APPLICATION OF CREDIBILITY THEORY TO MATERIAL ACCOUNTABILITY VERIFICATION

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1. Formulation of the problem

The nuclear materials safeguards system of the International Atomic Energy Agency (IAEA) in Vienna consists of two parts (see Reference [1]): the verification of the material flow and inventory data reported by the operator of a nuclear plant; the establishment of a material balance at the end of an inventory period with the help of the operator's reported data, which means that the book inventory (initial physical inventory plus receipts minus shipments) is compared with the ending physical inventory (see e.g., Reference [2]). By definition it is necessary that the plant operator maintains a complete measurement system for all nuclear materials processed in the plant.

In this paper, we consider an alternative inspection scheme which is based on material accountability too, but which does not make use of the data reported by the operator. Contrary to the IAEA safeguards system, the material balance in this system is closed only with the help of the data observed by the inspection team itself. Such a system could be important in situations where there is no reason for a plant operator to maintain a complicated measurement system, or where, for some reason, the records are not available.

It is clear that if the inspection team cannot measure the data of all material batches processed in the plant under consideration (e.g. if the inspection budget or time is limited), then some *prior information* about the average material contents of the different batches as well as the batch-to-batch variation have to be used. Therefore, a *Bayesian approach* seems to be natural for the treatment of problems of this kind. On the other hand, this prior information will not be very detailed, and so we will use the principles of *credibility theory* (see e.g., References [3],[4]) where only the first two moments of the prior distribution have to be known.

In the following, we first consider only one class of material, and then R different classes (inputs, outputs, etc.) with the problem of material balance closure. Finally we discuss the problem of optimization of a given inspection effort.

As the batch-to-batch variation of the true material contents within one class normally is much larger than the measurement variance, we will neglect the measurement errors here; they could easily be taken into account, if necessary.

2. One class of material

Let us consider one class of material consisting of N batches. An inspection team measures the material contents of n of these N batches precisely and wants to estimate the total material content of the class with the help of the n data. The true values of the material content of the batches vary from batch to batch; because of long term experience, however, the inspection team has a prior information about the average value and the batch-to-batch variation of the true material contents.

This prior information may be specified in the following way: the true material contents x_j of the ith batch is a random variable with a likelihood density $p(\tilde{x}_j|\theta)$, where θ is the parameter (possibly a vector), representing the unknown variation which has occurred in this production run. In Bayesian analysis, the parameter θ itself is considered as a random variable with a prior density $p(\tilde{\theta})$. We do not assume that the complete forms of $p(\tilde{x}_j|\theta)$ and $p(\theta)$ are known to the inspection team, but only the expectation value m,

$$m := \mathscr{E}\{\tilde{x}_j\} = \mathscr{E}\mathscr{E}\{\tilde{x}_j|\tilde{\theta}\} \quad , \qquad j = 1 \ldots N \tag{1}$$

and the two components of variance

$$E := \mathscr{E}\mathscr{V}\{\tilde{x}_j|\tilde{\theta}\} \tag{2}$$

$$D := \mathscr{V}\mathscr{E}\{\tilde{x}_j|\tilde{\theta}\} \quad , \qquad j = 1 \ldots N \ . \tag{3}$$

(As we have to differentiate carefully between random variables and their actual values, we indicate random variables by a tilde.) Notice that, even though the $\{\tilde{x}_j\}$ are independent, given θ, they are, a priori, dependent random variables; in other words, it is possible to make inferences about future values of the $\{\tilde{x}_j\}$ from observed values because they have the same (unknown) value of θ.

Assume that the inspection team has measured the material contents of $n \leq N$ batches (for simplicity we relabel the batches so that these are the first n batches); let $\underline{x} = (x_1 \ldots x_n)$ be the result of these measurements. The problem is to estimate the total material content of the class using these data and the prior information (1), (2), (3). Since we know $\underline{x}$, we must newly estimate $(\tilde{x}_{n+1} \ldots \tilde{x}_N)$.

The idea of the credibility approach is to take an estimate $f_n(\underline{x})$ for the material content x_{n+1} of the $n+1$st batch which is *linear in the data* and which minimizes the *preposterior variance* of the forecast error defined by

$$H_x = \mathscr{E}\left\{\left(\tilde{x}_{n+1} - f_n(\underline{\tilde{x}})\right)^2\right\} \quad . \tag{4}$$

(H_x is, in fact, a variance since $f_n(\underline{\tilde{x}})$ will be an unbiased estimate, i.e. $\mathscr{E}\left\{\tilde{x}_{n+1} - f_n(\tilde{x})\right\} = 0$.)

As a linear form, we take

$$f_n(\underline{\tilde{x}}) = z_0 + z_1 \cdot \frac{1}{n} \cdot \sum_{j=1}^{n} \tilde{x}_j \tag{5}$$

since there is no reason to use a different weighting factor for each x_j. Then H_x is given by

$$H_x = \mathscr{E}\left\{\tilde{x}_{n+1}^2\right\} + z_0^2 + \frac{z_1^2}{n^2}\mathscr{E}\left\{\left(\sum_{j=1}^{n} \tilde{x}_j\right)^2\right\} - 2z_0 \cdot \mathscr{E}\left\{\tilde{x}_{n+1}\right\} +$$

$$- 2\frac{z_1}{n}\mathscr{E}\left\{\tilde{x}_{n+1} \sum_{j=1}^{n} \tilde{x}_j\right\} + 2z_0\frac{z_1}{n}\mathscr{E}\left\{\sum_{j=1}^{n} \tilde{x}_j\right\} \tag{6}$$

and we get with

$$\mathscr{E}\left\{\tilde{x}_{n+1}^2\right\} = \mathscr{V}(\tilde{x}) + m^2 = D + E + m^2 \quad ,$$

$$\mathscr{E}\left\{\left(\sum_{j=1}^{n} \tilde{x}_j\right)^2\right\} = n(D + E + m^2) + n(n-1)(D + m^2) \quad ,$$

$$\mathscr{E}\left\{\tilde{x}_{n+1} \sum_{j=1}^{n} x_j\right\} = n(D + m^2) \quad ,$$

from equation (6)

$$H_x = D + E + m^2 + z_0^2 + \frac{z_1}{n} \cdot \left(n(D+E+m^2) + n(n-1)\cdot(D+m^2)\right)$$

$$- 2z_0 m - 2z_1(D+m^2) + 2z_0 z_1 m \quad .$$

The optimal values of z_0 and z_1 are determined by

$$\frac{\partial H_x}{\partial z_0} = 0 \quad , \qquad \frac{\partial H_x}{\partial z_1} = 0 \quad ,$$

which finally gives

$$z_1 = \frac{n}{n + E/D} \quad ; \qquad z_0 = m(1 - z_1) \quad . \tag{7}$$

Notice that (5),(7) can, in fact, be used to estimate any future $\{\tilde{x}_j\}$, $j = n+1 \ldots N$. The minimum of the preposterior variance of H_x is given by

$$\min H_x = D + E - \frac{n \cdot D}{n + E/D} = E + \left(nE^{-1} + D^{-1}\right)^{-1} \quad . \tag{8}$$

These results have an intuitive interpretation: for $nD >> E$ we obtain $z_1 \approx 1$, $z_0 \approx 0$ and therefore,

$$f_n(\underline{x}) \approx \frac{1}{n} \sum_{i=1}^{n} x_i \quad ,$$

i.e. we use primarily the information contained in the data. Note that this could happen either because the number of examples was very large, or because D, the variance for our prior information, was large. For $nD << E$ we obtain $z << 1$ and therefore,

$$f_n(\underline{x}) \approx m$$

i.e. we use primarily the prior information m.

We now estimate the sum S of all material in the class,

$$S = \sum_{j=1}^{n} x_j \qquad (9)$$

by the true values of the material contents in the first n batches, $\sum_{j=1}^{n} x_j$, plus the sum of the estimates of the remaining $N-n$ material contents, given by equation (5):

$$\sum_{j=1}^{n} x_j + (N-n)\cdot f_n(\underline{x}) \quad . \qquad (10)$$

Using (7), we obtain the following estimate $F^n(\underline{x})$ of the sum S:

$$F^n(\underline{x}) = (N-n)\cdot(1-z_1)\cdot m + \left(\frac{N-n}{n}\cdot z_1 + 1\right)\cdot \sum_{j=1}^{n} x_j \quad . \qquad (11)$$

The preposterior variance of the forecast error of this estimate, which is defined by

$$H_S = \mathscr{E}\left\{\left(\tilde{S} - F^n(\underline{\tilde{x}})\right)^2\right\} \qquad (12)$$

is not just the sum of $(N-n)$ terms H_x in (8), because the same value of θ applies throughout, and thus the error terms are correlated. However, it can be written in simplified form as:

$$H_S = \mathscr{V}\left\{\alpha \sum_{j=1}^{n} \tilde{x}_j + \sum_{j=n+1}^{N} \tilde{x}_j\right\} \qquad (13)$$

where

$$\alpha = -\frac{N-n}{n}\cdot z_1 \quad .$$

Therefore, we get

$$H_S = (\alpha^2\cdot n + (N-n))\mathscr{V}\{\tilde{x}_j\} + (n(n-1))\cdot\alpha^2 + (N-n)\cdot(N-n-1)$$
$$+ 2n(N-n)\cdot\alpha)\mathscr{C}\{\tilde{x}_j, \tilde{x}_{j'\neq j}\}$$

which gives with $\mathscr{C}\{\tilde{x}\ ,\tilde{x}_{j'\neq j}\} = D$ the final result

$$H_s = (N-n)\cdot\left[((N-n)\cdot z_1+1)\cdot E + (N-n)(1-z_1)^2 D\right] \quad . \quad (14)$$

For $n = N$ we get $H_s = 0$, since the "estimate" is

$$F^N = \sum_{j=1}^{N} x_j \quad ,$$

i.e. the true value of the total material content is known.
For $n = 0$ we get

$$H_s = N\cdot E + N^2\cdot D \quad ,$$

which shows that D behaves like the variance of a *systematic error*, which persists in all estimates because θ remains the same.

3. Several classes of material; no diversion of material

Let us consider now one inventory period and assume for simplicity that the physical inventories at the beginning and at the end of the inventory period are zero. The material flowing through the plant during this inventory period may be classified into R classes of material: R_1 input and $R-R_1$ output classes. Let x_{ij} be the true material content of the jth batch of the ith class which will be measured by the inspection team in case this batch is selected for measurement. x_{ij} is positive if i is an input class, negative otherwise.

In case that no material has been lost or diverted (null hypothesis H_0) the material balance principle postulates that at the end of the inventory period the algebraic sum of all throughputs must be zero; in other words:

$$\sum_{i=1}^{R}\sum_{j=1}^{N_i} x_{ij} = 0 \quad . \quad (15)$$

We assume that the random sampling scheme of the inspection team is to select n_i out of the N_i batches of each class at the end of the inventory period; for example, one may imagine a chemical plant, where samples from all batches are drawn and stored and where only a fraction of these samples is analyzed at the end of the inventory period.

Let the mean value, given θ, of the material contents of a batch of the ith class be defined as

$$m_i(\theta) := \mathscr{E}\{\tilde{x}_{ij} | \theta\} \quad , \qquad j = 1 \ldots N_i, \; i = 1 \ldots R \quad (16)$$

and let the covariance of material contents of the ith and the jth class given θ be defined as

$$C_{ii'}(\theta) := \mathscr{C}\left\{\tilde{x}_{ij} ; \tilde{x}_{i'j'} | \theta\right\} \quad ,$$

$$j = 1 \ldots N_i, \quad j' = 1 \ldots N_{i'}, \quad i,i' = 1 \ldots R \; . \quad (17)$$

We assume in the following

$$C_{ii'}(\theta) = 0 \qquad \text{for } i \neq i' \quad , \quad (18)$$

which means that the batch-to-batch variations between batches of *different* classes do not depend on each other.

Note: This assumption seems to contradict equation (15) where such a dependence is given explicitly. However, this equation is a material balance equation which may be interpreted in such a way that the last output batch can only contain the amount of material which has been left (and which may be excluded from the random sampling procedure.) This means that only the last batch depends on the foregoing batches; it does not imply a non zero correlation between all batches of the R classes under consideration.

Corresponding to the case of one class of material we now assume that the prior information available to the inspection team is the knowledge of the values of the parameters:

$$m_i = \mathscr{E}_\theta \, m_i\{\tilde{\theta}\} \quad (19a)$$

$$E_{ii'} = \mathscr{E}_\theta \, C_{ii'}(\tilde{\theta}) \quad (19b)$$

$$D_{ii'} = \mathscr{C}_\theta\left\{m_i(\tilde{\theta}) ; m_{i'}(\tilde{\theta})\right\} \quad . \quad (19c)$$

In the following, we denote the vector $(m_1 \dots m_R)'$ by $\underline{m}$ and the matrices corresponding to (19b) and (19c) by E and D, respectively. According to (18) E is a diagonal matrix; D is not assumed to be diagonal as one can imagine that disturbances of the plant operations (expressed by variations of the parameter θ may cause common changes to all class mean values $m_i(\theta)$.

Let $\bar{x}_i$ be the sample mean of the observed values of the ith class,

$$\bar{x}_i := \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij} \quad . \tag{20}$$

We then get

$$\mathscr{E}\left\{\tilde{\bar{x}}_i \mid \theta\right\} = m_i(\theta) \quad ; \quad \mathscr{E}\left\{\tilde{\bar{x}}_i\right\} = m_i \tag{21a}$$

$$\mathscr{V}\left\{\tilde{\bar{x}}_i \mid \theta\right\} = \frac{1}{n_i} C_{ii}(\theta) \quad ; \quad \mathscr{V}\left\{\tilde{\bar{x}}_i\right\} = \frac{1}{n_i} E_{ii} + D_{ii} \tag{21b}$$

and because of (19)

$$\mathscr{C}\left\{\tilde{\bar{x}}_i ; \tilde{\bar{x}}_{i'}\right\} = D_{ii'} \quad . \tag{21c}$$

We now consider a vector $\underline{x}_*$ of *unobserved values* of batch data. A credibility forecast for this vector $\underline{x}_*$ is given by

$$\underline{f}(\underline{\bar{x}}) = \underline{\underline{z}}_0 \cdot \underline{m} + \underline{\underline{z}}_1 \cdot \underline{\bar{x}} \quad , \tag{22}$$

where $\underline{\bar{x}}$ is the vector of the sample means (20).

Minimization of the trace of the preposterior variance matrix of the forecast error, H, defined by

$$\underline{\underline{H}} = \mathscr{E}\left\{(\underline{\tilde{x}}_* - \underline{f}(\underline{\bar{x}}))^2\right\} \tag{23}$$

gives after some calculations similar to those in the foregoing part (see e.g., Reference [3]).

$$\underline{\underline{z}}_1 = \left(\underline{\underline{N}}_0 + \underline{\underline{E}}\,\underline{\underline{D}}^{-1}\right)^{-1} \cdot \underline{\underline{N}}_0 ; \qquad \underline{\underline{z}}_0 = (\underline{\underline{I}}_R - \underline{\underline{z}}_1) \tag{24}$$

where the diagonal matrix is defined by

$$\underline{\underline{N}}_0 = \begin{pmatrix} n_1 & & 0 \\ & \ddots & \\ 0 & & n_R \end{pmatrix}$$

and where I_R is the $R \times R$ unit matrix. The preposterior variance of the forecast error then is given by

$$\underline{\underline{H}} = \underline{\underline{E}} + (\underline{\underline{I}}_R - \underline{\underline{z}}_1) \cdot \underline{\underline{D}} \quad . \tag{25}$$

In the same way we estimated in the foregoing part the sum of all material contents of one class we estimate now the sum S_i of all material contents of the ith class by

$$g_i(\underline{\bar{x}}) = n_i \cdot \bar{x}_i + (N_i - n_i) \cdot f_i(\underline{\bar{x}}) \quad . \tag{26}$$

which gives with (24) in explicit terms

$$g_i(\underline{\bar{x}}) = (N_i - n_i) \cdot \sum_{k=1}^{R} (I_{ki} - z_{ki}) \cdot m_k + \sum_{k=1}^{R} \sum_{j=1}^{n_k} \left((N_i - n_i) \cdot \frac{z_{ki}}{n_k} + I_{ik} \right) \cdot x_{kj} \quad . \tag{27}$$

Defining the diagonal matrix

$$\underline{\underline{N}}_1 = \begin{pmatrix} N_1 - n_1 & & 0 \\ & \ddots & \\ 0 & & N_R - n_R \end{pmatrix}$$

we get the vector forecast

$$\underline{g}(\underline{\bar{x}}) = \underline{\underline{N}}_0 \cdot \underline{\bar{x}} + \underline{\underline{N}}_1 \cdot \underline{f}(\underline{\bar{x}}) \quad ,$$

which gives with (22) and (24)

$$\underline{g}(\underline{\bar{x}}) = \underline{\underline{N}}_1 \cdot (\underline{\underline{I}}_R - \underline{\underline{z}}_1) \cdot \underline{m} + (\underline{\underline{N}}_0 + \underline{\underline{N}}_1 \cdot \underline{\underline{z}}_1) \, \underline{\bar{x}} \quad . \tag{28}$$

The preposterior covariance of the forecast error of the sums S_i is then given by

$$H_s = \mathscr{E}\left\{(\tilde{\underline{S}} - \underline{g}(\tilde{\underline{x}}))^2\right\}$$

where $\underline{S} = (S_1 \ldots S_R)$ which gives after some calculations

$$H_s = N_1 \cdot E + N_1(I_R - Z) \cdot D \cdot N_1 \quad . \qquad (29)$$

The elements of this covariance matrix are given by

$$H_{ij} = (N_i - n_i)E_{ij} + \sum_k \sum_\ell (N_i - n_i)(I_{k\ell} - Z_{k\ell}) \cdot D_{kj} \cdot (N_j - n_j) \quad . \qquad (30)$$

Finally, the preposterior variance of the forecast error of the sums is given by

$$H_{ss} = \mathscr{E}\left\{\left(\sum_i (S_i - g_i(\tilde{\tilde{\underline{x}}}))\right)^2\right\}$$

$$= \sum_{i,i'} \mathscr{E}\left\{(S_i - g_i(\tilde{\tilde{\underline{x}}}))(S_{i'} - g_{i'}(\tilde{\tilde{\underline{x}}}))\right\} \qquad (31)$$

$$= \sum_{i,i'} H_{ii'} \quad ,$$

where $H_{ii'}$ is given by (30).

4. Optimization of inspection effort

In the following we assume that for the inspection of the material flow during the inventory period under consideration there is only the amount C of inspection effort (given in manhours or in monetary terms) available. Furthermore, it is assumed that the observation of one batch datum of the ith class needs the effort $\mathscr{E}_i$. Therefore, the question arises how to distribute the effort among the different classes, in other words how to choose the class sample sizes n_i such that the boundary condition

$$C \geq \sum_{i=1}^{R} \mathscr{E}_i \cdot n_i \qquad (32)$$

is met.

In Reference [5] arguments have been given that the effort should be distributed in such a way that the *probability of detection in case the operator diverts the amount M of material* should be maximized. In case the plant operator wants to divert material during the inventory period under consideration (alternative hypothesis H_1), equation (15) does not hold any more. Let us assume that the operator does not change the number of batches in each class by simply taking away some of the batches but rather diverts from r_i batches of the ith class the amount μ_i of material. Let us assume furthermore, that the operator decides at the beginning of the inventory period whether or not he will divert any material. Finally, let us assume that the diversion takes place in the first R_1 classes *after* the inspection team's measurements, and in the remaining $R - R_1$ classes *before* the inspection team's measurements (the reason being that input batches are measured immediately after their arrival, and output batches immediately before their shipment). Then we have instead of equation (15) the following relation for the true material contents of the batches y_{ij} to be measured by the inspection team:

$$\sum_{i=1}^{R} \sum_{j=1}^{N_i} y_{ij} = \sum_{i=1}^{R} \mu_i \cdot r_i =: M \quad . \tag{33}$$

An example for this relation is given in Figure 1 for $R = 2$, $N_1 = 5$, $N_2 = 4$, $r_1 = 2$, $r_2 = 1$.

Let us define now the set A_i of batches of the ith class from which the operator diverts the amount μ_i of material. Then we have

$$y_{ij} = \begin{cases} x_{ij} - \mu_i & \text{for all batches from } A_i, \; i = R_1 + 1 \ldots R \\ x_{ij} & \text{otherwise} \end{cases} \tag{34}$$

where x_{ij} is the material content of the jth batch of the ith class to be measured by the inspection team in case of no diversion, and where accordingly

$$\mathcal{E}\{\tilde{x}_{ij}\} = m_i \quad . \tag{35}$$

As we get from (15)

$$\sum_{i=1}^{R} N_i m_i = 0 \quad ,$$

and as $|A_i|$, the number of elements of A_i, is hypergeometrically distributed, we have

$$\mathscr{E}\left\{|\tilde{A}_i|\right\} = \frac{r_i \cdot n_i}{N_i}$$

and the expectation value of the sum of the class sum forecasts is given by

$$\mathscr{E}\left\{\sum_i g_i(\tilde{\underline{x}}) \,|\, H_1\right\} = \sum_{i,k=1}^{R} \frac{r_i \cdot n_i}{N_i} \left((N_i - n_i) \cdot \frac{z_{ki}}{n_k} + I_{ik}\right) \mu_k \quad . \tag{36}$$

In the same way, we can calculate $\mathscr{E}\left\{\left(\sum_i g_i(\tilde{\underline{x}})\right)^2 |H_1\right\}$ and therefore the variance $\mathscr{V}\left\{\sum_i g_i(\tilde{\underline{x}})\,|H_1\right\}$ of the forecast $\sum_i g_i(\underline{x})$ under the *alternative hypothesis* H_1 (diversion of the amount M of material). Because of its length, and as we will not use it in the following, we will not give its explicit form here.

We now assume that the random variable $\sum_i g_i(\underline{x})$ is approximately normally distributed with expectation value and variance given as above. Then the probability of detection $1 - \beta$ based on a significance test for the null hypothesis $\mathscr{E}\left\{\sum_i g_i(\tilde{\underline{x}})\,|H_0\right\} = 0$, is given by the following expression

$$1 - \beta = \phi\left(\frac{\mathscr{E}\left\{\sum_i g_i(\tilde{\underline{x}})|H_1\right\} - U_{1-\alpha} \cdot \sqrt{\mathscr{V}\left\{\sum_i g_i(\tilde{\underline{x}})|H_0\right\}}}{\sqrt{\mathscr{V}\left\{\sum_i g_i(\tilde{\underline{x}})\,|H_1\right\}}}\right) \tag{37}$$

where α is the significance level, ϕ the normal distribution function and U its inverse.

According to the principle mentioned at the beginning of this chapter the optimal distribution of the inspection effort is determined by maximizing the probability of detection $1-\beta$ under the boundary condition (29) for the case that the operator wants to divert the amount M of material. As the inspection team does not know the 'diversion strategy' $(r_1 \ldots r_R)$ of the operator, and as one is furthermore interested in determining the *guaranteed probability of detection*, the inspection team will maximize the probability of detection for that case that the operator minimizes the probability of detection subject to the boundary condition

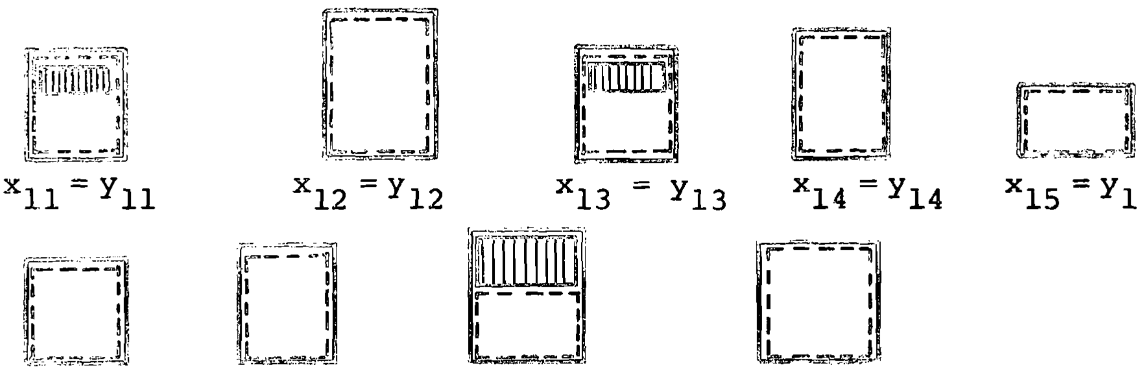
$$M << \sum_i \mu_i \cdot r_i \quad . \tag{39}$$

This means that the optimal distribution of inspection effort is gained by solving the following optimization problem

$$\max_{\substack{n_1 \ldots n_R: \\ C \geq \sum_i \mathscr{E}_i n_i}} \quad \min_{\substack{r_1 \ldots r_R: \\ M \leq \sum_i \mu_i r_i}} \quad 1-\beta(n_1 \ldots n_R; r_1 \ldots r_R) \quad . \tag{40}$$

Because of the complicated structure of $1-\beta$, given by equation (38) this problem can be solved only numerically.

Figure 1. <u>Illustration of the Material Balance</u>
(Zero beginning and ending physical inventories)



Input: $x_{11} = y_{11}$ $x_{12} = y_{12}$ $x_{13} = y_{13}$ $x_{14} = y_{14}$ $x_{15} = y_{15}$

Output: $x_{21} = y_{21}$ $x_{22} = y_{22}$ $x_{23} = y_{23} - \mu_2$ $x_{24} = y_{24}$

$(x_{2i} < 0, y_{2i} < 0, \mu_2 > 0)$

——— material contents x_{ij} measured by the inspection team in case of no diversion

– – – – material contents y_{ij} measured by the inspection team in case of diversion

 diverted material

Null hypothesis (no diversion):

$$\sum_{i=1}^{5} x_{1i} + \sum_{i=1}^{4} x_{2i} = 0$$

Alternative hypothesis (diversion):

$$\sum_{i=1}^{5} y_{1i} + \sum_{j=1}^{4} y_{2i} = 2 \cdot \mu_1 + 1 \cdot \mu_2$$

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