium of an underlying dynamical process, called the Walrasian tâtonnement: it is defined by the differential inclusion

$$p'(t) \in E(p(t))$$

where E is the *excess demand map* defined by

$$E(p) := \sum_{i=1}^{n} d_i(p) - M$$

We observe that if $p(t)$ is a price supplied by the Walras tâtonnement process and if it is not an equilibrium, *it cannot be implemented* because the associated demand is not necessarily available.

Hence, this model *forbids consumers to transact as long as the prices are not equilibria*. It is as if there was a super auctioneer calling prices and receiving offers from consumers. If the offers do not match, he calls another price according to the above dynamical process, but *does not allow transactions to take place* as long as the offers are not consistent.

Tâtonnement is therefore not viable. And it may be too much to ask the entity which regulates the price (the market, the invisible hand, the Gosplan, ...) to behave as a real decision-maker. It may be more reasonable to let the real decision-makers, the consumers, to govern the evolution of their consumption through differential equations parametrized (controlled) by prices:

$$x_i'(t) = c_i(x_i(t), p(t))$$

parametrized (or controlled) by the price $p(t)$, so that consumers change their consumptions knowing only the price $p(t)$ at each time t, without taking into account neither the behavior of the other consumers nor the knowledge of the set M of scarce resources. Hence it shares with the Walras static model its *decentralization property*.

The problem is then to *find a price function $p(t)$ such that the solutions $x_i(t)$ of the above differential equations do form an allocation at each time t.* We prove that this viability property holds true under a dynamical version of the Walras law and even prove the existence of an equilibrium, by using viability theorems are tehy can be found in DIFFERENTIAL INCLUSIONS by J.-P. Aubin & A. Cellina, Springer, 1984 and VIABILITY THEORY by J.-P. Aubin, to appear.

Actually, we would like to know more than a time-dependent price function (which can be regarded as an open loop control). We wish to obtain "closed loop controls", or, more generally, set-valued "regulation maps" associating with each allocation $x \in K$ the set $\Pi(x)$ of relevant messages, so that the evolution law of the relevant message is

$$\forall\, t, \;\; p(t) \;\in\; \Pi(x(t)) \;=\; \Pi(x_1(t), \ldots, x_n(t))$$

The set of viable prices (regarded as relevant messages) may contain more than one element. The question arises to select one of these prices , or, to shrink the set of viable prices by an adequate mechanism. This can be done by optimization techniques, or, more generally, by game theoretical methods.

In the dynamical case, this question splits in two: we have to distinguish between "intertemporal optimization" problems and "myopic or instantaneous optimization"problems.

In intertemporal optimization, we maximize intertemporal utility functions of the form

$$U(x(\cdot), p(\cdot)) \;:=\; \int_0^T u(t, x(t), p(t))dt + v(x(T), p(T))$$

under the constraint $(x(\cdot), p(\cdot)) \in \mathrm{Graph}(\Pi)$.

The use of such an intertemporal utility function assumes some knowledge of the future[3]. Furthermore, the choice of the optimal solution is made once and for all at the initial time, and cannot be corrected.

[3]This is this requirement of the use of optimal control theory which led to the popular theory of rational expectations. It shares with general equilibrium theory the feature of growing up from available mathematical theories and being transfered to economics. The pretty large consensus around these concepts make them "real" according to our definition of reality. But it should be time for this consensus to evolve by looking for economic facts to motivate new mathematical theories and not the other say around.

In myopic optimization, we use the feedback relation and we select for each allocation $x \in K$ a price $p \in \Pi(x)$ by a static optimization technique (or any other kind of technique). For instance, we can choose the element $\pi^0(x) \in \Pi(x)$ of minimal norm. Despite the lack of continuity of such a selection, we still can prove that the system of differential equations

$$x_i'(t) = c_i(x_i(t), \pi^0(x(t)))$$

has viable solutions, which are called "slow solutions".

However, this type of selection may not enjoy economic meaning. We propose another one which may be closer in spirits to economic mechanisms.

Actually, if the behavior of the consumers is well defined, what about either the market or the planning bureau, the task of which is to find the prices $p(t)$ in $\Pi(x(t))$? They do not behave as actual decision makers, knowing what is good or not (this is the case of even a planning bureau as soon as it involves more than three bureaucrats!). Hence, their role is only a *regulatory one*. If they are not able to optimize, we may assume that they only are able to correct the prices when the viability of the economic system is at stake, i.e., when the total consumption is no longer available.

Hence, we assume that the market (Adam Smith's "invisible hand") or the planning bureau are able to "pilot" or "act" on the system by choosing such controls according to the *inertia principle*:

Keep the price constant as long as the evolution provides allocations of available resources, and change them only when the viability is at stakes.

This is not enough to select an evolution of a relevant price, since we have to provide rules for choosing prices when viability is at stakes.

The simplest one (and most often, the most reasonable one) is to assume that at each instant, the prices are changed as slowly as possible.

We called evolutions obeying this principle "heavy[4] evolutions", in the sense of heavy trends.

Hence heavy evolution is obtained by requiring that at each instant, the (norm of the) velocity of the price is as small as possible.

[4] This is justified by the fact that the velocity of the price is related to the acceleration of the consumptions, a measure of which is then as small as possible.

Therefore, for implementing this inertia principle, we have to provide conditions under which relevant prices $p(\cdot)$ are differentiable (almost everywhere), to built the differential inclusion which governs the evolution of differentiable relevant prices and then, select a differential equation in this differential inclusion (called a "dynamical closed loop) which will obey the inertia principle.

The first task is solved (at least, partially) thanks to the concept of "viability kernel". We obtain the existence of a smaller regulation map $\Pi^c(\cdot) \subset \Pi(\cdot)$ such that the regulation law

$$\forall t, \ p(t) \ \in \ \Pi^c(x(t))$$

yields differentiable prices satisfying a rate of growth constraint $\|p'(t)\| \leq c\|p(t)\|$.

By differenting this regulation law, we obtain a differential inclusion governing the evolution of relevant prices of the form

$$p'(t) \ \in \ \Pi^{c'}(x(t), p(t))$$

The problem then is to select a particular solution by solving a differential equation

$$p'(t) \ = \ \pi^{c'}(x(t), p(t))$$

where $\pi^{c'}(\cdot,\cdot)$ is a selection of $\Pi^{c'}(\cdot,\cdot)$.

The one of interest is the selection $\varpi'(x,p) \in \Pi^{c'}(x,p)$ of minimal norm, which obeys the inertia principle.

In summary, *given the decentralized behavior of the consumers described by the differential equations* $x_i' = c_i(x_i, p)$ *and the set of scarce resources, we can built the dynamical behavior of the market*, so that the evolution of the economic system is described by the system of differential equations

$$\begin{cases} i) & x_i'(t) \ = \ c_i(x_i(t), p(t)) \quad (i = 1, \ldots, n) \\ ii) & p'(t) \ = \ \varpi^{c'}(x(t), p(t)) \end{cases}$$

Contrary to other dynamical models, this law governing the evolution of prices is not a modelization assumption, but a *consequence* of the modelization data of this elementary model.

1 Setting Up the Model

1.1 The allocation set

Our problem is to allocate scarce resources among n consumers, labeled $i = 1, \ldots, n$.

The set of scarce resources is a subset $M \subset Y$ of the *commodity space* Y. By assuming indivisibility of commodities[5], one can represent the commodity space by a finite dimensional vector-space $Y := \mathbf{R}^l$, where l denotes the number of commodities[6] (or services) considered in the model, where each commodity is endowed of a measure unit.

We assume mainly that M is a closed subset satisfying $M = M - \mathbf{R}^l_+$ (free disposal assumption). This means that any commodity $y \leq x$ smaller than or equal to an available commodity x is still available. We assume also, for simplicity, that M is convex. This is interpreted by economists by saying that decreasing return to scale prevails.

We begin now the mathematical description of a consumer i. It starts by her *consumption set* $L_i \subset Y$, which represents the set of potential consumptions. Actually, it is better to say that she will never accept a commodity outside her consumption set L_i. Most often, L_i is chosen to be the orthant $\mathbf{R}^l_+$. In a symmetric way, we assume that the consumption sets L_i are closed and satisfy $L_i = L_i + \mathbf{R}^l_+$. and, again for simplicity, that they are convex.

Hence, the subset M of scarce resources and the n consumption sets L_i being given, the *allocation set* $K \subset X := Y^n$ of resources to the n consumers is defined by

$$K := \{ x \in \prod_{i=1}^{n} L_i \mid \sum_{i=1}^{n} x_i \in M \}$$

(We use the same notation to represent a commodity $x \in Y$ and an allocation $x = (x_1, \ldots, x_i, \ldots, x_n) \in X$ of a commodity to the n consumers, hoping that the context will make clear which is which.)

[5] A way to accept that assumption is to represent a commodity by the services that it yields, since services are more divisible than physical goods

[6] The commodities we use are physical commodity, by opposition to fiduciary goods, whose scarcity is not set by physical considerations, but by social consensus

Prices are supposed to be nonnegative (this makes sense when free disposal prevails.) They can be normalized by fixing the value of a good chosen as a numéraire or the value of a commodity basket — an index— (there is no monetary illusion). We choose here the second normalization rule, by taking the commodity basket formed of one unit of each good and fixing its value equal to 1. By doing so, prices range over the *price simplex*

$$S^l := \{ p \in \mathbf{R}^l_+ \mid \sum_{h=1}^{l} p^h = 1 \}$$

We then translate the first law of economics

It is impossible to consume more physical goods than available

by saying the set K of allocations is a viability set.

1.2 Change Functions

Instead of describing the decentralized behavior of a consumer by a Walrasian demand function, which makes sense in the static case, we shall capture it to take into account the evolutionary aspect by a "change function"

$$(x,p) \mapsto c_i(x,p)$$

associating with each commodity x owned by consumer i and the price p she sees on the market the velocity with which she will change her commodity

Hence, the behavior of consumer i is described by the differential equation

$$x'_i(t) = c_i(x_i(t), p(t))$$

It is decentralized in the sense that the decision of consumer i does not involve the knowledge of the set M of available resources nor the behavior of her fellow consumers, but depends only upon her current consumption $x_i(t)$ and the "current price" (also called "spot price") $p(t)$ at time t.

Now, we have to introduce an a priori law for price behavior. In the simplest case, we can choose prices in the price simplex

$$S^l := \{ p \in \mathbf{R}^l_+ \mid \sum_{h=1}^{l} p^h = 1 \}$$

But we can take into consideration external laws or regulations, and for that purpose, introduce a set-valued map $P : K \leadsto \mathbf{R}^l_+$ associating to each allocation x a subset $P(x) \subset S^l$ of feasible prices (allowed by external regulations, for instance).

Hence, the prices are requested to obey the evolution law:

$$\forall\, t \geq 0, \;\; p(t) \;\in\; P(x(t))$$

By summarizing, the dynamics of the evolution of the consumption is described by

$$\forall\, t \geq 0, \;\; \begin{cases} i) & x_i'(t) \;=\; c_i(x_i(t), p(t)) \quad (i = 1, \ldots, n) \\ ii) & p(t) \;\in\; P(x(t)) \end{cases}$$

Equilibria of this dynamical system are solutions $(\bar{x}, \bar{p})$ to the system

$$\begin{cases} i) & c_i(\bar{x}_i, \bar{p}) \;=\; 0 \quad (i = 1, \ldots, n) \\ ii) & \bar{p} \;\in\; P(\bar{x}) \end{cases} \tag{1}$$

We first address the problem of finding viable allocations and/or viable equilibria, i.e., functions satisfying

$$\forall\, t \geq 0, \;\; \begin{cases} i) & x_i(t) \;\in\; L_i \quad (i = 1, \ldots, n) \\ ii) & \sum_{i=1}^n x_i(t) \;\in\; M \end{cases}$$

and/or equilibria satisfying

$$\bar{x}_i \;\in\; L_i \quad (i = 1, \ldots, n) \;\;\&\;\; \sum_{i-1}^n \bar{x}_i \;\in\; M$$

The behavior of the consumers being given, we associate with each set M of scarce resources the regulation map Π_M defined by

$$\forall\, x \in K, \; \Pi_M(x) \;:=\; \left\{ p \in P(x) \mid \sum_{i=1}^n c_i(x_i, p) \;\in\; T_M\Big(\sum_{i=1}^n x_i\Big) \right\}$$

where $T_M(y) := \overline{\bigcup_{h>0}(M - y)/h}$ is the *tangent cone* to the convex subset M at $y \in M$.

We posit now the assumptions we need to prove our fundamental theorem.

— Assumptions on the consumption and resource sets:

$$\left\{\begin{array}{ll} i) & M = M - \mathbf{R}^l_+ \text{ is a closed convex subset} \\ ii) & \forall\, i = 1,\ldots,n,\ \ L_i = L_i + \mathbf{R}^l_+ \text{ is closed and convex} \\ iii) & \exists\, x_i \in L_i\ (i = 1,\ldots,n) \mid \sum_{i=1}^n x_i \in \text{ Int}(\mathbf{R}^l_+) \\ iv) & M \subset \underline{y} - \mathbf{R}^l_+ \ \ \&\ \ \forall\, i = 1,\ldots,n,\ \ L_i \subset \underline{x}_i + \mathbf{R}^l_+ \end{array}\right. \quad (2)$$

— Assumptions on the feedback map and the change functions:

$$\begin{array}{l} \text{Graph}(P) \text{ is closed, the growth of } P \\ \text{is linear and the images of } P \text{ are convex} \end{array} \quad (3)$$

and

$$\left\{\begin{array}{ll} i) & c_i(x,p) := c_i(x) + C_i(x)p \text{ is affine, where} \\ ii) & c_i : L_i \mapsto Y \text{ is continuous} \\ iii) & C_i : L_i \mapsto \mathcal{L}(Y^\star, Y) \text{ is continuous} \\ iv) & \forall\, x_i \in L_i,\ p \in \text{ Im}(P),\ \ c_i(x_i,p) \in T_{L_i}(x_i) \end{array}\right. \quad (4)$$

We are now able to characterize the viability property, which says that for every initial allocation $x_0 \in K$, there exists a price function $p(\cdot)$ and a solution $x(\cdot) = (x_1(\cdot),\ldots,x_n(\cdot))$ to the system (1) which is viable (called a viable allocation starting at x_0), thanks to the Viability Theorem.

Theorem 1.1 *We posit assumptions (2), (3) and (4). Then the two following conditions are equivalent:*

$$\left\{\begin{array}{ll} a) & \forall\, x \in K,\ \ \Pi_M(x) \neq \emptyset \\ b) & \forall\, x_0 \in K,\ \text{ there exists allocations starting at } x_0 \end{array}\right.$$

In this case, the viable allocations are governed by the regulation law

$$\text{for almost all } t \geq 0, \quad p(t) \in \Pi_M(x(t)) \quad (5)$$

Furthermore, under these conditions, there exists at least a viable equilibrium $(\bar{x}_1,\ldots,\bar{x}_n,\bar{p})$.

Remark — Naturally, we can extend this basic result in many directions and relax some of the assumptions.

For instance, if we are not interested in the existence of an equilibrium, we can dispense of the convexity assumptions. In this case, we replace the tangent cone to a convex subset by the contingent cone $T_M(y)$ to any subset M at $y \in M$, defined as the subset of directions $v \in Y$ such that $\liminf_{h \to 0+} d_K(x+hv)/h = 0$. We say that M is *sleek* if the set-valued map $M \ni y \rightsquigarrow T_M(y)$ is lower semicontinuous.

We assume instead that

$$\begin{cases} i) & M = M - \mathbf{R}^l_+ \text{ is closed and sleek} \\ ii) & \forall\, i = 1, \ldots, n, \;\; L_i = L_i + \mathbf{R}^l_+ \text{ is closed and sleek} \\ iii) & \forall\, x \in K, \; \sum_{i=1}^n T_{L_i}(x_i) - T_M(\sum_{i=1}^n x_i) \; = \; Y \\ iv) & M \subset \underline{y} - \mathbf{R}^l_+ \;\; \& \;\; \forall\, i = 1, \ldots, n, \;\; L_i \subset \underline{x}_i + \mathbf{R}^l_+ \end{cases}$$

The first part of the theorem still holds true.

We observe also that condition (2)iv) is one among many which implies the compactness of K. Again, this compactness property is needed to obtain the existence of an equilibrium. For the first part of the theorem, we can relax it by assuming only that the functions $c_i : L_i \mapsto Y$ has linear growth and $C_i : L_i \mapsto \mathcal{L}(Y^\star, Y)$ is bounded. □

Hence viability of the system as well as the existence of an equilibrium follows from the nonemptiness of the images of the regulation map Π_M.

We thus have to characterize this property and/or provide sufficient conditions.

Proposition 1.2 *We posit the assumptions of Theorem 1.1. Then M is a viable available commodity domain*[7] *if*

$$\forall\, x \in K, \;\; \sup_{p \in S^l} \inf_{q \in P(x)} < p, \sum_{i=1}^n c_i(x_i, q) > \leq 0$$

In the case when the set-valued map P is the constant map S^l, a sufficient condition for the above property is the *collective instantaneous Walras law*

[7] A necessary and sufficient condition is

$$\forall\, x \in K, \quad \sup_{p \in N_M(\sum_{i=1}^n x_i)} \inf_{q \in P(x)} < p, \sum_{i=1}^n c_i(x_i, q) > \leq 0$$

where $N_M(y) := (T_M(y))^-$ denotes the *normal cone* to M at a point $y \in M$.
Since $M = M - \mathbf{R}^l_+$, we know that $N_M(\sum_{i=1}^n x_i) \subset \mathbf{R}^n_+$

$$\forall\, p \in S^l,\ \sum_{i=1}^{n} < p, c_i(x_i, p) > \leq\ 0$$

which itself can be decentralized by requiring the change functions c_i to obey the (individual) *instantaneous Walras law*

$$\forall\, p \in S^l,\ < p, c_i(x_i, p) > \leq\ 0$$

Indeed, we can interpret this property by saying that it is forbidden to spend more monetary units than earned in continuous transactions[8]. As we can see, the advantage of the Walras law is that it does not depend upon the set M of scarce resources, as long as it satisfies assumptions (2)i). Hence, the following corollary is the dynamical counterpart of the Arrow-Debreu theorem on the existence of an equilibrium (in the simple framework of an exchange economy):

Theorem 1.3 *We posit the assumptions (2) and (4) of Theorem 1.1. If the change functions c_i obey the instantaneous Walras law, then the economic system has viable allocations starting at any allocation x_0.*

Furthermore, under these conditions, there exists at least a viable equilibrium $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$.

Remark — When $P(\cdot)$ is no longer the constant map $P \equiv S^l$, we can assume that for all $x \in \prod_{i=1}^{n} L_i$, there exists a map $Q(x, \cdot) : S^l \mapsto P(x)$ satisfying the condition

$$\forall\, (x, p) \in \prod_{i=1}^{n} L_i \times S^l,\ < p, \sum_{i=1}^{n} c_i(x_i, Q(x, p)) > \leq 0$$

Then the viability condition holds true. □

[8]The Walras law implies that along solutions to the system of differential equations (1), we have $< p(t), x_i'(t) > \leq 0$, and thus, for all h small enough,

$$< p(t), x_i(t) > \leq < p(t), x_i(t-h) > +\varepsilon h$$

2 Slow Allocations

Since the subsets L_i and M have nonempty interiors (since $L_i = L_i + \mathbf{R}^l_+$ and $M = M - \mathbf{R}^l_+$), we can provide sufficient conditions for the regulation map Π_M to be lower semicontinuous.

Proposition 2.1 *We posit the assumptions of Theorem 1.1. We further suppose that*

$$\left\{ \begin{array}{ll} i) & P \text{ is lower semicontinuous} \\ ii) & \forall\, i = 1,\ldots,n,\ \forall\, x_i \in L_i,\ \forall\, p \in \operatorname{Im}(P), \\ & c_i(x_i,p) \in T_{L_i}(x_i) - \operatorname{Int}(\mathbf{R}^l_+) \end{array} \right. \tag{6}$$

If

$$\left\{ \begin{array}{l} \forall x \in K,\ \exists p \in P(x) \ \textit{such that} \\ \sum_{i=1}^n c_i(x_i,p) \in T_M(\sum_{i=1}^n x_i) - \operatorname{Int}(\mathbf{R}^l_+) \end{array} \right. \tag{7}$$

then the regulation map Π_M is lower semicontinuous.

We introduce now the minimal selection π°_M defined by

$$\pi^\circ_M(x) \in \Pi_M(x) \ \ \& \ \ \|\pi^\circ_M(x)\| = \inf_{v \in \Pi_M(x)} \|v\|$$

It is not continuous, but we still infer the existence of a "slow" viable allocation starting at x_0, i.e., a solution to the system of differential equations

$$x'_i(t) \ = \ c_i(x_i(t), \pi^\circ_M(x(t))) \ \ (i = 1,\ldots,n)$$

Theorem 2.2 *We posit the assumptions (2),(3),(4) of Theorem 1.1 and assumptions (6) and (7) of Proposition 2.1. Then, for all initial allocation $x_0 \in K$, there exists a slow viable allocation starting at x_0.*

We could as well derive other selection procedures of closed loop prices in the regulation map (for instance continuous closed-loop maps from the Michael Continuous Selection Theorem). But it is possible that a more realistic selection procedure operates on the *derivatives* of the price functions according to the inertia principle.

3 Evolution under Bounded Inflation

In order to obtain absolutely continuous price functions which regulate allocations of the economic system, we introduce a further restriction on some kind of inflation rate measured by $r(t) := \|p'(t)\|/\|p(t)\|$, by requiring that the inflation rate $r(\cdot)$ is bounded by a positive constant c.

Theorem 3.1 *We posit the following assumptions:*

$$\begin{cases} i) & \text{the subsets } M \text{ and } L_i \text{ are closed} \\ ii) & \text{the graph of } P \text{ is closed} \\ iii) & \text{the functions } c_i \text{ are continuous} \end{cases} \tag{8}$$

For any $c \geq 0$, there exists a largest closed graph regulation map $\Pi_M^c \subset \Pi_M$ having the following property:

For any initial allocation $x_0 \in K$ and any initial price $p_0 \in \Pi_M^c(x_0)$, there exists a viable solution $(x(\cdot), p(\cdot))$ to the system of differential inclusions

$$\begin{cases} i) & x_i'(t) = c_i(x_i(t), p(t)) \quad (i = 1, \ldots, n) \\ ii) & r(t) := \|p'(t)\|/\|p(t)\| \leq c \end{cases} \tag{9}$$

regulated by the law

$$p(t) \in \Pi_M^c(x(t))$$

Naturally, the above theorem does not state that the regulation maps Π_M^c are not empty, or that their domains coincide with the allocation set K. This explains why the assumptions are weaker than the ones of Theorem 1.1.

We observe right away that if $0 \leq c_1 \leq c_2$, we have the inclusions

$$\Pi_M^0 \subset \Pi_M^{c_1} \subset \Pi_M^{c_2} \subset \Pi_M$$

The regulation map Π_M^0 plays an interesting role regarding the inertia principle, since it is related to the evolution of the economic system under *constant price*.

Indeed, if we choose some initial allocation x_0 in the domain of the regulation map Π_M^0 assumed to be nonempty, and if we take $p_0 \in \Pi_M^0(x_0)$, the above theorem shows that there exists a solution to the system of differential equations

$$x_i'(t) \; = \; c_i(x_i(t), p_0) \quad (i = 1, \ldots, n)$$

which remains in the viability cell $\Pi_M^{0^{-1}}(p_0)$ associated to the punctuated equilibrium p_0. If the evolution of the price hits a punctuated equilibrium p_0, the solution may remain forever in the viability niche $\Pi_M^{0^{-1}}(p_0)$.

Naturally, any equilibrium price $\bar{p}$ associated to an equilibrium $(\bar{x}, \bar{p}) = (\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is a punctuated equilibrium and we have $\bar{x} \in \Pi_M^{0^{-1}}(\bar{p})$.

We want now to derive a differential inclusion which governs the evolution of the allocation-price pair.

The idea is then to differentiate the regulation law

$$\forall\, t \geq 0, \;\; p(t) \in \Pi_M^c(x(t))$$

4 Heavy Allocations

We first recall what is the *contingent derivative* of a set-valued map $\Pi : X \rightsquigarrow Y$ at a point (x, y) of its graph. It is a set-valued map $D\Pi(x, y) : X \rightsquigarrow Y$ whose graph is the contingent cone to the graph of Π at (x, y). One can check that $v \in D\Pi(x, y)(u)$ if and only if

$$\liminf_{h \to 0+, u' \to u} d(v, \frac{\Pi(x + hu') - y}{h}) = 0$$

Therefore, since Theorem 3.1 provides absolutely continuous solutions, we can differentiate the regulation law

$$\forall\, t \geq 0, \;\; p(t) \in \Pi_M^c(x(t))$$

and obtain the following differential inclusion

$$p'(t) \; \in \; D\Pi_M^c(x(t), p(t))(c(x(t), p(t)))$$

(since $x'(t) = c(x(t), p(t))$), where we set for simplicity

$$c(x, p) \; := \; (c_1(x_1, p), \ldots, c_n(x_n, p))$$

From now on, it will be handy to introduce a notation for the right-hand side of the above differential inclusion: we set

$$\Pi_M^{c'}(x,p) \;:=\; D\Pi_M^c(x,p)(c(x,p)) \tag{10}$$

We observe that the smooth viable solutions to the system (9) are the solutions to the system of differential inclusions

$$\left\{ \begin{array}{lll} i) & x_i'(t) & = \; c_i(x_i(t),p(t)) \quad (i=1,\ldots,n) \\ ii) & p'(t) & \in \; \Pi_M^{c'}(x(t),p(t)) \end{array} \right. \tag{11}$$

Therefore, finding smooth allocations of our model with an inflation rate bounded by c amounts to solve the above system of differential inclusions.

We have to assume some regularity properties on the set-valued map $\Pi_M^{c'}(x,p)$ to go further, and in particular, to find heavy evolutions.

We say that our system is "dynamically regular" if

$$\left\{ \begin{array}{ll} i) & \text{the domain of } \Pi_M^c \text{ is equal to } K \\ ii) & \Pi_M^c \text{ is sleek} \\ iii) & \forall\,(x,p) \in \mathrm{Graph}(\Pi_M^c),\;\; \mathrm{Dom}(D\Pi_M^c(x,p)) \;=\; X \end{array} \right. \tag{12}$$

We know that the under these conditions, the set-valued map $\Pi_M^{c'}(x,p)$ *is lower semicontinuous with closed convex images.*

This allows to find continuous selections $\pi_M^{c'}(\cdot,\cdot)$ of this map thanks to Michael's Theorem, but our task is to select the element of minimal norm for obtaining an heavy allocation.

We denote by $\varpi_M^{c'} \in \Pi_M^{c'}(x,p)$ the element of minimal norm:

$$\varpi_M^{c'}(x,p) \in \Pi_M^{c'}(x,p) \;\;\&\;\; \|\varpi_M^{c'}(x,p)\| = \inf_{v\in\Pi_M^{c'}(x,p)} \|v\| \tag{13}$$

Although the map $\varpi_M^{c'}$ is not continuous (because $\Pi_M^{c'}(x,p)$ is not continuous, but only lower semicontinuous), we can still prove the existence of heavy allocation.

Theorem 4.1 *We posit he assumptions of Theorem 3.1 and we assume that the system is dynamically regular.*

Then, for any initial allocation $x_0 \in K$ *and any initial price* $p_0 \in \Pi^c_M(x_0)$*, there exists a viable heavy solution* $(x(\cdot), p(\cdot))$ *to the system of differential equations*

$$\begin{cases} i) & x_i'(t) = c_i(x_i(t), p(t)) \quad (i = 1, \ldots, n) \\ ii) & p'(t) = \varpi^{c'}_M(x(t), p(t)) \end{cases} \tag{14}$$

Naturally, much work remains to be done to check that a system is dynamically regular.

It is interesting to consider the *viability niches*

$$N^c_M(p) \ := \ \{ \ x \mid 0 \in \Pi^{c'}_M(x, p) \ \}$$

The heavy allocation $x(\cdot)$ starting at some $x_0 \in N^c_M(p_0)$ will be regulated by the constant control p_0 as long as $x(t)$ remains in $N^c_M(p_0)$. If $x_0 \in N^0_M(p_0)$, then it will remain in this viability niche for ever.

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