WORKING PAPER

ON A UNIFIED FRAMEWORK FOR DETERMINISTIC & STOCHASTIC TREATMENT OF IDENTIFICATION PROBLEMS

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January 1989
WP-89-013

IIASA
International Institute for Applied Systems Analysis

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Foreword

This paper deals with the conventional problem of identifying a matrix parameter on the basis of observations corrupted by an uncertainty in the measurements. Recalling two basic approaches to this problem – the stochastic scheme when the error in observation is treated as a gaussian noise and the deterministic approach with only a set-membership description of the unknown variables, the paper indicates the connections and interactions in the techniques involved in the respective solutions.

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1. INTRODUCTION

The applications of elementary identification theory have indicated the relevance of two basic approaches to the problem: the conventional statistical approach with measurement noise modelled by probabilistic techniques such as gaussian or other types of noise [1-6] and the approach based on guaranteed estimates with undefined parameters taken to be unknown but bounded and with a set-membership description of the estimates [7-12].

The first problem is resolved by conventional statistical techniques while the second model formally requires the application of elementary set-valued calculus and nonlinear analysis [13, 14]. It is shown however that the solutions to these problems may be treated within a common framework - the problem of identification under statistical uncertainty with measurement noise taken for example to be gaussian with unknown but bounded mean values [15, 16]. The statistical solution (with an additional extremal procedure) may then be used to solve the deterministic problem (§ § 8,9) [16, 17]. On the other hand, the deterministic solution will be consistent if applied to certain types of statistical models (§ 5) [17-19]. A sequential and a multistage ellipsoidal approximation scheme may then be formally applied for ensuring numerical results.

The problems under discussion are related in general to nonquadratic constraints on the unknowns.

2. NOTATION

Here we list some conventional notations adopted in this paper:

$\mathbf{R}^n$ will stand for the n-dimensional vector space, while $\mathbf{R}^{m \times n}$ - for the space of $m \times n$ - dimensional matrices, I_n will be the unit matrix of dimension n, $A \otimes B$ - the *Kronecker product* of matrices A , B, so that $(A \otimes B)$ will be the matrix of the form

$$\begin{pmatrix} a_{11}B\,, & \cdots, & a_{1n}B \\ \cdots & \cdots & \cdots \\ a_{n1}B\,, & \cdots, & a_{nn}B \end{pmatrix}$$

The prime will stand for the transpose and $\bar{A}$ - for an mn - dimensional vector obtained by stacking the matrix $A = \{a^{(1)}, \ldots, a^{(n)}\}$, with columns $a^{(i)} \in \mathbf{R}^m$ $(a_j^{(i)} = a_{ij})$, so that $a_{(i-1)m+j} = a_j^{(i)}$, $(i = 1, \ldots, n)$, $(j = 1, \ldots, m)$, or in other terms

$$\bar{A} = \sum_{i=1}^{n} (e^{(i)} \otimes (A\ e^{(i)}))$$

where $e^{(i)}$ is a unit orth within $\mathbf{R}^n$ $(e_j^{(i)} = \delta_{ij}$, with δ_{ij} the Kronecker delta : $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j)$.

If $\mathbf{C} = \{C\}$ is a set of $(m \times n)$-matrices C, then $\bar{\mathbf{C}}$ will stand for the respective set of mn-vectors $\bar{C} : \bar{\mathbf{C}} = \{\bar{C}\}$.

The few basic operations used in this paper are as follows:

If $<A\ , B> = tr\ AB'$ is the *inner product of matrices* $A\ , B \in \mathbf{R}^{m \times n}$ and $(p\ , q)$ - the *inner product* of vectors $p\ , q \in \mathbf{R}^n$, then for $x \in \mathbf{R}^n\ , y \in \mathbf{R}^m$ we have

$$y \otimes x' = yx' \in \mathbf{R}^{m \times n}$$
$$<A\ , y \otimes x'> = (A\ x\ , y) \tag{2.1}$$

Other matrix equalities used here are

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

$(A\ , B$ are $n \times n$ dimensional and their determinants $|\ A\ | \neq 0\ , |\ B\ | \neq 0)$

$$(A \otimes B)' = A' \otimes B'$$
$$(A \otimes B)\ \bar{K} = \overline{BKA'} \tag{2.2}$$

A sequence of integers $i = k, \ldots, s$ will be $[k\ , s]$. A finite sequence of vectors $\{\xi(i) : i = k, \ldots, s\}$ will be denoted as $\xi\ [k\ , s]$, while an infinite one $\{\xi(i)\ , i = s, \ldots, \infty\}$ as $\xi\ [s\ ;\ \cdot]$ with $\xi\ [1\ , \cdot] = \xi[\cdot]$. Similar notations will be used for sequences of sets. For example $R[k\ , s]$ will stand for a sequence of sets $R(i)\ , k \leq i \leq s$.

Symbols *conv* $\mathbf{R}^n$ and *co* $\mathbf{R}^n$ will denote the varieties of all *convex compact* and *closed convex* subsets of $\mathbf{R}^n$ respectively, and

$$\rho(\ell\ |\ Q) = \sup\ \{(\ell\ , q)\ |\ q \in Q\}$$

will be the *support function* of set $Q \subseteq \mathbf{R}^n$.

With $Q \in conv\ \mathbf{R}^n$ the operation of *sup* in the definition of $\rho(\ell \mid Q)$ may be substituted for *max*.

$$S_r(x_0) = \{x : ||\ x - x_0\ || \leq r\ ;\ x\ ,\ x_0 \in \mathbf{R}^n\}$$

will denote the *Euclidean ball* with center x_0 and radius r, $(||\ x\ || = (x\ ,\ x)^{1/2})$, while $h(P\ ,\ Q)$ will stand for the *Hausdorff distance* between sets $P\ ,\ Q \in conv\ \mathbf{R}^n$. Namely

$$h(P\ ,\ Q) = \min\ \{\ r : P \subseteq Q + S_r(0)\ ,\ \ Q \subseteq P + S_r(0)\ \}\ .$$

The "time interval" is denoted as $\{1\ ,\ldots,\ N\} = T_N$

3. THE IDENTIFICATION PROBLEM: A DETERMINISTIC MODEL

Consider a system

$$y(k) = C\ p(k) + v(k) \tag{3.1}$$
$$k \in T_s$$

where $y(k)$ is the *available measurement*, $p(k)$ is a *given input*, C is the *matrix parameter to be identified* and $v(k)$ is the *unknown disturbance*. We further assume $p \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, hence $v \in \mathbf{R}^m$, $C \in \mathbf{R}^{m \times n}$.

The available additional information on C, $\xi\ [1\ ,\ s]$ is given through *geometrical restrictions* on these values which are taken to be specified in advance. These are

$$C \in \mathbf{C}_0\ ,\ v(k) \in Q(k) \tag{3.2}$$

where $\mathbf{C}_0$, $Q(k)$ are assumed to be convex and compact in $\mathbf{R}^{m \times n}$ and $\mathbf{R}^m$ respectively.

With measurement $y[1\ ,\ s]$ given, the aim of the solution will be to find the set of all pairs $\varsigma[1\ ,\ s] = \{C, v[1,s]\}$ consistent with (3.1), (3.2) and with given $y\ [1\ ,\ s]$. More precisely the solution will be given through the notion of *the informational domain.*

Definition 3.1. The informational domain $\mathbf{C}[s]$ *consistent with measurement* $y[1\ ,\ s]$ *and restriction (3.2) will be defined as the set of all matrices* C *for each of which there exists a corresponding sequence* $v[1\ ,\ s]$ *such that the pair* $\varsigma[1\ ,\ s] = \{C\ ,\ v[1\ ,\ s]\}$ *satisfies* *both restriction (3.2) and equation (3.1) (for the given* $y[1\ ,\ s]$ *).*

The idea of the solution of the estimation problem is to find the set $\mathbf{C}[s]$ of all the possible values of C each of which (together with an adequate $v[1\ ,\ s]$) could generate the given measurement sequence $y[1\ ,\ s]$.

It is obvious that set $\mathbf{C}[s]$ now contains the unknown actual value $C = C^{\circ}$ which is to be estimated.

If s varies and even $s \to \infty$ it makes sense to consider the *evolution* of $\mathbf{C}[s]$ and its *asymptotic behaviour* in which case the estimation process may turn to be *consistent*, i.e.

$$\lim_{s \to \infty} \mathbf{C}[s] = \{C^{\circ}\} \tag{3.3}$$

The convergence here is understood in the sense that

$$\lim_{s \to \infty} h\ (\mathbf{C}[s]\ ,\ C^{\circ}) = 0$$

where C° is a singleton in $\mathbf{R}^{m \times n}$.

In some particular cases the equality (3.3) may be achieved in a finite number s_0 of stages s when for example

$$\mathbf{C}[s_0] = C^{\circ},\ s_0 > 1\ .$$

4. THE INFORMATIONAL DOMAIN

Returning to equation (3.1) the *informational domain* $\mathbf{C}[s]$. Using standard techniques of convex analysis and matrix algebra we come to the following sequence of operations.

The system equations (3.1), (3.2) may be transformed into

$$y(k) \in (p'(k) \otimes I_m)\bar{C} + Q(k)$$

since $I_m\ C\ p = (p' \otimes I_m)\ \bar{C}$ according to (2.2).

The set $\mathbf{C}[s]$ will then consist of all matrices C such that for every $k \in [1\ ,\ s]$ we have

$$\psi'(k)(p'(k) \otimes I_m)\ \bar{C} \leq (\psi(k)\ ,\ y(k)) + \rho(\psi(k) \mid -\ Q(k))\ , \tag{4.1}$$

together with

$$(\bar{\wedge}\ ,\ \bar{C}) \leq \rho(\bar{\wedge} \mid \mathbf{C}_0) \tag{4.2}$$

for any $\psi(k) \in \mathbf{R}^m$, $\bar{\wedge} \in \mathbf{R}^{mn}$. This leads to the inequality

$$\sum_{k=1}^{s} \psi'(k)(p'(k) \otimes I_m)\bar{C} + (\bar{\wedge}\ ,\ \bar{C}) \leq$$

$$\leq \sum_{k=1}^{s} \{(\psi(k)\ ,\ y(k)) + \rho(\psi(k) \mid -\ Q(k))\} + \rho(\bar{\wedge} \mid \bar{\mathbf{C}}_0)$$

for any $\psi(k) \in \mathbf{R}^m$, $\bar{\wedge} \in \mathbf{R}^{mn}$. Therefore, with $\bar{\wedge} \in \mathbf{R}^{mn}$ given we have

$$(\bar{\wedge}\ ,\ \bar{C}) \leq \rho(\bar{\wedge} - \sum_{k=1}^{s} (p(k) \otimes I_m)\psi(k) \mid \bar{\mathbf{C}}_o) + \tag{4.3}$$

$$+ \sum_{k=1}^{s} ((\psi(k)\ ,\ y(k)) + \rho(\psi(k) \mid -\ Q(k)))$$

For an element $C \in \mathbf{C}[s]$ it is necessary and sufficient that relation (4.3) is true for any $\psi(k) \in \mathbf{R}^m$, $k \in [1\ ,\ s]$.

Hence we come to the following assertion.

Lemma 4.1. The informational domain $\mathbf{C}[s]$ *consistent with measurement* $y[1\ ,\ s]$ *and with restrictions (3.1), (3.2) is defined by the following support function.*

$$\rho(\wedge \mid \mathbf{C}\ [s]) = f(\wedge) \tag{4.4}$$

where

$$f(\wedge) = inf\ \{\rho(\bar{\wedge} - \sum_{k=1}^{s} (p(k) \otimes I_m)\ \psi(k) \mid \mathbf{C}_0) +$$

$$+ \sum_{k=1}^{s} \{\psi'(k)\ y(k) + \rho(\psi(k) \mid -Q(k))\} \mid \psi(k) \in \mathbf{R}^m\ ,\ k = [1\ ,\ s]\ \}$$

The proof of Lemma 4.1 follows from (4.3) and from the fact that $f(\wedge)$ is a convex, positively homogeneous function, [14].

A special case arrives when there is no information on C at all and therefore $\mathbf{C}_0 = \mathbf{R}^{m \times n}$. Following the previous schemes we come to

Lemma 4.2. Under restrictions (3.2), $\mathbf{C}_0 = \mathbf{R}^{m \times n}$, *the set* $\mathbf{C}[s]$ *is given by the support function*

$$\rho(\bar{\wedge} \mid \bar{\mathbf{C}}[s]) = \tag{4.5}$$
$$= inf \left\{ \sum_{k=1}^{s} \{ \rho(-\psi(k) \mid Q(k)) + \psi'(k)\, y(k) \} \right\}$$

over all vectors $\psi(k)$ *that satisfy*

$$\sum_{k=1}^{s} \psi'(k)\, (p'(k) \otimes I_m) = \bar{\wedge}' \tag{4.6}$$

A question may however arise whether in the last case the set $\mathbf{C}[s]$ is bounded.

Lemma 4.3. Suppose $\mathbf{C}_0 = \mathbf{R}^{m \times n}$ *and rank* $P(s) = n$ *for the matrix* $P(s)=\{p(1), \ldots, p(s)\}$. *Then the set* $\mathbf{C}[s]$ *is bounded.*

Taking equation (4.6) it is possible to solve it in the form

$$\psi(k) = (p'(k) \otimes I_m)\, (W[s] \otimes I_m)^{-1}\, \bar{\wedge} \tag{4.7}$$

where

$$W[s] = \sum_{k=1}^{s} (p(k) \otimes p'(k))$$

Indeed (4.6) may be transposed into

$$\sum_{k=1}^{s} (p(k) \otimes I_m)\psi(k) = \bar{\wedge} \tag{4.8}$$

and the solution may be sought for in the form

$$\psi(k) = (p'(k) \otimes I_m)\ell \tag{4.9}$$

In view of (4.6) this yields equation

$$(W[s] \otimes I_m)\ell = \bar{\wedge} \tag{4.10}$$

where the matrix $W[s]$ is invertible (the latter condition is ensured by the condition of $P(s)$. Equations (4.8)–(4.10) produce the solution (4.7).

Substituting $\psi(k)$ of (4.7) into (4.5) it is possible to observe that the support function $\rho(\wedge \mid \mathbf{C}[s])$ is equibounded in $\bar{\wedge}$ over all $\bar{\wedge} \in \mathbf{S}_1^{mn}(0)$ where $\mathbf{S}_1^{mn}(0)$ is a unit ball in

$\mathbf{R}^{mn}$. This proves the boundedness of $\mathbf{C}[s]$.

5. RECURRENCE EQUATIONS AND CONSISTENCY CONDITIONS

The next step will be to derive recurrence evolution equations of the set $\mathbf{C}[s]$.

Starting with relation (4.3), substitute

$$\psi'(k) = \bar{\Lambda}' \, M(k)$$

where $M(k) \in \mathbf{R}^{mn \times m}$, $1 \leq k \leq s$.

Then (4.3) will be transformed into the following inequality

$$(\bar{\Lambda} , \bar{C}) \leq \rho(\bar{\Lambda} \mid (I_{mn} - \sum_{k=1}^{s} M(k)(p'(k) \otimes I_m))\bar{\mathbf{C}}_0) + \qquad (5.1)$$

$$+ \sum_{k=1}^{s} \{(\bar{\Lambda} , M(k) \, y(k)) + \rho(\bar{\Lambda} \mid M(k)(-Q(k)))\}$$

Denote the sequence of matrices $M(k) \in \mathbf{R}^{mn \times m}$, $k \in [1,\ldots,s]$ as $M[1 , s]$.

Lemma 5.1 In order that $C \in \mathbf{C}[s]$ it is necessary and sufficient that (5.1) would hold for any $\bar{\Lambda} \in \mathbf{R}^{mn}$, and any sequence $M[1 , s] \in \mathbf{M}[1 , s]$.

The proof is obvious from (4.3), (5.1) and Lemma 4.1. Hence in view of the properties of support functions for convex sets we come to the following assertion.

Lemma 5.2 In order that the inclusion

$$C \in \mathbf{C}[s]$$

would be true it is necessary and sufficient that

$$\bar{C} \in \mathbf{C}(s , \bar{\mathbf{C}}_0 , M[1 , s])$$

for any sequence $M[1 , s] \in \mathbf{M}[1 , s]$ where

$$\mathbf{C}(s , \bar{\mathbf{C}}_0 , M[1 , s]) = (I_{mn} - \sum_{k=1}^{s} M(k) \, (p'(k) \otimes I_m)) \, \bar{\mathbf{C}}_0 +$$

$$+ \sum_{k=1}^{s} M(k) \, (y(k) - Q(k))$$

From Lemma 5.2 it now follows

Lemma 5.3. The set $\mathbf{C}[s]$ *may be defined through the equality*

$$\bar{\mathbf{C}}[s] = \bigcap \{ \mathbf{C}(s , \bar{\mathbf{C}}_0 , M[1 , s]) \mid M[1 ,s] \in \mathbf{M}[1 , s] \}$$

In a similar way, assuming the process starts from set $C[s]$ at instant s, we have

$$\bar{\mathbf{C}}[s+1] \subseteq (I_{mn} - M(s+1)(p'(s+1) \otimes I_m))\, \bar{\mathbf{C}}[s] + \\ + M(s+1)(y(s+1) - Q(s+1)) = \mathbf{C}(s+1 , \bar{\mathbf{C}}[s] , M(s+1)) \tag{5.2}$$

and that we have

$$\bar{\mathbf{C}}[s+1] \subseteq \mathbf{C}(s+1 , \bar{\mathbf{C}}[s] , M(s+1))$$

for any $M(s+1) \in \mathbf{R}^{mn \times m}$ and further on

$$\bar{\mathbf{C}}[s+1] = \bigcap \{ \mathbf{C}(s+1 , \bar{\mathbf{C}}[s] , M) \mid M \in \mathbf{R}^{mn \times m} \} \tag{5.3}$$

This allows us to formulate

Theorem 5.1 The set $\mathrm{C}[s]$ *satisfies the recurrence inclusion*

$$\bar{\mathbf{C}}\,[s+1] \subseteq \mathbf{C}(s+1 , \bar{\mathbf{C}}[s] , M), \mathrm{C}[0] = \mathrm{C}_0 \tag{5.4}$$

- whatever is the matrix $M \in \mathbf{R}^{mn \times m}$ *- and also the recurrence equation (5.3).*

The relations of the above allow to construct numerical schemes for approximating the solutions to the guaranteed identification problem.

Particularly, (5.4) may be decoupled into a variety of systems

$$\bar{\mathbf{C}}_M\,[s+1] \subseteq \mathbf{C}(s+1 , \bar{\mathbf{C}}_M[s] , M(s)) , \mathbf{C}[0] = \mathbf{C}_0 \tag{5.5}$$

each of which depends upon a given sequence $M[1 , s]$ of "decoupling parameters". It therefore makes sense to consider

$$\mathbf{C}_U\,[s] = \{\bigcap \mathbf{C}_M[s] \mid M[1 , s]\} \tag{5.6}$$

Obviously $\mathbf{C}\,[s] \subseteq \mathbf{C}_U\,[s]$

From the linearity of the right-hand side of (5.2) and the convexity of sets $\mathbf{C}_0$, $Q(s)$ it follows that actually $\mathbf{C}[s] = \mathbf{C}_U[s]$.

Lemma 5.4 The set $\mathbf{C}[s] = \mathbf{C}_U[s]$ *may be calculated through an intersection (5.6) of solutions* $\mathbf{C}_M[s]$ *to a variety of independent inclusions (5.5) parametrized by sequences* $M[1 , s]$.

This fact indicates that $\mathbf{C}[s]$ may be reached by *parallel computations* due to equations (5.5). The solution to each of these equations may further be substituted by approximative set-valued solutions with ellipsoidal or polyhedral values.

An important question to be studied is whether the estimation procedures given here may be consistent. It will be shown in the sequel that there exist certain classes of identification problems for which the answer to this question is affirmative.

We will discuss this problem assuming $\mathbf{C}_0 = \mathbf{R}^{m \times n}$. Then the support function $\rho(\wedge \mid \mathbf{C}[s])$ for set $\mathbf{C}[s]$ is given by (4.5), (4.6).

The measurement $y(k)$ may be presented as

$$y(k) = (p'(k) \otimes I_m)\, \bar{C}^* + v^*(k) \, , \, (k = 1 \, , \ldots, \, s) \tag{5.7}$$

where $\bar{C}^*$ is the actual vector to be identified, $v^*(k)$ is the unknown actual value of the disturbance.

Substituting (5.7) into (4.5), (4.6) we come to

$$\rho(\wedge \mid \bar{\mathbf{C}}[s]) = inf \left\{ \sum_{k=1}^{s} \rho(\psi(k) \mid v^*(k) - Q(k)) + \sum_{k=1}^{s} \psi'(k)(p'(k) \otimes I_m)\, \bar{C}^* \right\},$$

over all vectors $\psi(k)$ that satisfy

$$\psi[1 \, , \, s] \in \Psi[s \, , \wedge] \tag{5.8}$$

where

$$\Psi[s \, , \wedge] = \{\psi[1 \, , \, s] : \sum_{k=1}^{s} \psi'(k)(p'(k) \otimes I_m) = \bar{\wedge}' \}$$

This is equivalent to

$$\rho(\wedge \mid \bar{\mathbf{C}}[s]) = (\bar{\wedge}, \bar{C}^*) + \rho(\wedge \mid R^*[s]) \, ,$$

where

$$\rho(\bar{\wedge} \mid R^*[s]) =$$

$$= inf \left\{ \sum_{k=1}^{s} \rho(\psi(k) \mid v^*(k) - Q(k)) \mid \psi[1 \, , \, s] \in \Psi[s \, , \wedge] \right\} = \varphi(\wedge) \tag{5.9}$$

In other terms

$$\bar{\mathbf{C}}[s] \subseteq \bar{C}^* + R^*[s]$$

where $R^*[s]$ is the *error set* for the estimation process. The support function for $R^*[s]$ is given by (5.9).

Since $v^*(k) \in Q(k)$ we have

$$\rho(\bar{\wedge} \mid R^*[s]) \geq 0 \; , \; \forall \; \bar{\wedge} \in \mathbf{R}^{mn}$$

Hence every sequence $\psi^0 \; [1 \; , \; s] \in \Psi \; (s \; , \; \wedge)$ that yields

$$\sum_{k=1}^{s} \rho(\psi(k) \mid v^*(k) - Q(k)) = 0$$

will be a minimizing element for problem (5.9).

The estimation process will be consistent within the interval $[1 \; , \; s]$ if

$$R^*[s] = \{0\}$$

or, in other terms, if

$$\rho(\bar{\wedge} \mid R^*[s]) = 0 \; , \; \forall \; \bar{\wedge} \in \mathbf{R}^{mn} \; . \tag{5.10}$$

The proof of the following assertions may be found in [17] (see also [16, 18]). We will now indicate particular classes of problems when the inputs and the disturbances are such that they ensure the conditions for *consistency* to be fulfilled.

Condition 5.A

(i) *The disturbances $v^*(k)$ are such that they satisfy the equalities*

$$(v^*(k) \; , \; \psi^*(k)) = \rho(\psi^*(k) \mid Q(k))$$

for a certain r-periodic function $\psi^*(k)$ $(r \geq m)$ that yields

$$\text{Rank } \{\psi^*(1) \; , \ldots , \; \psi^*(r)\} = m \; .$$

(ii) *The input function $p(k)$ is q-periodic with $q \geq n + 1$*

Among the vectors $p(k)$, $(k = 1 \; , \ldots , \; q)$ one may select a simplicial basis in $\mathbf{R}^n$, i.e. for any $x \in \mathbf{R}^n$ there exists an array of numbers $\alpha_k \geq 0$ such that

$$x = \sum_{k=1}^{q} \alpha_k \, p(k)$$

(iii) *Numbers r and q are relative prime.*

Lemma 5.5 Under Condition 5.A the error set $R^*[s] = 0$ for $s \geq rq$.

Condition 5.B

(i) *function $p(k)$, $k = 1 , \ldots, \infty$, is periodic with period $q \leq n$; the matrix $W[q]$ is nonsingular.*

(ii) *the sequence $v(i)$ is formed of jointly independent random variables with identical nondegenerate probabilistic densities, concentrated on the set*

$$Q(k) \equiv Q \ , \ Q \in comp\ \mathbf{R}^m \ , \ int\ Q \neq \emptyset$$

Condition (ii) means in particular that for every convex compact subset $Q_\epsilon \subseteq Q$, ($Q_\epsilon \in comp\ \mathbf{R}^m$) of measure $\epsilon > 0$ the probability

$$P\{v(k) \in Q_\epsilon\} = \delta > 0 \ , \forall\ k \in [1 \ , \infty]$$

At the same time it will not be necessary for values of the distribution densities of the variables $v(i)$ to be known.

Lemma 5.6 Under Condition 5.B the relation

$$h(R^*[s] \ , \{0\}) \longrightarrow 0 \ , \ s \longrightarrow \infty$$

holds with probability 1.

The examples indicate two important classes of disturbances $v(k)$ where one consists of *periodic functions* and the other of *a sequence of equidistributed independent random variables.* In both cases one may ensure consistency of the identification process. However this requires some additional assumptions on the inputs $p(k)$. Basically this means that function $p(k)$ should be periodic and its "informational matrix" should be nondegenerate as indicated in the precise formulations, (see also [18, 19]). We shall now pass to the discussion of some statistical estimation schemes.

6. THE STANDARD STOCHASTIC MODEL WITH GIVEN STATISTICS

Consider a *linear regression model*

$$y(k) = C_* \, p(k) + \xi(k) \, , \, k \in T_N \, , \tag{6.1}$$

where $C_* \in \mathbf{R}^{m \times n}$, $\xi \in \mathbf{R}^m$ and C_* , $\xi(k)$ are random gaussian variables. Following (2.2) we have

$$y(k) = (p'(k) \otimes I_m) \, \bar{C}_* + \xi(k)$$

where the stacked vector $\bar{C}_* \in \mathbf{R}^{mn}$. The mean values for $\bar{C}_*$, $\xi(k)$ are taken to be C , $v(k)$:

$$E \, C_* = C \, , \, E \, \xi(k) = v(k) \tag{6.2}$$

and the respective covariance matrices to be L^{-1} and $N^{-1}(k)(L \in \mathbf{R}^{mn \times mn}$, $N(k) \in \mathbf{R}^{m \times m})$.

For a one-stage process

$$y = (p' \otimes I_m) \, \bar{C}_* + \xi$$

with measurement y, mean values C,v and covariances L , N being given, a standard calculation yields

$$E(\bar{C}_* \mid y \, , \, C,v) = \bar{C} + \mathbb{P}^{-1}(p \otimes I_m) \, N \, (y - Cp - v) \tag{6.3}$$

where

$$\mathbb{P} = L + (p \otimes I_m) N \, (p' \otimes I_m)$$

If one denotes

$$C[s] = E(C_* \mid y(s) \, , \, C[s-1])$$
$$(p(s) \otimes I_m) \, N(s) \, (p'(s) \otimes I_m) = P(s)$$
$$(p(s) \otimes I_m) = G(s)$$

then the formula (6.3) will lead to a recurrence equation

$$\bar{C}[s] = (I_{mn} - \mathbb{P}^{-1}(s) \, P(s)) \, \bar{C}[s-1] + \mathbb{P}^{-1}(s) \, G(s) N(s) (y(s) - v(s)) \tag{6.4}$$
$$\bar{C}[0] = \bar{C}$$

where the matrices $\mathbf{P}(s)$, $\mathbb{P}^{-1}(s)$ follow the equations

$$\begin{gathered}\mathbf{P}(s) = \mathbf{P}(s-1) + P(s) \; , \; \mathbb{P}(0) = L \\ \mathbb{P}^{-1}(s) = \mathbb{P}^{-1}(s-1) - \mathbb{P}^{-1}(s-1) G(s) \; K^{-1}(s) \; G'(s) \; \mathbb{P}^{-1}(s-1) \\ K(s) = N^{-1}(s) + G'(s) \, \mathbf{P}^{-1}(s-1) \; G(s)\end{gathered} \tag{6.5}$$

It is well known that $C[s]$ gives ***the best quadratic estimate*** for C_*. Namely

$$E\{ \| \, \bar{C}_* - \bar{C}[s] \, \|^2 \mid y[1,s] \; , \; C,v[1,s]\} \leq E\{\| \bar{C}_* - \bar{X} \, \|^2 \mid y[1,s] \; , \; C,v[1,s]\}$$

whatever is the vector $\bar{X} \in \mathbf{R}^{mn}$. In other words we come to

Lemma 6.1 The conditional mean value $C[s]$ for the estimate of C_* due to the linear-gaussian-quadratic model (6.1), (6.2) is given by relations (6.4), (6.5).

The given well-known relations may be used as a complementary tool for some further problems.

7. UNCERTAINTIES IN THE MEAN VALUES

Assume that in the standard model (6.1) the mean values C , $v(k)$ are unknown in advance and the only information on these is given by a ***set-membership constraint*** (3.2), namely

$$C \in \mathbf{C}_0 \; , \; v(k) \in Q(k) \tag{7.1}$$

with $\mathbf{C}_0$, $Q(k)$ convex and compact.

Assuming

$$\mathbf{C}_*[s] = \{\bigcup C[s] \mid C \in \mathbf{C}_0 \; ; \; v(k) \in Q(k) \; , \; k = 1 \, ,\ldots,s\}$$

we come to a recurrent equation with set-valued variable $\mathbf{C}_*[s]$. This is

$$\begin{gathered}\bar{\mathbf{C}}_*[s] = (I_{mn} - \mathbb{P}^{-1}(s) \; P(s)) \; \bar{\mathbf{C}}_*[s-1] + \\ + \mathbf{P}^{-1}(s) \; G(s) \; N(s) \; (y(s) - Q(s)) \; , \\ \bar{\mathbf{C}}_*[0] = \bar{\mathbf{C}}_0\end{gathered} \tag{7.2}$$

with $\mathbf{P}(s)$, $P(s)$, $G(s)$ as in (6.5).

Lemma 7.1 The set $\mathbf{C}_*[s]$ of conditional mean values of the estimates of $\mathbf{C}_*$ after s measurements satisfies equation (7.2).

Substituting

$$\mathbb{P}^{-1}(s)\ G(s) = R(s) \tag{7.3}$$

into (7.2) we obtain

$$\bar{\mathbf{C}}_*[s+1] = \bar{\mathbf{C}}_*(s+1\ ,\ \bar{\mathbf{C}}_*[s]\ ,\ R(s+1))\ , \tag{7.4}$$

where

$$\bar{\mathbf{C}}_*(s+1\ ,\ \bar{\mathbf{C}}_*[s]\ ,\ R(s+1)) = (I_{mn} - R(s+1)N(s+1)G'(s+1))\ \bar{\mathbf{C}}_*[s] + \\ + R(s+1)\ N(s+1)(y(s+1) - Q(s+1))$$

with support function

$$\rho(\ell \mid \bar{\mathbf{C}}_*[s+1]) = \rho(\ell \mid (I_{mn} - R(s+1)\ N(s+1)G'(s+1))\ \bar{\mathbf{C}}_*[s]) + \\ + \rho(\ell \mid R(s+1)\ N(s+1)\ (y(s+1) - Q(s+1)))$$

Due to a conventional matrix transformation given in (6.5), relation (7.4) may be also rewritten as

$$\bar{\mathbf{C}}_*[s+1] = (I_{mn} - \mathbb{P}^{-1}(s)\ G(s+1)\ K^{-1}(s+1)G'(s+1))\ \bar{\mathbf{C}}_*[s] + \\ + \mathbb{P}^{-1}(s)\ G(s+1)\ K^{-1}(s+1)(y(s+1) - Q(s+1))\ , \\ \bar{\mathbf{C}}_*[0] = \bar{\mathbf{C}}_0$$

or, taking the notation

$$\mathbb{P}^{-1}(s)\ G(s+1)\ K^{-1}(s+1) = S(s+1) \tag{7.5}$$

$$\bar{\mathbf{C}}_*[s+1] = (I_{mn} - S(s+1)\ G'(s+1))\bar{\mathbf{C}}_*[s] + S(s+1)\ (y(s+1) - Q(s+1))\ . \tag{7.6}$$

8. STOCHASTIC VERSUS DETERMINISTIC SCHEMES (THE ONE-STAGE CASE)

Let us compare the results of the identification procedure within the models (3.1) and (6.1). Suppose that in s stages the measurement $y[1\ ,\ s]$ is the same for both models and the restrictions (3.2), (7.1) are also the same. We recall however that in (3.1) the problem is *deterministic with set-membership bounds (3.2) on the unknown values of* C , $v(k)$, while in (7.1) it is *stochastic* with C , $v(k)$ being the unknown mean values for C_* , $\xi(k)$. The set-membership bound (7.2) on the latter is however the same as in (3.1).

Comparing (7.4) with (5.2) and taking $M(s+1) = R(s+1)$ we observe that

$$\bar{\mathbf{C}}[s+1] \subseteq \bar{\mathbf{C}}_*(s+1\ ,\ \bar{\mathbf{C}}[s]\ ,\ R(s+1)) \tag{8.1}$$

for any L , $N[1\ ,\ s+1]$ provided

$$\bar{\mathbf{C}}_*[s] \supseteq \bar{\mathbf{C}}[s]$$

Since $\mathbf{C}_0 = \mathbf{C}[0] = \mathbf{C}_*[0]$ the latter inclusion holds for any s , $K(s)$.

Therefore the following assertion is true.

Lemma 8.1 Assume that in the models (3.1), (6.1) the measurement $y[1\ ,\ s]$ *is the same and the restrictions (3.2), (7.1) do coincide. Then the set*

$$\mathbf{C}[s] \subseteq \mathbf{C}_*[s] \tag{8.2}$$

whatever is the realization $\mathbf{C}_*[s]$ *generated by equation (7.4) with any* L , $N[1\ ,\ s]$.

As it was indicated in (5.3), with *set* $\mathbf{C}[s] = \mathbf{W}$ given, the set $\mathbf{C}[s+1] = \mathbf{C}(s+1\ ,\ s\ ,\ \mathbf{W})$ for the system

$$y(s+1) \in (p'(s+1) \otimes I_m)\ \bar{C} + Q(s+1)\ ,\ \bar{C} \in \bar{\mathbf{W}}$$

may be given as

$$\mathbf{C}[s+1] = \mathbf{C}(s+1\ ,\ s\ ,\ \mathbf{W}) = \{\bigcap \mathbf{C}(s+1\ ,\ \mathbf{W}\ ,\ M) \mid M \in \mathbf{R}^{mn \times m}\} \tag{8.3}$$

On the other hand, due to (5.2) (8.1) we have

$$\mathbf{C}[s+1] \subseteq \mathbf{C}_*(s+1\ ,\ \mathbf{W}\ ,\ R) \tag{8.4}$$

for any $R \in \mathbf{R}$ where $\mathbf{R}$ is the set of matrices of the special structure given in (7.3) or due to (5.2), (8.1), (7.6).

$$\mathbf{C}[s+1] \subseteq \mathbf{C}_*(s+1\ ,\ \mathbf{W}\ ,\ S) \tag{8.5}$$

for any $S \in \mathbf{S}$ where $\mathbf{S}$ is the set of matrices of structure (7.5). Then a question does arise whether (8.4), (8.5) may be transformed into equalities

$$\mathbf{C}[s+1] = \{\bigcap C_*(s+1\ ,\ \mathbf{W}\ ,\ M) \mid M\} \tag{8.6}$$

over all $M \in \mathbf{R}$ or over all $M \in \mathbf{S}$. The structure of the problem will be shown to be such that relation (8.6) would be already true with the intersection taken only over the subclass $\mathbf{R}$ or $\mathbf{S}$ rather than over the whole space $\mathbf{R}^{mn \times m}$.

We shall prove that (8.6) is true with the intersection taken over all $M \in \mathbf{S}$. Assume $\mathbf{C}[s]$ for the model (3.1), (3.2) to be given.

Let us consider a complementary model taken in the form

$$y(s+1) = (p'(s+1) \otimes I_m)\ \bar{C} + \xi(s+1) + v(s+1) \tag{8.7}$$

with unknown deterministic variable

$$v(s+1) \in Q(s+1)$$

and with

$$E\ C \in \mathbf{C}[s]\ ;\ E\ \xi(s+1) = 0\ ,$$
$$E\ (\bar{C} - E\bar{C})\ (\bar{C} - E\bar{C})' = L^{-1}\ ;\ E\ \xi(s+1)\ \xi'(s+1) = N^{-1}(s+1)\ ,\ L > 0\ ,\ N(s+1) > 0\ ,\ \forall s\ .$$

With $y(s+1)$ being given, consider a one-stage process due to

$$\bar{\mathbf{C}}_*[s+1] = \bar{\mathbf{C}}_*(s+1\ ,\ \mathbf{C}[s]\ ,\ S)$$

with set $\mathbf{S}$ consisting of all the matrices S of the form

$$S = L^{-1}\ G(s+1)(N^{-1} + G'(s+1)\ L^{-1}\ G(s+1))^{-1}$$

and with N , L arbitrary.

Lemma 8.2. The following equality is true

$$\mathbf{C}[s+1] = \{\bigcap \bar{\mathbf{C}}_*(s+1\ ,\ \mathbf{C}[s]\ ,\ S) \mid S \in \mathbf{S}\}\ . \tag{8.8}$$

Before proving this assertion we introduce several additional propositions.

Lemma 8.3. With $L = I_{mn}$, $N^{-1}(s+1) = \alpha I_m$, $\alpha > 0$, $p(s+1) \neq 0$ we have

$$(p'(s+1) \otimes I_m)\ L^{-1}(p(s+1) \otimes I_m)\ K^{-1}(s+1) \longrightarrow I_m\ ,\ (\alpha \longrightarrow 0) \tag{a}$$

$$K^{-1}(s+1) \longrightarrow 0\ (\alpha \longrightarrow \infty) \tag{b}$$

Indeed, taking the given values for L , $N^{-1}(s+1)$ and seeing that

$$(p'(s+1) \otimes I_m)\ (p(s+1) \otimes I_m) = p'(s+1)\ p(s+1)\ I_m\ ,$$

we come to the relation

$$(p'(s+1) \otimes I_m)\ L^{-1}\ (p(s+1) \otimes I_m)\ K^{-1}(s+1) =$$
$$= p'(s+1)\ p(s+1)\ (\alpha\ I_m + p'(s+1)\ p(s+1)\ I_m)^{-1} = \varphi(\alpha)$$

$$\{\lim \varphi(\alpha) \mid \alpha \longrightarrow 0\} = I_m$$

this proves the assertion (a) of the Lemma. The proof of assertion (b) is obvious.

Lemma 8.4. In order that

$$\bar{C} \in \bar{\mathbf{C}}[s+1] \tag{8.9}$$

it is necessary and sufficient that the inequality

$$(\ell\,,\,\bar{C}) \le \rho(\ell \mid \mathbf{C}_*[s+1]) = \tag{8.10}$$
$$= \rho(\ell \mid (I_{mn} - S(s+1)\,G'(s+1))\,\bar{\mathbf{C}}_*[s]) + \rho(\ell \mid S(s+1)(y(s+1) - Q(s+1))) =$$
$$= \Psi(\ell, S(s+1))$$

would be true for any values of $\mathbf{P}(s) = L > 0$, $N(s+1) = N > 0$.

In order to prove Lemma 8.4, recall that

$$\bar{\mathbf{C}}[s+1] = \bar{\mathbf{C}}[s] \bigcap \bar{\mathbf{C}}_y[s+1]$$

where $\mathbf{C}_y[s+1]$ is the set of matrices $\mathbf{C}$ that satisfy the inclusion

$$y(s+1) \in \mathbf{C}\,p(s+1) + Q(s+1)$$

for the given value of $y(s+1)$.

If $\mathbf{C} \in \mathbf{C}[s+1]$ then (8.10) is always true due to (8.5). Let us therefore prove that if (8.10) is true then (8.9) does hold. Suppose that (8.10) is true for L , N but (8.9) is false. Then there exists a vector $\bar{C}^*$ that satisfies (8.10) for any L , N but either $\bar{C}^* \bar{\in} \bar{\mathbf{C}}[s]$ or $\bar{C}^* \bar{\in} \bar{\mathbf{C}}_y[s+1]$.

If $\bar{C}^* \bar{\in} \bar{\mathbf{C}}_y[s+1]$, then one can specify a vector q^* such that

$$(-q^*\,,\,y) + (q^*\,,\,p'(s+1) \otimes I_m)\,\bar{C} \ge \rho(q^* \mid -Q) + \epsilon \tag{8.11}$$

for a certain $\epsilon > 0$. Taking

$$\ell^* = (p(s+1) \otimes I_m)\,q^*$$

and calculating the support function of

$$\mathbf{C}_*[s+1] = \mathbf{C}_*(s+1\,,\,\bar{\mathbf{C}}_*[s]\,,\,S(s+1))$$

in (7.6) (with $\mathbf{P}(s)$ substituted for an arbitrary $L > 0$) we have

$$(\ell^*\,,\,\bar{\mathbf{C}}^*) \le \rho(\ell^* \mid \bar{\mathbf{C}}_*[s+1]) = \tag{8.12}$$

$$= \rho(q^* \mid (p'(s+1) \otimes I_m) - (p'(s+1) \otimes I_m)\, L^{-1}\, (p(s+1) \otimes I_m)\, K^{-1}(s+1) \times$$
$$\times\, (p'(s+1) \otimes I_m))\, \mathbf{C}_*[s]) +$$
$$+\, \rho(q^* \mid (p'(s+1) \otimes I_m)\, L^{-1}(p(s+1) \otimes I_m)\, K^{-1}(s+1)(y(s+1) - Q(s+1)))$$
$$= \Phi(L\,,\, N)$$

Take

$$L = I_{mn}\,,\; N^{-1}(s+1) = \alpha I_m\,, \tag{8.13}$$

Substituting (8.13) into (8.12) and using the result of Lemma 8.3 (a) we come to the assertion that there exists for a given $\epsilon > 0$ a number $\alpha_0 > 0$ such that for $\alpha < \alpha_0$, we have

$$\mid \Phi(I_{mn}\,,\, \alpha^{-1}\, I_m) - ((q^*\,,\, y(s+1)) + \rho(q^* \mid -Q(s+1))) \mid \leq \frac{\epsilon}{2}$$

Hence, with substitution (8.13), for $\alpha \leq \alpha_0$ and for $\ell^* = (p(s+1) \otimes I_m) q^*$ we have

$$(q^*\,,\, p'(s+1) \otimes I_m)\; \bar{C} \leq \rho(q^* \mid y(s+1) - Q(s+1)) + \frac{\epsilon}{2}$$

This contradicts with (8.11). Therefore $\bar{C}^* \in \bar{\mathbf{C}}_y[s+1]$.

Let us now suppose that $\bar{C}^* \bar{\in} \bar{\mathbf{C}}[s]$. Then there exists an $\ell^0 \in \mathbf{R}^{mn}$ and a $\delta > 0$ such that

$$(\ell^0 \mid \bar{C}^*) \geq \rho(\ell^0 \mid \bar{\mathbf{C}}[s]) + \delta \tag{8.14}$$

Taking $L = I_{mn}\,,\, N^{-1}(s+1) = \alpha I_m$ we observe from Lemma 8.3(b) that $S(s+1) \longrightarrow 0\,,\, \alpha \longrightarrow \infty$ and therefore from (7.5), (8.10) it follows that there exists an α_1 such that with $\alpha > \alpha_1$ we have

$$(\ell^0\,,\, \bar{C}^*) \leq \rho(\ell^0 \mid \bar{C}_*[s]) + \frac{\delta}{2}$$

This contradicts with (8.14).

Hence if (8.10) is fulfilled for any $L > 0\,,\, N > 0$ then the inclusion (8.9) will also be true. Lemma 8.4 is therefore verified. Lemma 8.2 is now a direct consequence of Lemma 8.4. The result given by Lemma 8.4 may be used for *sequential estimates* in the identification process.

9. STOCHASTIC VERSUS DETERMINISTIC SCHEMES (THE MULTI-STAGE CASE)

It was shown in the previous section that in each stage of the identification process one may use a relation between the solution to the deterministic estimation problem and the solution to a related stochastic estimation scheme. This allows some sequential estimation procedures.

A similar property is however true for a *multistage scheme*. Namely, consider the model (3.1), (3.2) and the related complementary model (8.7). These could be reshaped to the form

$$\mathbf{y}(s) = \mathbf{T}(s)\ \bar{C} + \mathbf{V}(s) \tag{9.1}$$

for the deterministic system (3.1), (3.2) and

$$\mathbf{y}(s) = \mathbf{T}(s)\ \bar{C} + \mathbf{V}(s) + \Xi(s) \tag{9.2}$$

for the stochastic model (8.7).

Here

$$\mathbf{y}(s) = \begin{bmatrix} y(1) \\ \vdots \\ y(s) \end{bmatrix}, \quad \mathbf{T}(s) = \begin{bmatrix} p'(1) \otimes I_m \\ \vdots \\ p'(s) \otimes I_m \end{bmatrix},$$

$$\mathbf{V}(s) = \begin{bmatrix} v(1) \\ \vdots \\ v(s) \end{bmatrix}, \quad \Xi(s) = \begin{bmatrix} \xi(1) \\ \vdots \\ \xi(s) \end{bmatrix}.$$

The set-membership constraint is

$$\mathbf{V}(s) \in \mathbf{Q}(s)\ , \quad \mathbf{Q}(s) = \begin{bmatrix} Q(1) \\ \vdots \\ Q(s) \end{bmatrix},$$

$$E\ \bar{C} \in \bar{\mathbf{C}}[0], \ E\ \Xi(s) = 0$$

and the covariances are

$$E(\bar{C} - E\bar{C})\ (\bar{C} - E\ \bar{C})' = L^{-1}\ , \quad E\ \Xi(s)\ \Xi'(s) = \mathbb{N}^{-1}(s)$$

where $L > 0$, and the $ms \times ms$ matrix $\mathbb{N}(s) > 0$ is diagonal

$$\mathbb{N}(s) = \begin{bmatrix} N(1) & & 0 \\ 0 & \ddots & \\ & & N(s) \end{bmatrix}$$

(the variables $\xi(1),\ldots,\xi(s)$ are taken to be non correlated).

The result of Lemma 5.3 may now be reformulated in a form that corresponds to a one-stage problem (similar to (5.2)).

Lemma 9.1. The following inclusion is true

$$\mathbf{C}[s] \subseteq (I_{mn} - \mathbf{M}(s)\ \mathbf{T}(s))\ \bar{\mathbf{C}}_0 + \mathbf{M}(s)(\mathbf{y}(s) - \mathbf{Q}(s)) = \mathbf{R}(\mathbf{M}(s)) \tag{9.3}$$

for any $mn \times ms$-matrix

$$\mathbf{M}(s) = (M(1),\ldots,M(s)) \ .$$

The equality

$$\mathbf{C}[s] = \{\bigcap \mathbf{R}\ (\mathbf{M}(s)) \mid \mathbf{M}(s)\}$$

is true.

It is clear that (9.3) now coincides with the basic relation for Lemma 5.2.

On the other hand, considering the estimation of $\bar{C}$ through model (9.2), and applying a formula similar to (6.3) we come to the equality

$$\begin{aligned} &E(\bar{C}[s] \mid \mathbf{y}(s),\, E\ \bar{C}, \mathbf{V}(s)) = \\ &\quad = (I_{mn} - \mathbb{P}^{-1}\ \mathbf{T}'(s)\ \mathbb{N}\ \mathbf{T}(s))\ E\ \bar{C} + \mathbb{P}^{-1}\ \mathbf{T}'(s)\ \mathbb{N}\ (\mathbf{y}(s) - \mathbf{V}(s)) \end{aligned} \tag{9.4}$$

with $mn \times mn$-matrix

$$\mathbb{P} = L + \mathbf{T}'(s)\ \mathbb{N}\ \mathbf{T}(s) = \mathbb{P}^{-1}(s)$$

where $\mathbb{P}(s)$ was defined by (6.5).

Denoting

$$\bar{\mathbf{C}}_*[s] = \{\bigcup E(\bar{C}[s] \mid \mathbf{y}(s),\, E\ \bar{C}, \mathbf{V}(s)) \mid E\ \bar{C} \in \mathbf{C}_0,\, \mathbf{V}(s) \in \mathbf{Q}(s)\}$$

and making a transformation similar to (6.5) we observe

$$\bar{\mathbf{C}}_*[s] = (I_{mn} - \mathbf{S}\ \mathbf{T}(s))\bar{\mathbf{C}}_0 + \mathbf{S}(\mathbf{y}(s) - \mathbf{Q}(s)) \tag{9.5}$$

where

$$\begin{aligned} \mathbf{S} &= L^{-1}\ \mathbf{T}'(s)\ \mathbb{K}^{-1}(s)\ , \\ \mathbf{K}(s) &= \mathbb{N}^{-1}(s) + \mathbf{T}(s)\ L^{-1}\ \mathbf{T}'(s)\ . \end{aligned}$$

Clearly from (9.3), (9.5) we have

$$\mathbf{C}[s] \subseteq \mathbf{C}_*[s] \tag{9.6}$$

for any $\mathbf{S}$ derived through any pair $L > 0, \mathbb{N}(s) > 0$. Assuming that the sequence $N[1,s] = \{N(1), \ldots, N(s)\}$ is generated by the diagonal elements of $\mathbb{N}(s)$ and applying Lemma 8.2 to system (9.5), (9.6) we come to the assertion:

Lemma 9.2. Provided $\mathbf{y}(s)$ *is the same for both (9.1) and (9.2), the deterministic set*

$$\mathbf{C}[s] \subseteq \mathbf{C}_*[s]$$

for any pair $\{L, N[1,s]\}$ *that generates* $\mathbf{C}_*[s]$. *Moreover*

$$\mathbf{C}[s] = \{\bigcap \mathbf{C}_*[s] \mid L > 0 , N[1 , s] > 0\} . \tag{9.7}$$

Finally, a direct calculation shows that $\mathbf{C}_*[s]$ may also be achieved through the equation

$$\bar{\mathbf{C}}_*[s] = \Phi(s,1)\bar{\mathbf{C}}_0 + \sum_{i=1}^{s} \Phi(s , i + 1)\, S(i)\, (y(i) - Q(i)) \tag{9.8}$$

where

$$\Phi(s , s + 1) = I$$
$$\Phi(s , i) = \Phi(s , i + 1)\, (I_{mn} - S(i)G'(i)) \quad i = s , s - 1 ,\ldots, 1 .$$

Lemma 9.3. The expressions (9.5) and (9.8) are equivalent.

Proof. Starting with (9.4) we have

$$(9.4) = (I_{mn} - \mathbb{P}^{-1}(s) \sum_{i=1}^{s} P(i))\, \bar{\mathbf{C}}_0 + \mathbb{P}^{-1}(s) \sum_{i=1}^{s} G(i)\, N(i)(y(i) - Q(i)) \tag{9.9}$$

Suppose (9.9) and (9.8) are equivalent at stage k, i.e.

$$\Phi(k , 1) = I - \mathbb{P}^{-1}(k) \sum_{i=1}^{k} P(i)$$
$$\Phi(k , i + 1)S(i) = \mathbb{P}^{-1}(k)\, G(i)N(i)$$

Then from the relations (6.4), (6.5) and definition of Φ,

$$\begin{aligned}
\Phi(k + 1 , 1) &= (I_{mn} - S(k + 1)G'(k + 1))\, \Phi(k , 1) \\
&= (I_{mn} - \mathbb{P}^{-1}(k + 1)\, P(k + 1))\, (I_{mn} - \mathbb{P}^{-1}(k) \sum_{i=1}^{k} P(i)) \\
&= I_{mn} - \mathbb{P}^{-1}(k + 1)P(k + 1) - (I_{mn} - \mathbb{P}^{-1}(k+1)P(k+1))\, \mathbb{P}^{-1}(k)\sum_{i=1}^{k} P(i) \\
&= I_{mn} - \mathbb{P}^{-1}(k + 1) \sum_{i=1}^{k+1} P(i)
\end{aligned}$$

and

$$\Phi(k+1,\, i+1)S(i) = (I_{mn} - \mathbb{P}^{-1}(k+1)P(k+1))\Phi(k,i+1)S(i)$$
$$= \mathbf{P}^{-1}(k+1)G(i)N(i) \ .$$

This completes the proof.

In order to ensure numerical results one may apply an approximation technique. A convenient scheme is based on ellipsoidal approximations [11, 20, 21].

10. UNCERTAINTY IN MEAN VALUES: ELLIPSOIDAL APPROXIMATIONS (THE ONE-STAGE CASE)

With covariances L,$N[1,\cdot]$ given, the recursion (7.6)

$$\overline{\mathbf{C}}_*[s+1] = (I_{mn} - S(s+1)G'(s+1))\overline{\mathbf{C}}_*[s] + S(s+1)(y(s+1) - Q(s+1))$$

allows to be computed.

Assume that $\overline{\mathbf{C}}_*[s]$, $Q(s+1)$ are *ellipsoids* expressed by

$$\overline{\mathbf{C}}_*[s] = \{\bar{C}_*(s) \mid (\bar{C}_*(s) - \bar{C}_*^0(s))' \Sigma_1^{-1}(s)(\bar{C}_*(s) - \bar{C}_*^0(s)) \le 1\,,\, \bar{C}_*(s) \in \mathbf{R}^{mn}\} \quad (10.1)$$

$$Q(s) = \{v(s) \mid (v(s) - v^0(s))' \,\Sigma_2^{-1}\,(s)(v(s) - v^0(s)) \le 1,\, v(s) \in \mathbf{R}^m\} \ . \quad (10.2)$$

Then it is well known that the support functions of these sets are given by

$$\rho(\ell_1 \mid \overline{\mathbf{C}}_*[s]) = (\ell_1,\bar{C}_*^0(s)) + (\Sigma_1(s)\ell_1,\ell_1)^{1/2} \quad \ell_1 \in \mathbf{R}^{mn} \quad (10.3)$$

$$\rho(\ell_2 \mid Q(s)) = (\ell_2,v^0(s)) + (\Sigma_2(s)\ell_2,\ell_2)^{1/2} \quad \ell_2 \in \mathbf{R}^m \ . \quad (10.4)$$

And from (7.6),

$$\begin{aligned}\rho(\ell \mid \overline{\mathbf{C}}_*[s+1]) = {} & (\ell,A(s+1)\bar{C}_*^0(s)) + (A(s+1)\Sigma_1(s)A'(s+1)\ell,\ell)^{1/2} + \\ & + (\ell,S(s+1)(y(s+1) - v^0(s+1))) + \\ & + (S(s+1)\Sigma_2(s+1)S'(s+1)\ell,\ell)^{1/2}\end{aligned} \quad (10.5)$$

where

$$A(s+1) = I_{mn} - S(s+1)G'(s+1) \ . \quad (10.6)$$

Clearly the set $\overline{\mathbf{C}}_*[s+1]$ is not an ellipsoid.

We could then observe from relation (6.5), that $A(s+1)$ is nonsingular, and therefore that $A(s+1)\Sigma_1(s)A'(s+1) > 0$, if $\Sigma_1(s) > 0$.

On the other hand, $S(s+1)\Sigma_2(s)S'(s+1)$ turns to be singular due to the dimension of the respective matrices $(m < n)$. Therefore, we would have to consider the approximation of the Minkowski-sum of a nondegenerate and a degenerate ellipsoids.

For two given ellipsoids $E_1(\alpha_1,R_1)$, $E_2(\alpha_2,R_2)$ with support functions

$$\rho(\ell \mid E_1(\alpha_1,R_1)) = (\ell,\alpha_1) + (R_1\ell,\ell)^{1/2}, \quad R_1 > 0$$
$$\rho(\ell \mid E_2(\alpha_2,R_2)) = (\ell,\alpha_2) + (R_2\ell,\ell)^{1/2}, \quad R_2 \geq 0$$

define a new ellipsoid $E[x_1,x_2]$ with support function

$$\rho(\ell \mid E[x_1,x_2]) = (\ell,\alpha_1 + \alpha_2) + (R(x_1,x_2)\ell,\ell)^{1/2} \tag{10.7}$$

where $x_1,x_2 \in (0,\infty)$ and

$$R(x_1,x_2) = (x_1+x_2)\left[\frac{1}{x_1}R_1 + \frac{1}{x_2}R_2\right]. \tag{10.8}$$

Then we can find that $E[x_1,x_2]$ has the following properties:

Lemma 10.1. For any $x_1,x_2 \in (0,\infty)$,

$$E_1(\alpha_1,R_1) + E_2(\alpha_2,R_2) \subseteq E[x_1,x_2] \ .$$

This follows from

$$(R(x_1,x_2)\ell,\ell) = (R_1\ell,\ell) + (R_2\ell,\ell) + \frac{x_2}{x_1}(R_1\ell,\ell) + \frac{x_1}{x_2}(R_2\ell,\ell)$$

$$\geq (R_1\ell,\ell) + (R_2\ell,\ell) + 2(R_1\ell,\ell)^{1/2}(R_2\ell,\ell)^{1/2}$$

$$= ((R_1\ell,\ell)^{1/2} + (R_2\ell,\ell)^{1/2})^2 , \quad \forall \ell \in \mathbf{R}$$

Lemma 10.2. The equality

$$\rho(\ell \mid E_1(\alpha_1,R_1) + E_2(\alpha_2,R_2)) = \rho(\ell \mid \bigcap_{x_1,x_2} E[x_1,x_2]) = \inf_{x_1,x_2} \rho(\ell \mid E[x_1,x_2])$$

holds for any $\ell \in \mathbf{R}^{mn}$.

Proof. From Lemma 10.1 and an obvious inequality, it follows

$$\rho(\ell \mid E_1(\alpha_1,R_1) + E_2(\alpha_2,R_2)) \leq \rho(\ell \mid \bigcap_{x_1,x_2} E[x_1,x_2]) \leq \inf_{x_1,x_2} \rho(\ell \mid E[x_1,x_2]) \ .$$

Therefore it suffices to prove that this relation turns to an equality for any ℓ.

Since R_1 is nondegenerate, $(R_1\ell,\ell) \neq 0$ for $|\ell| \neq 0$. If $(R_2\ell,\ell) \neq 0$, the inequality in the proof of Lemma 10.1 turns to an equality with $x_i = (R_i\ell,\ell)$, $i = 1,2$ while with $(R_2\ell,\ell) = 0$ we have

$$\lim_{x_2 \to 0} (R(x_1,x_2)\ell,\ell) = (R_1\ell,\ell) \ .$$

This completes the proof.

The assertion of Lemma 10.2 means that the exact set

$$\mathbb{E} = E_1(\alpha_1,R_1) + E_2(\alpha_2,R_2) = \{\bigcap E(x_1,x_2) \mid x_1 > 0,\ x_2 > 0\}$$

could be obtained by the intersection of the bounding ellipsoids of the form $E[x_1,x_2]$, each of which contains no other ellipsoid that contains $\mathbb{E}$ and therefore is one of the minimal ellipsoids with respect to inclusion (or in other words, each is a Pareto-ellipsoid).

Hence we could 'select' one of the $E[x_1,x_2]$ which has a given optimality property while the optimal criterion φ should satisfy

$$\varphi(\Sigma_1) \leq \varphi(\Sigma_2) \ \text{ if } \ E_1(0,\Sigma_1) \subset E_2(0,\Sigma_2) \ .$$

A simple example occurs with $\varphi(\Sigma) = Tr[\Sigma]$ where $Tr[\Sigma]$ is the trace of Σ (the sum of semi–axes).

Lemma 10.3. The ellipsoid $E[x_1^*,x_2^*]$ that minimizes the function

$$f(x_1,x_2) = Tr[R(x_1,x_2)]$$

is generated by the values

$$x_i^* = Tr^{1/2}[R_i] \quad i = 1,2 \ .$$

Note that $\dfrac{\partial}{\partial x_i} f(x_1,x_2) = 0$, $i = 1,2$, yields

$$\frac{x_i^{*2}}{(\sum_i x_i^*)^2} = \frac{Tr[R_i]}{Tr[R(x_1,x_2)]}$$

and therefore

$$x_i^* = T_r^{1/2}[R_i] \ .$$

(The φ-optimal ellipsoid is tangential to the true set.)

Returning to (10.5), we obtain

Lemma 10.4. Suppose $\bar{\mathbf{C}}_*[s]$, $Q(s+1)$ *are defined by (10.1), (10.2). Then the bounding ellipsoid* $\bar{\mathbf{C}}_*^b[s+1]$ *of* $\bar{\mathbf{C}}_*[s+1]$ *with a minimal sum of semi-axes* $(TrR(x_1,x_2))$ *is given by*

$$\rho(\ell \mid \bar{\mathbf{C}}_*^b[s+1]) = (\ell,\bar{C}_*^0(s+1)) + (\Sigma_1^b(s+1)\ell,\ell)^{1/2} \tag{10.10}$$

where

$$\bar{C}_*^0(s+1) = A(s+1)\bar{C}_*^0(s) + S(s+1)(y(s+1) - v^0(s+1)) \tag{10.11}$$

$$\Sigma_1^b(s+1) = (x_1 + x_2)(\frac{1}{x_1}A(s+1)\Sigma_1(s)A'(s+1) + \tag{10.12}$$

$$+ \frac{1}{x_2}S(s+1)\Sigma_2(s+1)S'(s+1))$$

$$x_1 = Tr^{1/2}[A(s+1)\Sigma_1(s)A'(s+1)] \tag{10.13}$$

$$x_2 = Tr^{1/2}[S(s+1)\Sigma_2(s+1)S'(s+1)] \tag{10.14}$$

We can obtain a recursive scheme by defining $\Sigma_1(s+1) = \Sigma_1^b(s+1)$, but it should be noted that the error between the bounding ellipsoid and the true set would accumulate with the number of steps so that the obtained ellipsoid would be larger than a bounding ellipsoid of the true set after many recursions.

11. UNCERTAINTY IN MEAN VALUES: ELLIPSOIDAL APPROXIMATIONS (THE MULTISTAGE CASE)

The recursive scheme of the previous section is convenient to update the set, but the recursive approximation would yield a set estimate which may clearly be larger than the true set in φ-optimal sense. Here, we would consider the nonrecursive case when C is to be estimated at a certain fixed time s.

Denoting $\bar{\mathbf{C}}_*[s]$ with $\bar{\mathbf{C}}_0$ and $Q[1\ ,\ s]$ where $\bar{\mathbf{C}}_0$, $Q[1\ ,\ s]$ are ellipsoids defined by $\bar{\mathbf{C}}_0 = E_0(\bar{C}^0\ ,\ \Sigma_0)$, $Q(k) = E_k(v^0(k)\ ,\ \Sigma_2(k))$, $k = 1\ ,\ldots,\ s$, we obtain from (7.6)

$$\bar{\mathbf{C}}_*[s] = \Phi(s\ ,\ 1)\bar{\mathbf{C}}_0 + \sum_{i=1}^{s} \Phi(s\ ,\ i+1)\ S(i)\ (y(i) - Q(i)) \tag{11.1}$$

where

$$\Phi(s\ ,\ s+1) = I$$

$$\Phi(s\ ,\ i) = \Phi(s\ ,\ i+1)\ (I_{mn} - S(i)G'(i))\quad i = s\ ,\ s-1\ ,\ldots,\ 1\ .$$

Then (11.1) is a Minkowski-sum of $s+1$ ellipsoids, among which $\Phi(s\ ,\ 1)\ \bar{\mathbf{C}}_0$ is a nondegenerate ellipsoid. Consider the approximation of $\bar{\mathbf{C}}_*[s]$ by a bounding ellipsoid.

The case of the sum of two ellipsoids may now be extended to the sum of $s+1$-ellipsoids.

Lemma 11.1. Define $E[x_0\ ,\ldots,\ x_s]$ as an ellipsoid with support function

$$\rho(\ell \mid E[x_0,\ldots,x_s]) = (\ell, \Phi(s\ ,1)\bar{C}^0 + \sum_{i=1}^{s} \Phi(s\ ,i+1)\ S(i)\ (y(i) - v^0(i))) + \tag{11.2}$$
$$+\ (R(x_0\ ,\ldots,\ x_s)\ell\ ,\ \ell)^{1/2}$$

where $x_0\ ,\ldots,\ x_s \in (0\ ,\ \infty)$, and

$$R(x_0\ ,\ldots,\ x_s) = \sum_{j=0}^{s} x_j(\frac{1}{x_0}\ \Phi(s\ ,\ 1)\ \Sigma_0\ \Phi'(s\ ,\ 1) \tag{11.3}$$
$$+ \sum_{i=1}^{s} \frac{1}{x_i}\ \Phi(s\ ,\ i+1)\ S(i)\ \Sigma_2(i)\ S'(i)\ \Phi'(s\ ,\ i+1))\ .$$

Then for any sequence $x[0\ ,\ s]$

$$E[x_0\ ,\ldots,\ x_s] \supset \bar{\mathbf{C}}_*[s]\ . \tag{11.4}$$

Lemma 11.2. The equality

$$\inf_{x_0\ ,\ldots,\ x_s} \rho(\ell \mid E\ [x_0\ ,\ldots,\ x_s]) = \rho(\ell \mid \bar{\mathbf{C}}_*[s])$$

holds for any $\ell \in \mathbf{R}^{mn}$. Therefore,

$$\bigcap_{x_0\ ,\ldots,\ x_s} E[x_0\ ,\ldots,\ x_s] = \bar{\mathbf{C}}_*[s]\ . \tag{11.5}$$

The proof is similar to that of Lemma 10.2.

Lemma 11.3. The bounding ellipsoid which is tangential to $\bar{\mathbf{C}}_*[s]$ to the direction ℓ is given by $E[x_0(\ell)\ ,\ldots,\ x_s(\ell)]$ where

$$x_0(\ell) = (\Phi(s\ ,\ 1)\ \Sigma_0\ \Phi'(s\ ,\ 1)\ \ell\ ,\ \ell)^{1/2} \tag{11.6}$$

$$x_i(\ell) = (\Phi(s\ ,\ i+1)\ S(i)\ \Sigma_2(i)\ S'(i)\ \Phi'(s\ ,\ i+1)\ \ell\ ,\ \ell)^{1/2} \qquad i = 1\ ,\ldots,\ s \tag{11.7}$$

except for ℓ^* which yields $x_i(\ell^*) = 0$.

Lemma 11.4. The bounding ellipsoid for $\bar{\mathbf{C}}_*[s]$ *which has the minimal sum of the semi-axes is given by* $E[x_0^*, \ldots, x_s^*]$ *where*

$$x_0^* = Tr^{1/2}\,[\Phi(s\,,1)\,\Sigma_0\,\Phi'(s\,,1)] \tag{11.8}$$

$$x_i^* = Tr^{1/2}\,[\Phi(s\,,i+1)\,S(i)\,\Sigma_2(i)\,S'(i)\,\Phi'(s\,,i+1)] \quad i = 1\,,\ldots, s\,. \tag{11.9}$$

12. APPROXIMATION OF THE DETERMINISTIC SOLUTION

In order to approximate the deterministic solution $\mathbf{C}[s]$ one may apply formula (9.6) so that

$$\mathbf{C}[s] = \{\bigcap \mathbf{C}^{\text{♀}}_*(s \mid L\,, N[1\,,s]) \mid L\,, N[1\,,s]\}$$

where

$$\mathbf{C}^{\text{♀}}_*(s \mid L\,, N[1\,,s]) = \mathbf{C}^{\text{♀}}_*[s] \tag{12.1}$$

is the set given by (11.1) for a fixed pair $L\,, N[1\,,s]$.

On the other hand each of the latter sets may be approximated by ellipsoids as in (11.5) so that

$$\bar{\mathbf{C}}_*[s] = \{\bigcap E(x[0\,,s] \mid L\,, N[1\,,s]) \mid x[0\,,s]\} \tag{12.2}$$

where

$$E(x[0\,,s] \mid L\,, N[1\,,s]) = E[x_0\,,\ldots, x_s]$$

is the ellipsoid of (11.2) calculated for a fixed pair $L\,, N[1\,,s]$.

Combining (12.1), (12.2) we have

$$\bar{\mathbf{C}}[s] = \{\bigcap\{\bigcap E(x[0\,,s] \mid L\,, N[1\,,s]) \mid x[0\,,s]\} \mid L\,, N[1\,,s]\} \tag{12.3}$$

and obviously

$$\bar{\mathbf{C}}[s] \subseteq \{\bigcap E(x^*[0\,,s] \mid L\,, N[1\,,s]) \mid L\,, N[1\,,s])\} \tag{12.4}$$

where $x^*[0\,,s] = \{x_0^*, \ldots, x_s^*\}$ is calculated due to (11.8), (11.9).

Formulae (12.3), (12.4) allow to decouple the estimation process into independent "parallel" procedures.

13. COMPUTER SIMULATION

We will now give an example following assertions of this paper. The scalar observation $y(k)$ is generated by the deterministic model

$$y(k) = (c_1^* \; c_2^*) \begin{bmatrix} p_1(k) \\ p_2(k) \end{bmatrix} + v(k) \qquad k = 1$$

where $c_1^* = 10$, $c_2^* = 5$. The uncertainty is defined by

$$\bar{\mathbf{C}}_0 = \{\bar{C} \mid (\bar{C} - \bar{C}^0)' \, \Sigma_0^{-1} \, (\bar{C} - \bar{C}^0) \le 1\}$$
$$Q(k) = \{v(k) \mid v'(k) \, \Sigma_2^{-1}(k) \, v(k) \le 1\} = \{v(k) \mid \mid v(k) \mid \le \mu\}$$

where

$$\bar{C}^0 = (11.8, 3.9) \; , \Sigma_0 = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix} , \mu = 2$$

Figure 1 shows the informational domain $\mathbf{C}[1]$ after observation $y(1)$, where $p(1) = (-0.94 \; , 0.22)$ and the noise value $v(1) = 1.31$. Figures 2-4 show

$$\{\bigcap E(x[0\;,1] \mid L \;, N(1)) \mid x[0\;,1]\} = S[L^{-1} \;, N^{-1}(1)]$$

with $L = I$ and $N^{-1}(1) = 10^{-2}$, 1 , 10 respectively. It can be seen that the intersection of the sets tends to the informational domain

$$\lim_{\substack{\alpha \to +0 \\ \beta \to \infty}} S[I,\alpha] \bigcap S[I,\beta] = \mathbf{C}[1] \; .$$

Figure 5 shows the bounding ellipsoids of minimum semi-axes with $L = I$, $N^{-1} = 0, 10^i$, $i = 4, -3.5, \cdots, 2$, and the shaded portion expresses the set given by the right hand side of (12.4). From here it can be seen that in this case (12.4) is a strict inclusion.

We shall pass to the multi-stage case. The informational domain $\mathbf{C}[3]$ is shown in Figure 6, where $p(k)$, $v(k)$, $k = 1,\ldots,3$ are selected randomly. Figure 7 shows the ellipsoids tangential to $\mathbf{C}_*[3]$ and henceforth approximately expresses $\mathbf{C}_*[3]$ when $L = I$, $N^{-1}(k) = 10^{-4}$, $k = 1,\ldots,3$. It can be seen that in this case we would not obtain the true set by simply taking the extreme value $\mathbb{N} = \alpha I$, $\alpha \to 0$. Figure 8 shows the trace-minimal ellipsoids obtained by the recursive scheme (R) and the multi-stage scheme

(M) with $L = I$ and $N^{-1}(k) = 10^{-4}$, $k = 1,\ldots,3$. The informational domain $\mathbf{C}[3]$ can be obtained in the following way. Consider the estimate $\mathbf{C}_*[s]$ of the form yielded by (9.4)

$$\mathbf{C}_*[s] = (I_{mn} - \mathbb{P}^{-1}(s)\mathbf{T}'(s)\mathbb{N}\mathbf{T}(s))\mathbf{C}_0 + \mathbb{P}^{-1}(s)\mathbf{T}'(s)\mathbb{N}(y(s) - \mathbf{Q}(s)) \; . \quad (13.1)$$

Taking $N^{-1}(k) = \alpha_k I$, $\alpha_k \to \infty$, $k \neq j$, we obtain

$$\mathbf{C}_*[s] = (I_{mn} - (L + P(j))^{-1}P(j))\mathbf{C}_0 + (L + P(j))^{-1}G(j)N(j)(y(j) - Q(j)) \quad (13.2)$$

which is the same form as (7.2), and therefore the discussion of the one-stage problem can be applied. That is, by varying L and $N^{-1}(j)$, we would obtain a set which is the intersection of $\mathbf{C}_0$ and $\mathbf{C}_y[j]$ where

$$\mathbf{C}_y[j] = \{C \mid Cp(j) \in y(j) - Q(j)\} \; .$$

The shaded portions of Figures 9-11 show the sets obtained by taking $L = I$, $N^{-1}(j) = 10^{-4}$, $N^{-1}(k) = 10^2$, $k \neq j$, $j = 1,\ldots,3$, respectively.

14. CONCLUSION

This paper indicates a unified framework for the treatment of the standard identification problem under uncertainty in the measurements which could be modelled by both stochastic and set-membership techniques. It is shown that the deterministic techniques could be used to prove consistency for some probabilistic models while the stochastic identification scheme may be relevant for approximating the deterministic solution. Ellipsoidal approximations may be appropriate for numerical simulations although the consistency of approximate solutions should be a separate theme for investigation. It is important to underline that the topic of this paper is also closely linked to the issues discussed in [22, 23].

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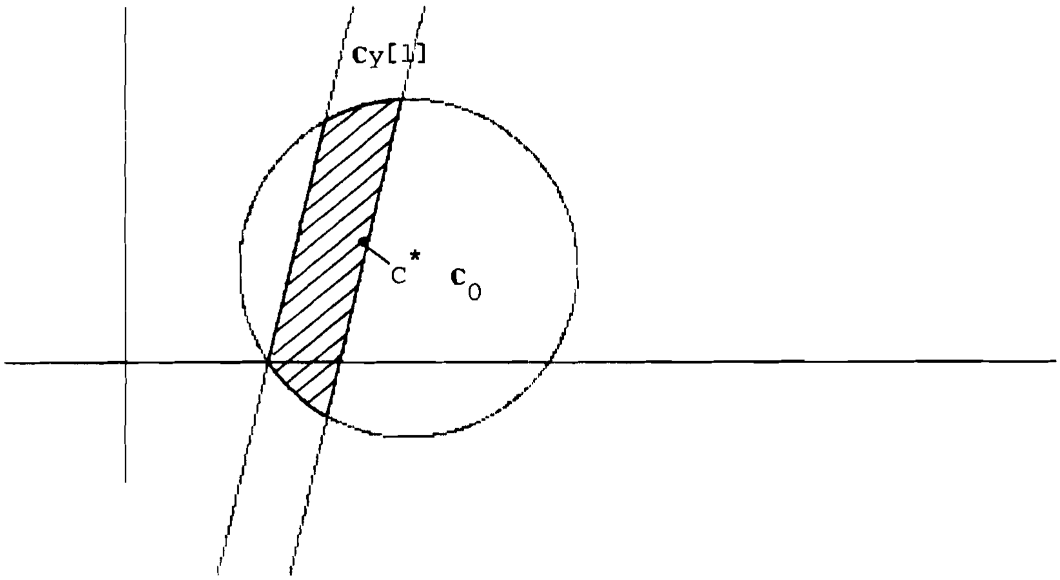

Figure 1 Informational domain **C**[1]

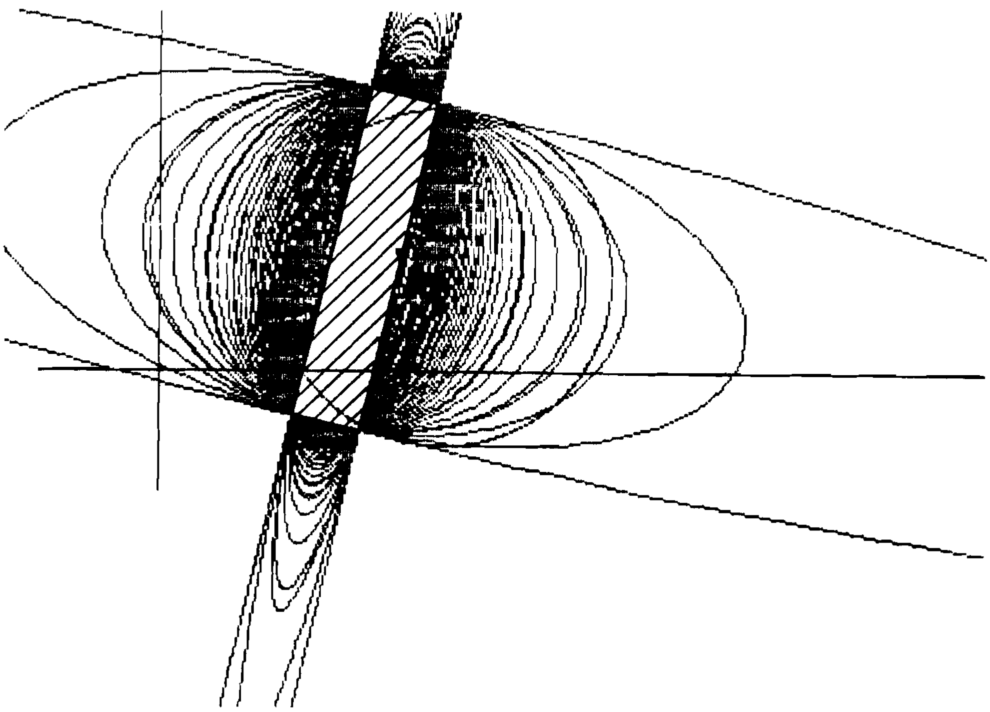

Figure 2 Stochastic solution $\mathbf{C}_*[1]$ when $L = I$, $N^{-1}(1) = 10^{-2}$

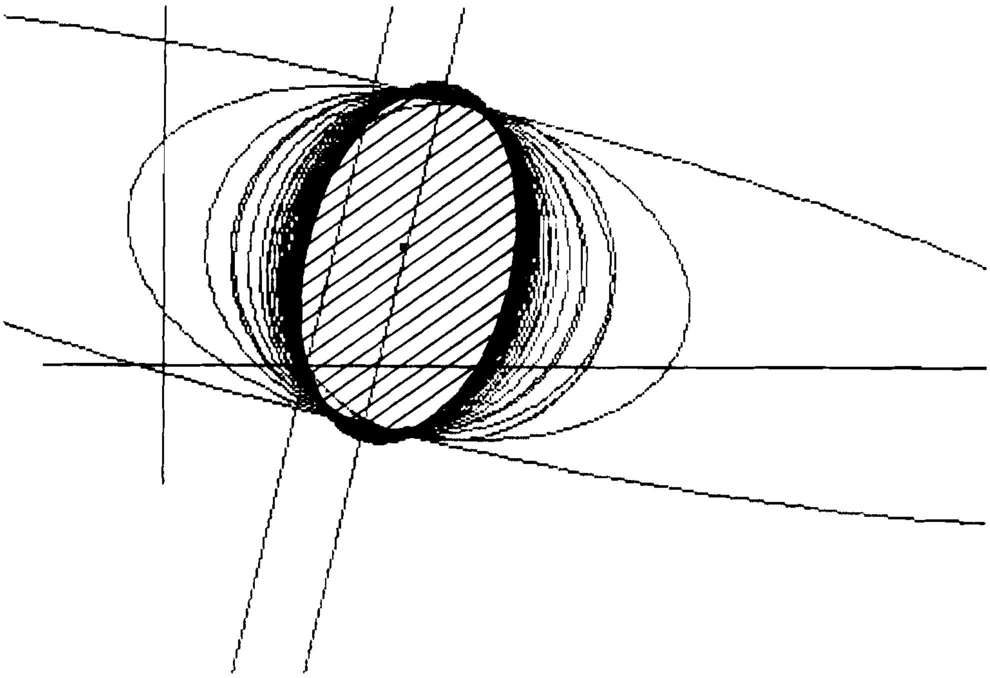

Figure 3 Stochastic solution $\mathbf{C}_*[1]$ when $L = I$, $N^{-1}(1) = 1$

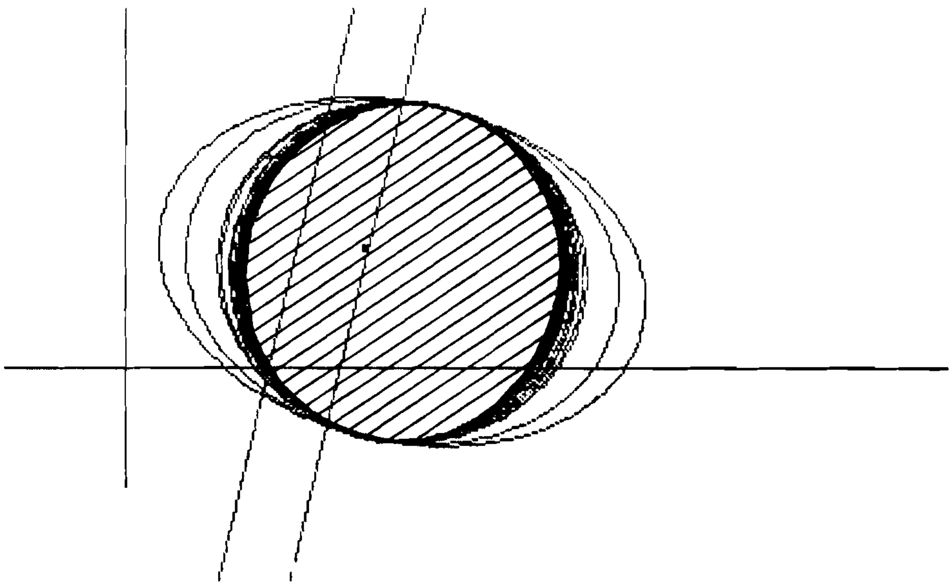

Figure 4 Stochastic solution $\mathbf{C}_*[1]$ when $L = I$, $N^{-1}(1) = 10$

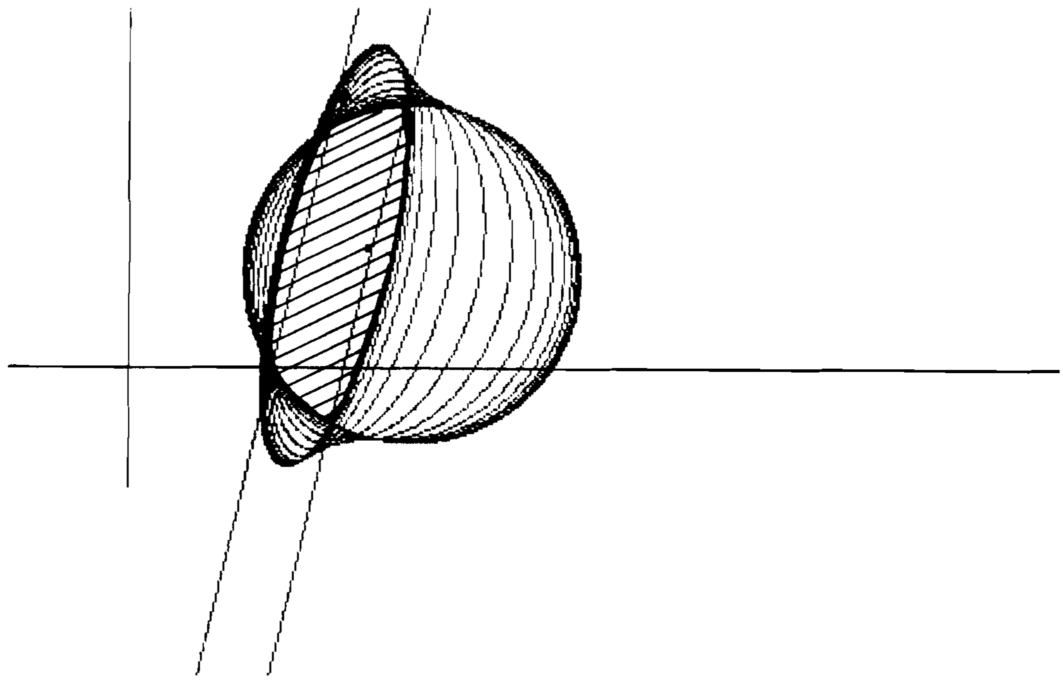

Figure 5 Intersection of bounding ellipsoids of minimal semi-axes

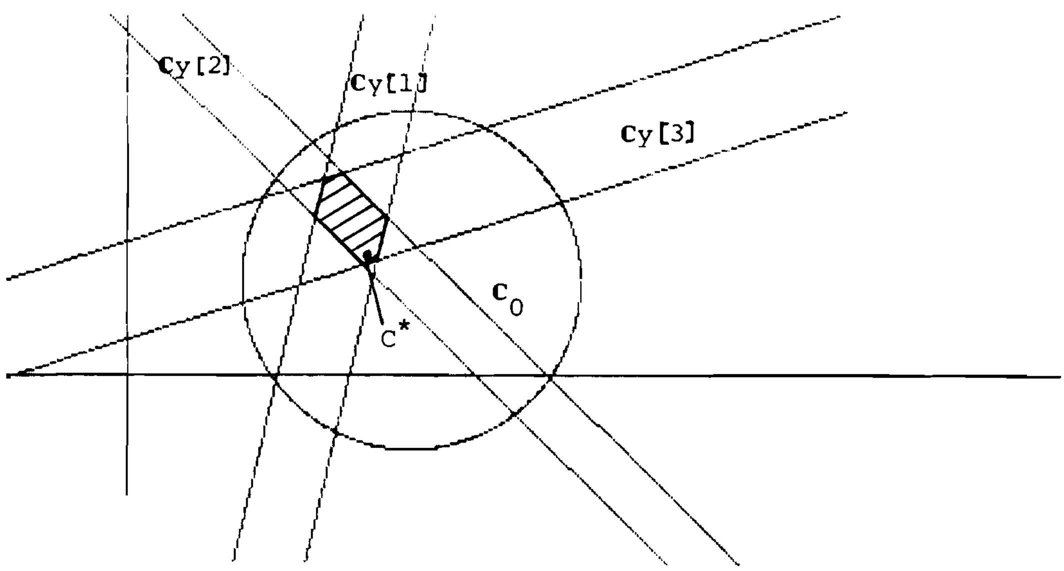

Figure 6 Informational domain **C**[3]

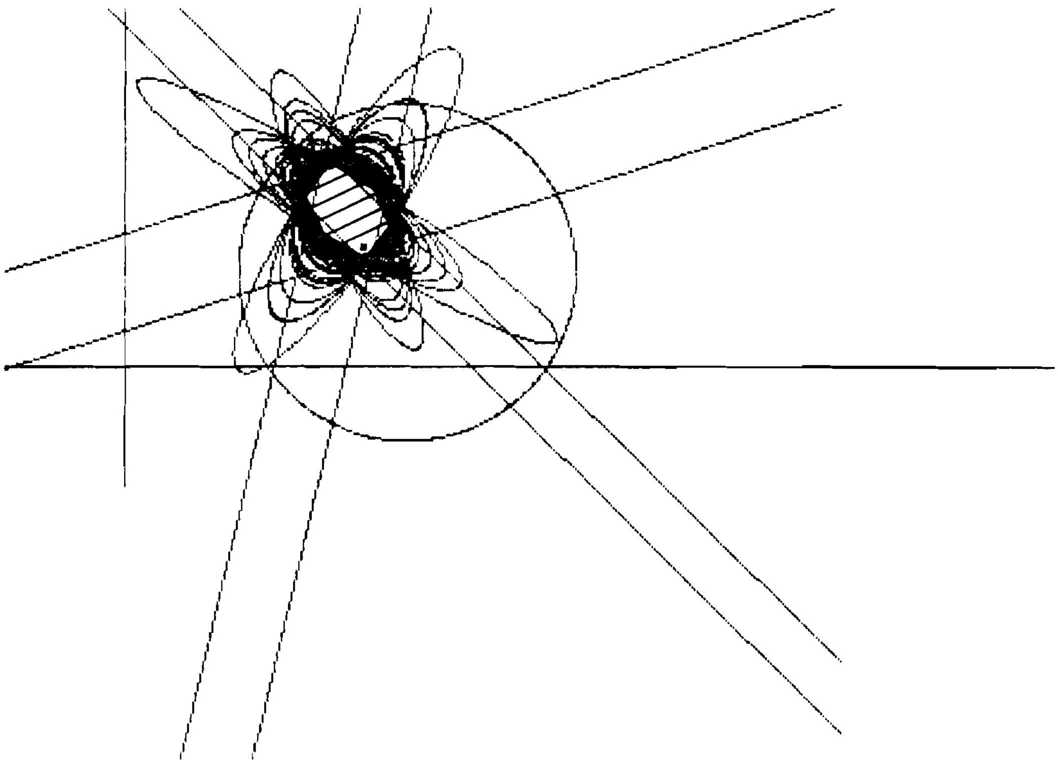

Figure 7 Stochastic solution $\mathbf{C}_*[3]$

when $L = I$, $N^{-1}(k) = 10^{-4}$, $k = 1,..,3$.

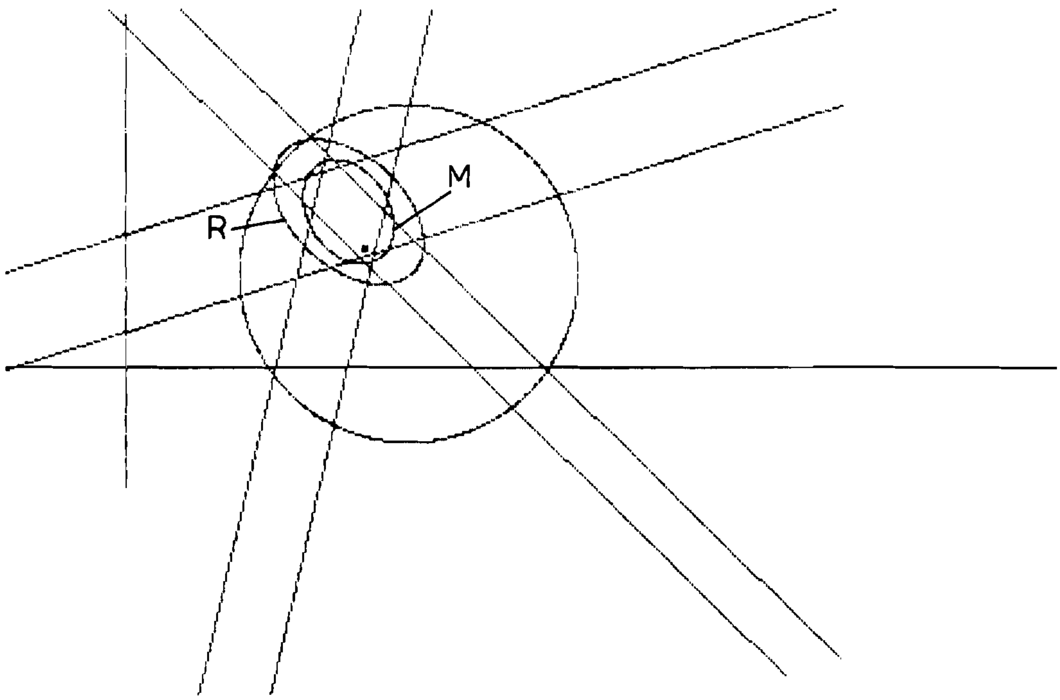

Figure 8 Trace-minimal ellipsoids by the recursive scheme (R)

and the multi-stage scheme (M)

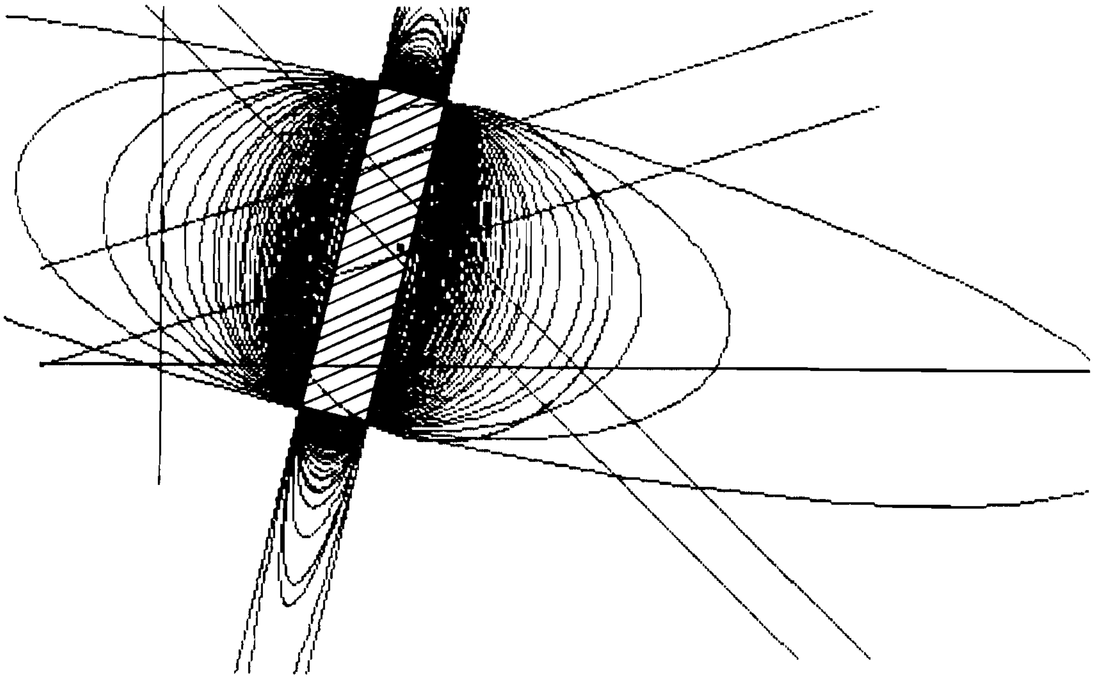

Figure 9 Stochastic solution $\mathbf{C}_*[3]$

when $L = I$, $N^{-1}(1) = 10^{-4}$, $N^{-1}(k) = 10^2$, $k \neq 1$.

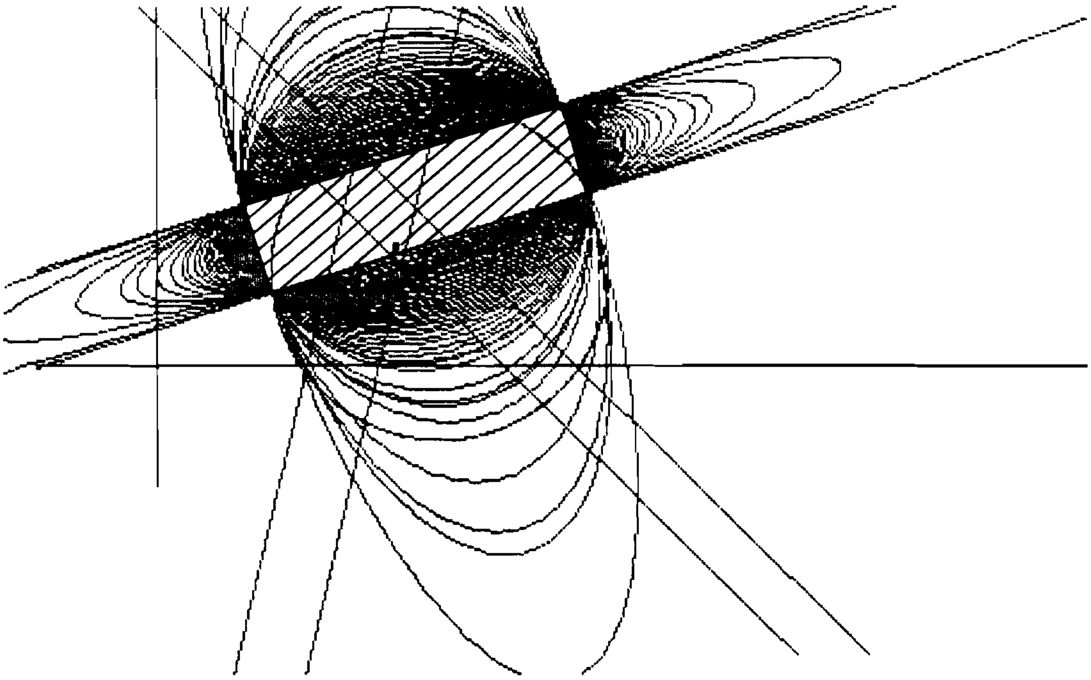

Figure 10 Stochastic solution $\mathbf{C}_*[3]$

when $L = I$, $N^{-1}(2) = 10^{-4}$, $N^{-1}(k) = 10^2$, $k \neq 2$.

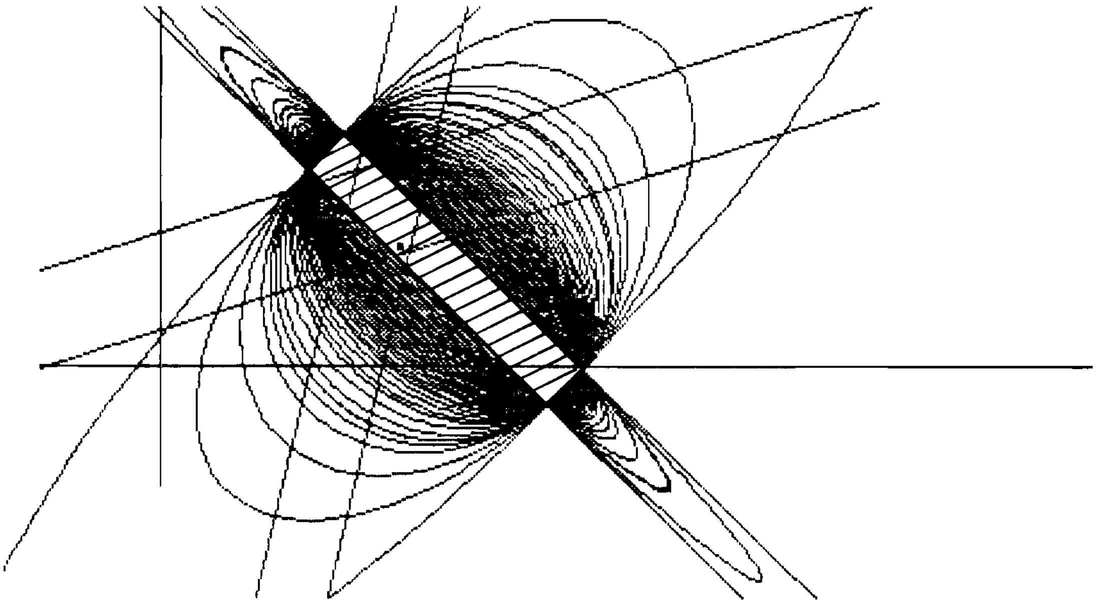

Figure 11 Stochastic solution $\mathbf{C}_*[3]$

when $L = I$, $N^{-1}(3) = 10^{-4}$, $N^{-1}(k) = 10^2$, $k \neq 3$.