ANALYTICAL CARTOGRAPHY

W. R. Tobler

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Analytical Cartography*

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An understanding of the situation in geographical cartography in the United States during the early 1960's is helpful as a background. Cartography as such has only been a university subject in the U.S. since the late thirties of this century. Of the 3,000 or so colleges in the United States, cartography is taught in most of those which offer geography instruction, but less than a dozen have developed specializations in this area. To these one should add a small number of engineering schools which have professional programs in surveying, photogrammetry, or geodesy. The relation between official governmental cartography and academic cartography is not very close. The governmental cartographic establishments are bureaucratic factories and only seldom do theoretical innovations arise from these sources. They are generally technically competent and do some developmental work, and of course produce most of the actual maps. By contrast, it is dangerous for a university to become engaged in active map production. The governmental agencies are mostly concerned with topographical mapping and, although geography was once associated with geology, it has now (at least in North America) moved away from this focus to become a social science. This has led to a further separation of geographical cartography and topographical mapmaking.

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*An abbreviated translation of a lecture, "Das Wesen und die Bedeutung der Analytischen Kartographie," presented under the auspices of Dr.-Ing. h.c., Prof. Dr. E. Arnberger at the Institut für Geographie und Kartographie of the Universität Wien on 6 May 1975.

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Geography in the late fifties and early sixties went through the so-called "quantitative revolution," in which statistical description replaced verbal description, and formal abstract model building replaced anecdotal explanation. Many of the students graduating in this period wound up teaching cartography. The production of schools specializing in cartography at the graduate level was insufficient to fill all the undergraduate instructional courses in cartography. As a consequence, the newest staff member often ended up by teaching the introductory cartography course, in which he neither had great interest nor training. But he knew a lot about multiple regression and factor analysis, and related statistical techniques. Thus, courses called "Statistical Cartography" came into being. Here students learned correlation techniques, trend analysis, and so on, all legitimate topics, but hardly cartography. Treatment of the subject as "technique," to be taught by the lowest man on the academic totem pole, certainly did not lead to the expansion of the theoretical core of the field. At the same time, it was clear that computers would play a crucial role in the future of cartography, the first computer maps having been produced in 1951, and that many changes were needed in traditional cartography. The world seems to be changing so fast that fifty percent of what one learns is obsolete within five years. I had the hope that my lectures would have a half-life of twenty years. The eventual result was Analytical Cartography, as described in Figure 1.

A popular title would have been Computer Cartography. This did not appeal to me because it is not particularly critical which production technology is used. Such a title would also imply a course on "Handicraft Cartography" or "Pen and Ink Cartography." The substance is the theory which is more or less independent of the particular devices; these become obsolete rather quickly anyway.

Mathematical Cartography could have been used, but this already has a definite meaning and I had in mind more than is usually covered under this heading. Cartometry is another available term, but this has a very narrow meaning. One could also speak of Theoretical Cartography. This did not appeal to me on two grounds. It would frighten students who always are concerned that they learn something do-able. Secondly, the precedent is not very attractive. Max Eckert, for example, wrote a great deal but did not solve many problems. I wished to emphasize that mathematical methods are involved, but also that an objective is the solution of concrete problems. As regards the substance, rather than the title, the major difference is perhaps only that a somewhat more general view is taken of the subject. Embarrassingly frequently cartographers claim as unique problems which also occur in other fields, and which may even have been solved there. All professions must constantly fight their myopia.

It is appropriate to introduce students to what is already in the literature, and to how similar concepts occur in other fields, and to suggest new directions. Thus a simple introduction to Analytical Cartography is through Photogrammetry and Geodesy. These fields have a long mathematical tradition and healthy literature to which the student need only be referred. The principle equations of the method of least squares and its newer derivatives, of the theory of errors, the theorems of projective geometry and of potential theory, etc., are all easily established. One can also see the tendencies; the direct production of mosaics and photographic recognition of objects by computer; the replacement of triangulation by trilateration, to take only two examples. In a short lecture such as this today, I cannot cover all the details, but as an example of how a more general view is useful, consider the last topic. This is, of course, well-known to this audience.

Suppose that we have identified n points on the surface of the earth. Between these there exist n(n-1)/2 distances, d_{ij}. Let us assume that all of these distances have been measured and that the locations of the points are to be found. This means that 2n coordinates must be determined. In a Euclidean space (leaving the earth) one learns in high school that

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \rightarrow d_{ij} \quad .$$

The surveying problem reverses the arrow. Since n(n-1)/2 grows much faster than 2n, there are more equations than unknowns. This has three consequences: (1) no solution satisfies all equations (since all empirical measurements have error) and this leads naturally to a discussion of least squares methods, error ellipses, and iterative solution techniques; (2) one need only know the ordinal relations of the distances in order to obtain a solution (see Kendall), which comes as quite a shock to the cartographer who so strongly believes in numbers; (3) only 2n measurements are really necessary, and one is then led to consider "optimal" positioning of surveying measurements, a rather recent development in the literature. This latter question also arises in psychology where one attempts to measure separations between personality types.

Returning to the course outline, one knows, a priori, that all maps which can be drawn by hand can also be drawn by computer controlled devices. This follows from Turing's (1936) theorem. Of course, can does not imply should. Students are given a short introduction to the equipment which provides realization of these ideas, and where they can obtain geographical data tapes and computer programs. The equipment at the University of Michigan consists of a large central computer connected to telephones. Teletype terminals and graphic devices may be attached to telephones for interactive use. Students write one program to draw a simple map of their choice in one of the programming languages. The available data tapes include world outlines (Figure 2 is an example), county boundaries (Figure 3), street patterns, topographic elevations, etc.

The user of geographical data is, in principle, indifferent as to whether the data are on a geographical map or on magnetic tape. From a pedagogical point of view, these are simply two alternate methods of storing geographical information. One of the principal uses of geographical maps is just that of a graphical data storage device. We can assert that, when one has enough information on a magnetic tape to be able to draw a geographical map, one also has enough information to solve all of the problems which could be solved using that map. But to store geographical information in electronic form in such a manner that a geographical map can be drawn requires that the substantive data be given geographical referencing.

The usual procedure is to reference points, lines, and areas by coordinates: geographical latitude and longitude, or Gauss-Kruger coordinates, etc. But this is really too narrow a point of view. Call the fire department and announce that there is a fire in this room, at

$$48^{\circ}15'22"\ N \quad , \quad 16^{\circ}23'10"\ E \quad .$$

It would never be done; one would use the street address, or the building name. But show me the cartography book which describes the street naming/numbering system. If I can locate a house using the street address then this label must contain exactly the same amount of information as does the latitude/longitude designation. The telephone area code number in Austria, 0222 = Vienna, locates this place to circa ± 20 km. If I call from the United States to a phone in Vienna, I need to dial twelve digits, and these locate a place to ± one meter. If I know the postal code 2361, I have located a region to ± 5 km.

Equivalently, Gauss-Kruger coordinates can be calculated from latitude and longitude

$$\varphi,\lambda \rightarrow G,K$$

and this is invertible

$$G,K \rightarrow \varphi,\lambda \quad .$$

Make a list of as many ways as you can recall of how locations are identified. Now form a table by repeating this list in the orthogonal direction. Now consider this table as a transformer: place name to latitude and longitude, and the inverse, might be an example of one transformation. The concern in the literature with computerized address coding, the DIME system, point-in-polygon programs, etc., all relate to these transformations. More exactly considered, coordinates are a way of naming places which, inter alia, allow all places to be given a unique name, and which allow relations between places to be deduced from their names. The North American telephone area codes, for example, have the property that if two area codes are similar, the places are widely separated, and the converse. Thus, we could draw a map solely by knowing area codes: Interpret "A is near or far from B" as "adjacent or non-adjacent" in the sense of Kendall, and compute the 2n coordinates from the set of these n(n-1)/2 relations.

When one puts geographical information into a computer, one finds that it is extremely voluminous. A four color map of size 10 x 10 cm contains perhaps 100 x 100 x 4 = 40,000 elements, not a great amount for a computer, but this is a small map. One can ask whether all of these elements are necessary. It is clear that not all of the 4^{10000} possible maps of this type can occur, because of the inherent geographical structure. One could throw away a goodly proportion of the 4×10^4 elements and still have a very useful map. I use the word "goodly" for lack of a numerical estimate.

This leads into the topic of geographical map simplification. This is often considered the most difficult topic in automated cartography today, and goes by the title of "map generalization." I have a few examples, but first remark that I avoid the term generalization because it has a rather mystical connotation in cartography; at a minimum it seems to mean several different things. I believe that automatic map simplification is not as difficult as was once thought, and it does contain some interesting aspects. The condensed book, the overture to a musical work, and a caricature, are all similar modifications of an original. A very similar topic is also treated in other fields under the heading of "aggregation," and it would pay cartographers to look at the econometric literature on this topic.

Figure 4 shows some examples of a simplification algorithm applied to an outline of Michigan. The method is simple, rapid, and seems effective. Figure 5 shows an application to contours. One of the more appropriate methods for such surfaces seems to me to be that of two-dimensional filtering. This method, although not the only one available, is exactly controllable and invertible. It can also be applied to choropleth maps; Figure 6 is an example.

As an example of problem solving, given geographical data in a computer, one can take the case of map overlays, e.g., given a soils map of Michigan and a geological map of Michigan, find the logical intersection of the two. Conceptually not a difficult problem (see Figure 7). One can store the soil regions in various ways in the computer. These become quite technical, but for example the area can be written as

$$F(x,y) = \begin{cases} 1 \text{ if in } R \\ 0 \text{ otherwise} \end{cases}$$

or the boundary can be described as an equation

$$x(s) + iy(s) \quad ,$$

or store the skeleton of the region, etc. From all of these representations (and there are more), one can compute the area of the region, one can calculate whether or not a point lies inside of the region, whether two regions overlap, and one can draw maps. It turns out, however, that some representations are more convenient for particular purposes than are others, even though they are algebraically equivalent (each can be converted into all of the others). A comparison to two methods of solving a pair of simultaneous linear equations may be appropriate. If one remembers that linear equations have the form

$$y = a_1 + b_1x$$
$$y = a_2 + b_2x$$

and a bit of algebra, then the intersection point can be found. But sometimes it is easier to plot the lines and read off the coordinates of the intersection. Many uses of maps are of this nomographic nature. Recall Mercator's projection, a graphic solution to the problem of finding the intersection angle between a great circle and a logarithmic spiral on a sphere. The subject matter of cartometry is the study of the accuracy of such graphical calculations. Or consider the problem of finding the nearest gas station when the automobile gauge indicates "nearly empty." With the entire street system stored in the core of a pocket calculator, and all the gas stations which accept my credit card also stored, and a minimal path algorithm which is efficient for networks of this size, it seems like a trivial computation. What is easy, convenient, or difficult depends on the technology, circumstances, and problem.

The coin also has an obverse. There are many cartographic concepts which can be put to good use in approaching geographical problems. The examples I have chosen all stem from the subject of map projections, but there are many other possibilities.

In the 1880's, Francis Galton invented the geographical isochrone, a line connecting all points which can be reached in a given time. Isochronic maps are now quite popular, and the concept has been generalized to include travel costs (isotims). These are really geographical circles, recalling that a circle is the locus of points equidistant from a center point. Measure distance in units of time and you have a geographical circle. But what curious circles. They have holes in them, and disjoint pieces, and the ratio of circumference to radius is hardly 2π. What a curious geometry--it makes Einstein seem simple! Or consider a set of concentric isochrones. Now draw in the orthogonal trajectories and one has the equivalent of a set of polar coordinates; technically polar geodesic coordinates (Gauss, 1820) for which the metric takes on a particularly simple form. One can draw maps of this geographical geometry by using the usual ideas from the study of map projections, but the reference object is no longer a sphere or ellipsoid, rather it is more like a pulsating Swiss cheese.

If one is successful in getting people to complete a questionnaire in which they are required to estimate the distance between, say, prominent buildings in Vienna, then the result is similar to the measurements obtained from a geodetic survey. One can compute coordinates and their standard errors, thus obtaining a type of mental map and its degree of variance. By comparison to geodetic triangulations one can, using Tissot's theorem, measure the amount of distortion of these mental maps. Data over time may reveal the rate of spatial learning.

As a final example, consider the problem of dividing the U.S.A. into compact cells each of which contains the same number of people. Figure 8 shows how this can be approached as a map projection problem.

One can anticipate many useful technical innovations in cartography in the future: wristwatch latitude/longitude indicators, for example, and pocket calculators with colored LED maps. But most critical for the development of the subject would be a useful model of the functioning of the human brain. The design of maps cannot be improved without such a standard against which to test visual effectiveness.

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Figure 1

Analytical Cartography. Geography 482. Prof. Waldo R. Tobler
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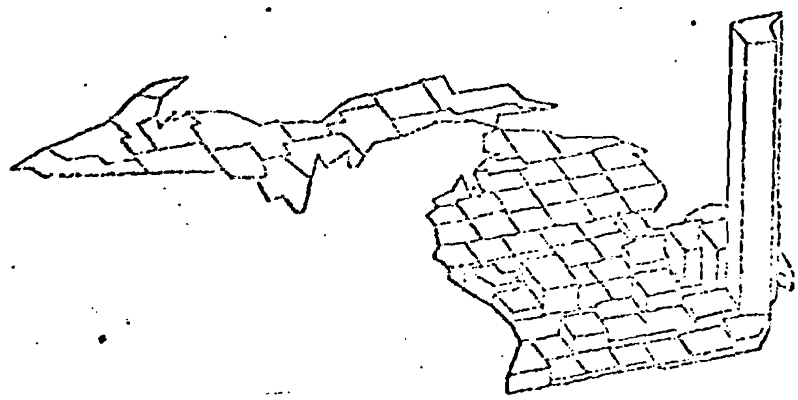
Week I. Introduction: Relation to mathematical geography, geodesy, photogrammetry, remote sensing. Replacement of map data storage by computer data storage. Technological change and the need for a theoretical approach. Historical perspective.
Week II: Computer Graphics: Turing's theorem in relation to cartography. Output devices: lines, halftones, color. Sources of programs and algorithms. Dynamic cartography and computer movie making. Interactive graphics in cartography and geography.
Week III: Geographical Matrices: Triagonal, quadrilateral, hexagonal, and Escher types. Notation, neighborliness property, topological invariance. The varieties of geographical data: nominal, binary, scalar, complex, colored, N-valued, and infinite-valued matrices. Isomorphism to the surface of the earth.
Week IV: Geographical Matrix Operators: Functions of matrices: algebraic, logical, differentiable, invertible; linear, local, spatially invariant (translationally and rotationally). Parallel processing, windows, edge effects. Finite difference calculations.
Week V: Response Functions: Fourier and other orthogonal series. Operations in the frequency domain. Two-dimensional transforms.
Week VI: Sampling and Resolution: Fourier interpretations of aliasing, band limited functions, Nyquist limit, comb functions. The sampling theorem, random plane sampling, invisible distributions.
Week VII: Quantization and Coding. Analogue and digital processing. Quantization error, reduction of. Information theory: how many aerial photographs are there? Huffman coding, higher order statistics, spatial autocorrelation functions. Television and choropleth maps.
Week VIII: Map Generalization: Textual, acoustical, visual abstractions: smoothing and reconstruction, spread functions and inverses. Information loss. Point, line, network, binary to N-valued matrix generalization. Digital implementation, optical data processing. How the brain works: Limulus, frog, cat, human.
Week IX: Pattern Recognition: Preprocessing, enhancement: feature extraction; discrimination and classification (linear, Gaussian); signal-to-noise ratios; perceptrons.
Week X: Generalized Spatial Partitionings: Census tracts and the like, *ad nauseum*. Point functions versus interval functions, a false dichotomy. Spatial resolution redefined. Generalized neighbors in a point set: epsilon neighborhood, K^{th} surround, minimal triangulation, Sokal contiguity, Thiessen polygons. Higher order neighbors. Interval sets associated with a point set; point sets associated with an interval set. Higher dimensional cases.
Week XI: Generalized Geographical Operators. Expansion of matrix operators to irregular point sets, to interval data, in such a manner as to include matrix as a special case. Generalized two-dimensional sampling theorem and reconstructions from sampled data.
Week XII: Geographical Coding. Information theoretical content of Latitude/Longitude, street address, ZIP code, telephone number, Public Land Survey, and the like. Topological and metrical properties of place naming schemes. Gaussian coordinates. A variety of plane coordinate schemes. Formulae for working on sphere and ellipsoid.
Week XIII: Geographical Code Conversions. Complete-partial, redundant-optimal, invertible-non-invertible codes. Blum geometry and skeletal invariants. Point-point, point-interval, interval-interval conversions and their inverses. Polygonal and skeletal approaches; error measures. Street address, Latitude/Longitude, and so forth.
Week XIV: Map Projections. The classical theory: Ptolemy, Mercator, Lambert, Euler, Gauss, Airy, Chebyshev, Tissot. Finite and differential measures of distortion. Applicability to "mental maps." Simplifying computations by using map projections. Some new ways of inventing projections. Computation of cartograms.
Week XV: Geographical Information Systems. Band width requirements; dollar requirements; hardware and software. Input schemes, manipulation algorithms, output schemes. Historical overview and examples: TIROS-ERTS, CATS-PJ-BATS, CLI-MLADS-DIME. Analytical approaches to using geographical data: optimization techniques, sensitivity testing, regionalization, spatial trend analysis, dynamic simulation, growth models, regional forecasting.

Lecture Outline 1969

FIGURE 2

FIGURE 3

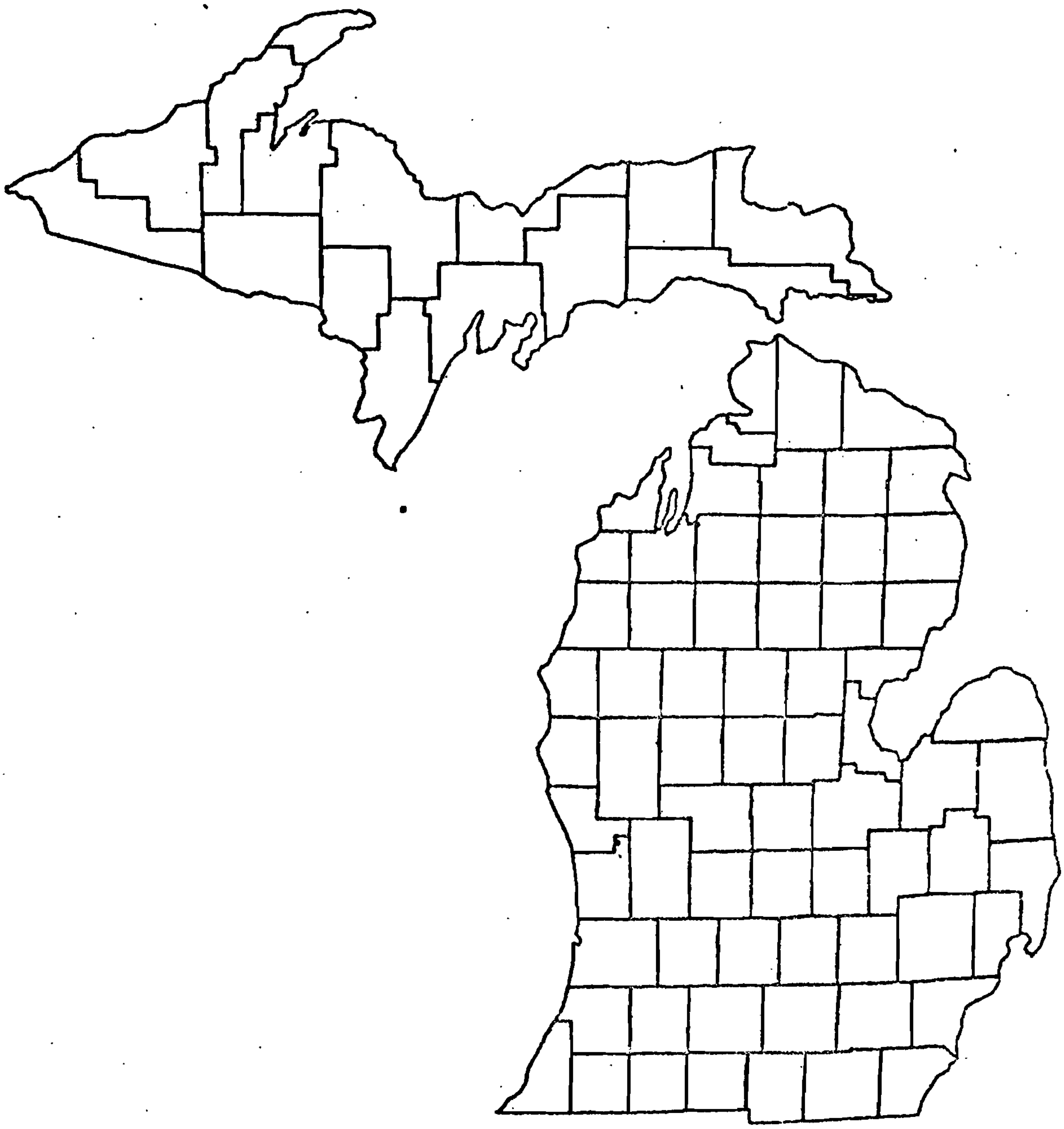


MICHIGAN 1970 POPULATION BY COUNTIES

[illegible] [illegible]

FIGURE 4a

MICHIGAN DESCRIBED BY 1028 POINTS



(due to the plotter resolution only about 950 points are discernable)

FIGURE 4b

MICHIGAN DESCRIBED BY 403 POINTS
epsilon = 2 km

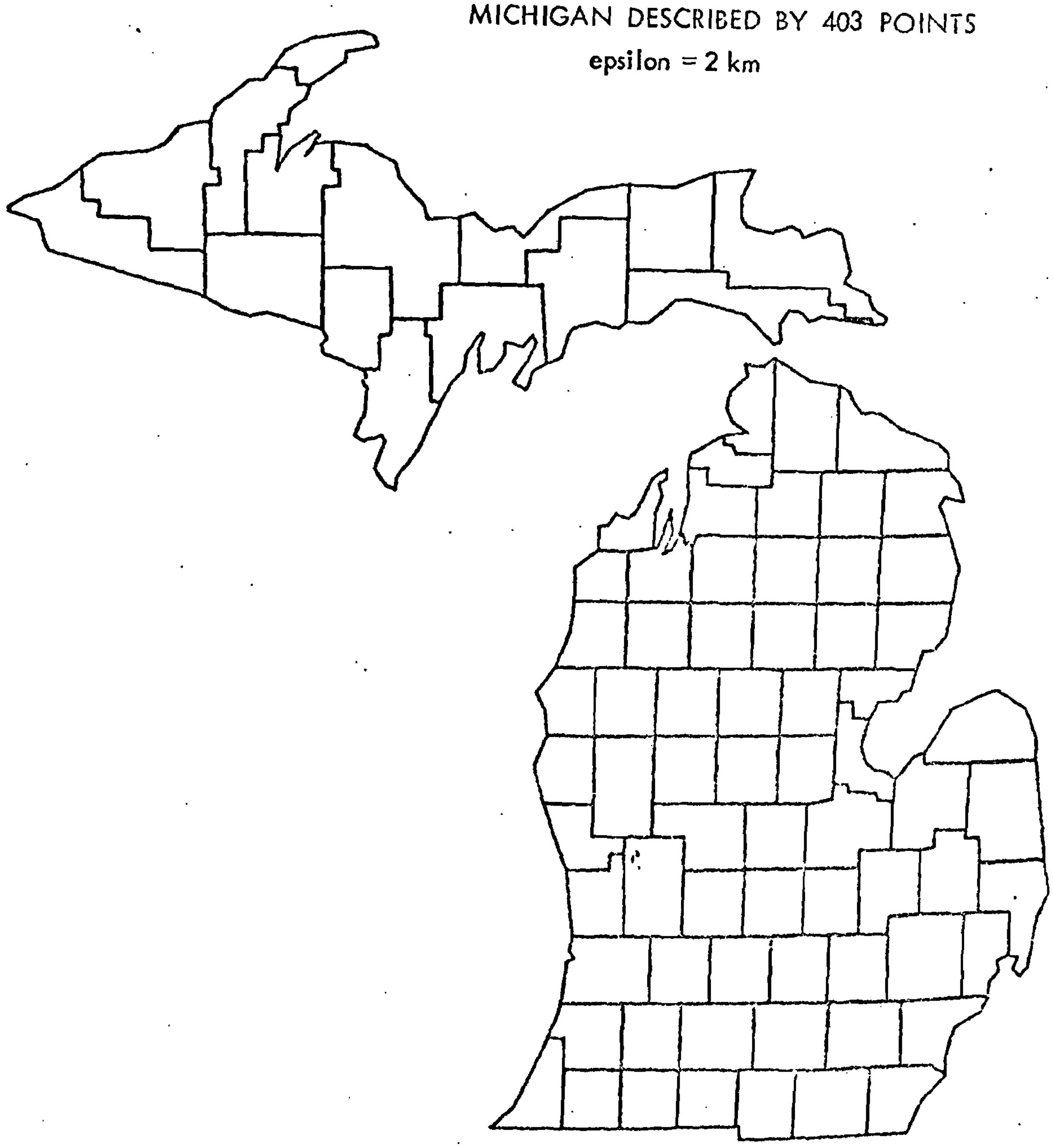


FIGURE 4c

MICHIGAN DESCRIBED BY 296 POINTS

epsilon = 4 km

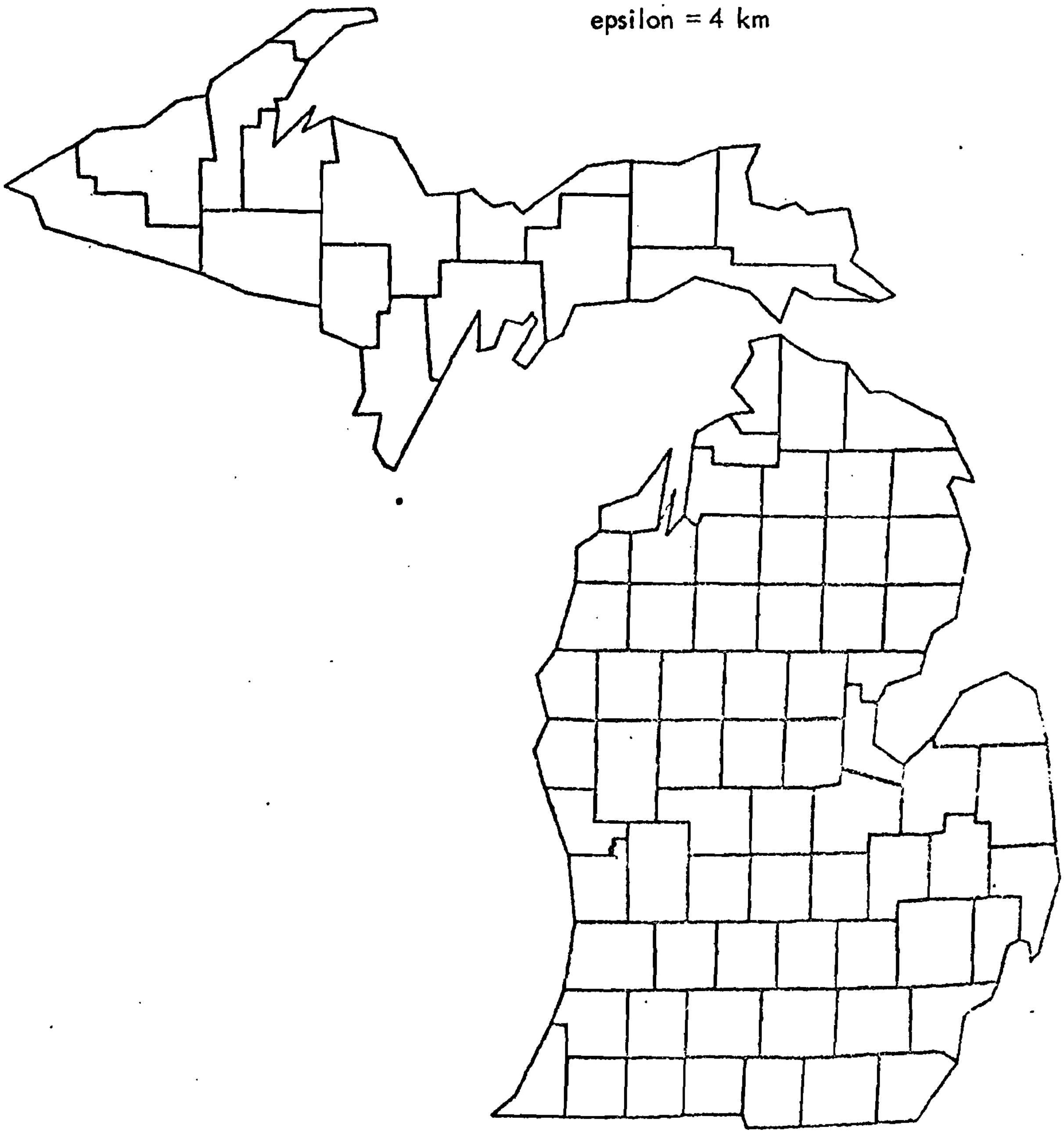


FIGURE 4d

MICHIGAN DESCRIBED BY 240 POINTS

epsilon = 6 km

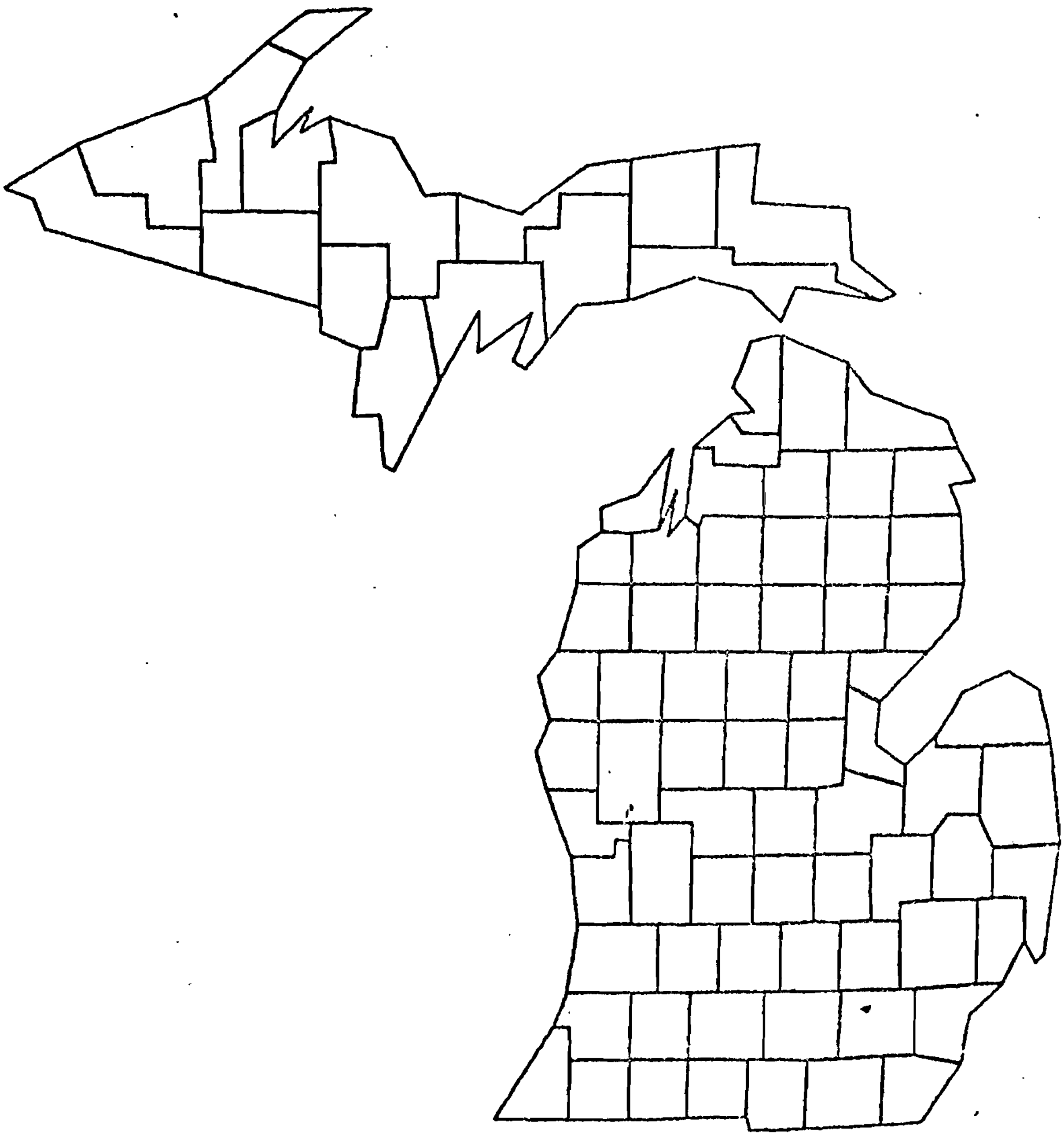


FIGURE 4e

MICHIGAN DESCRIBED BY 179 POINTS

epsilon = 12 km

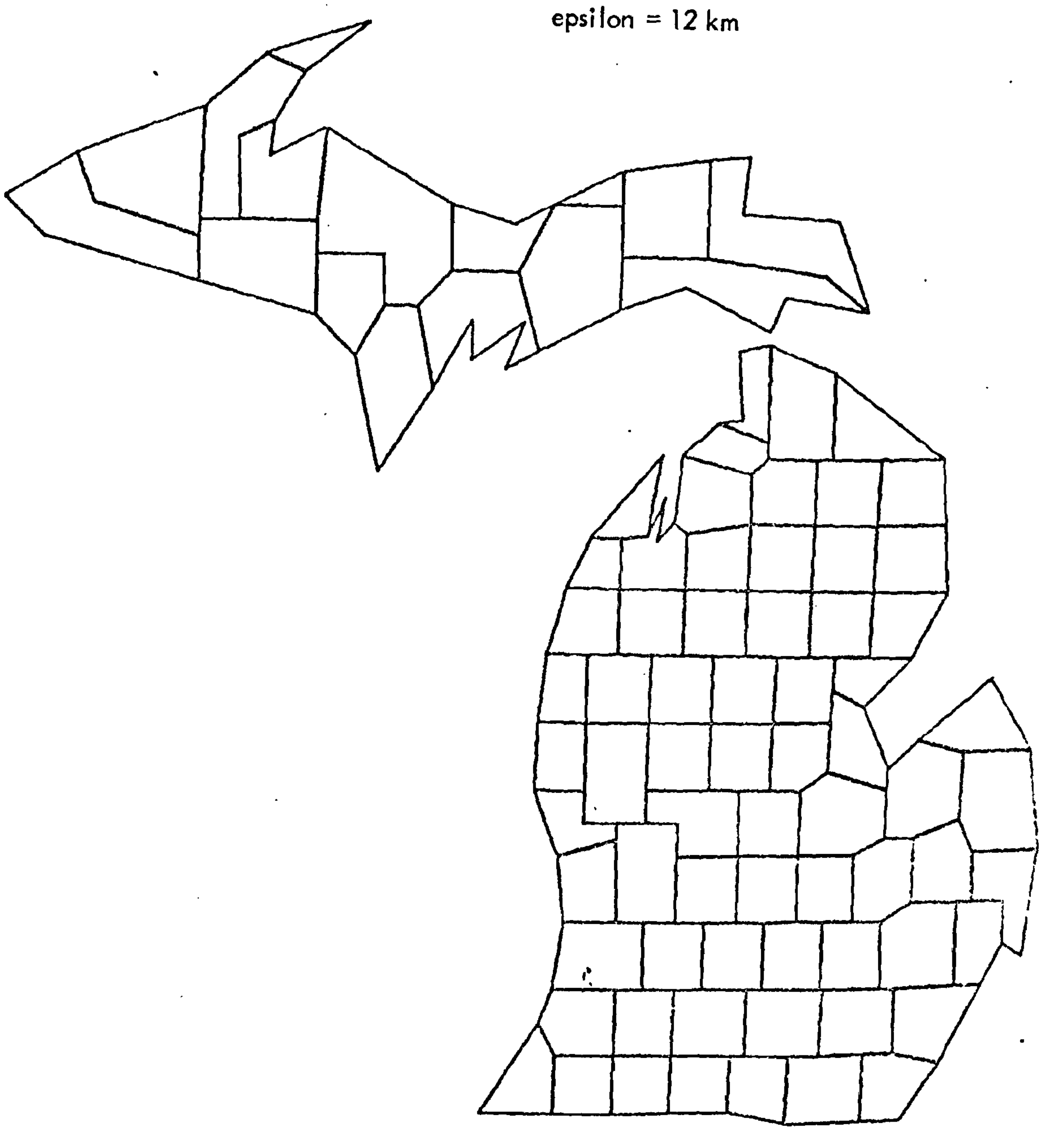
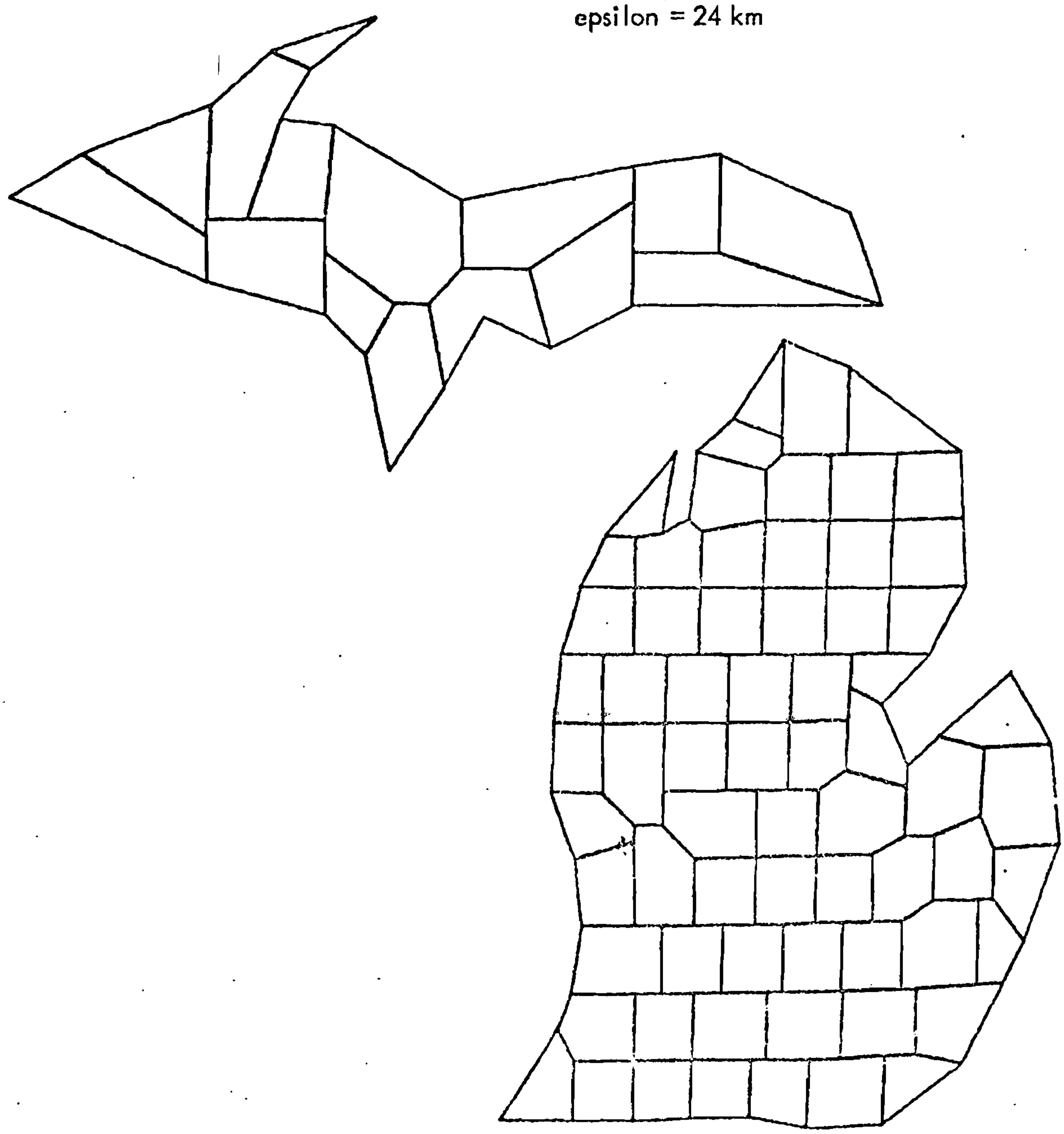


FIGURE 4f

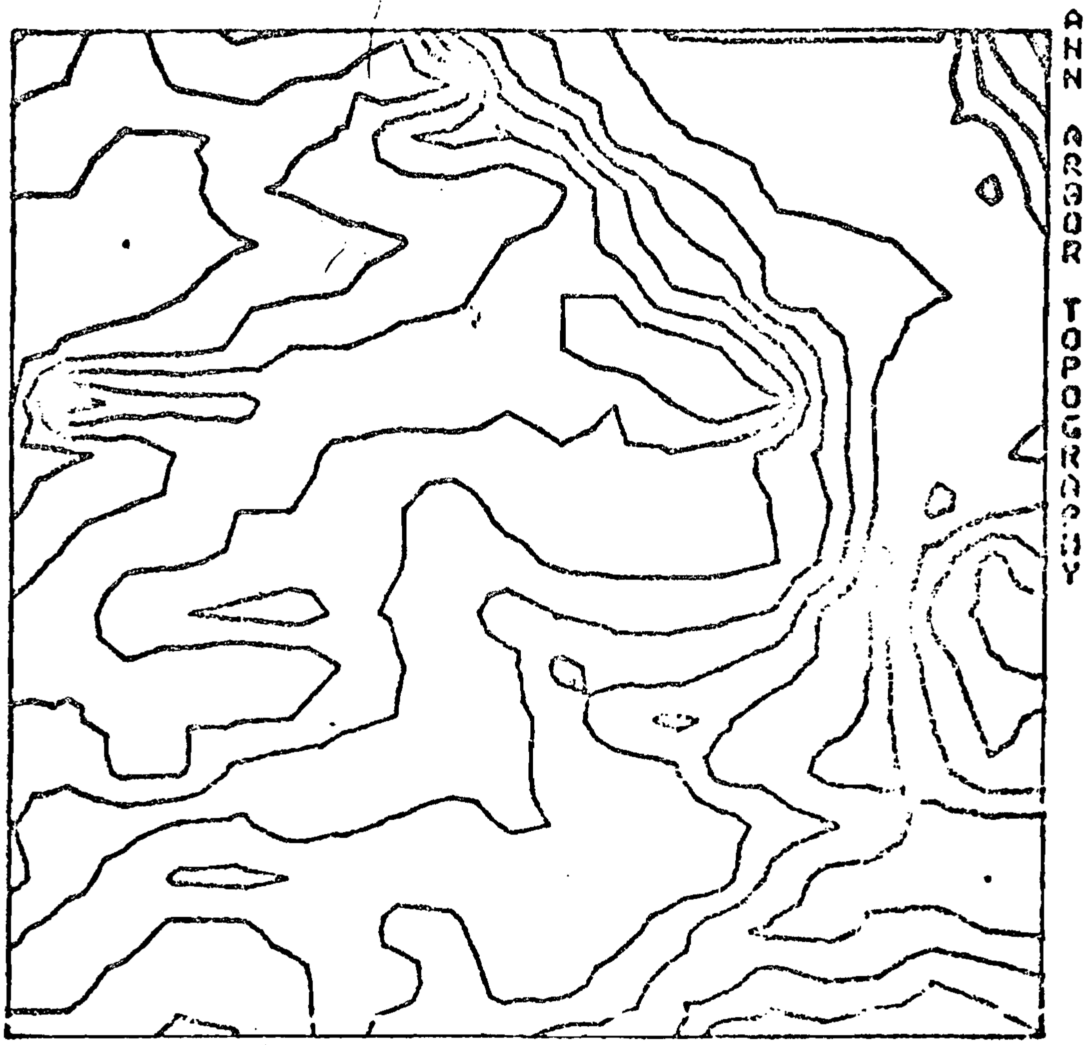
MICHIGAN DESCRIBED BY 154 POINTS

epsilon = 24 km



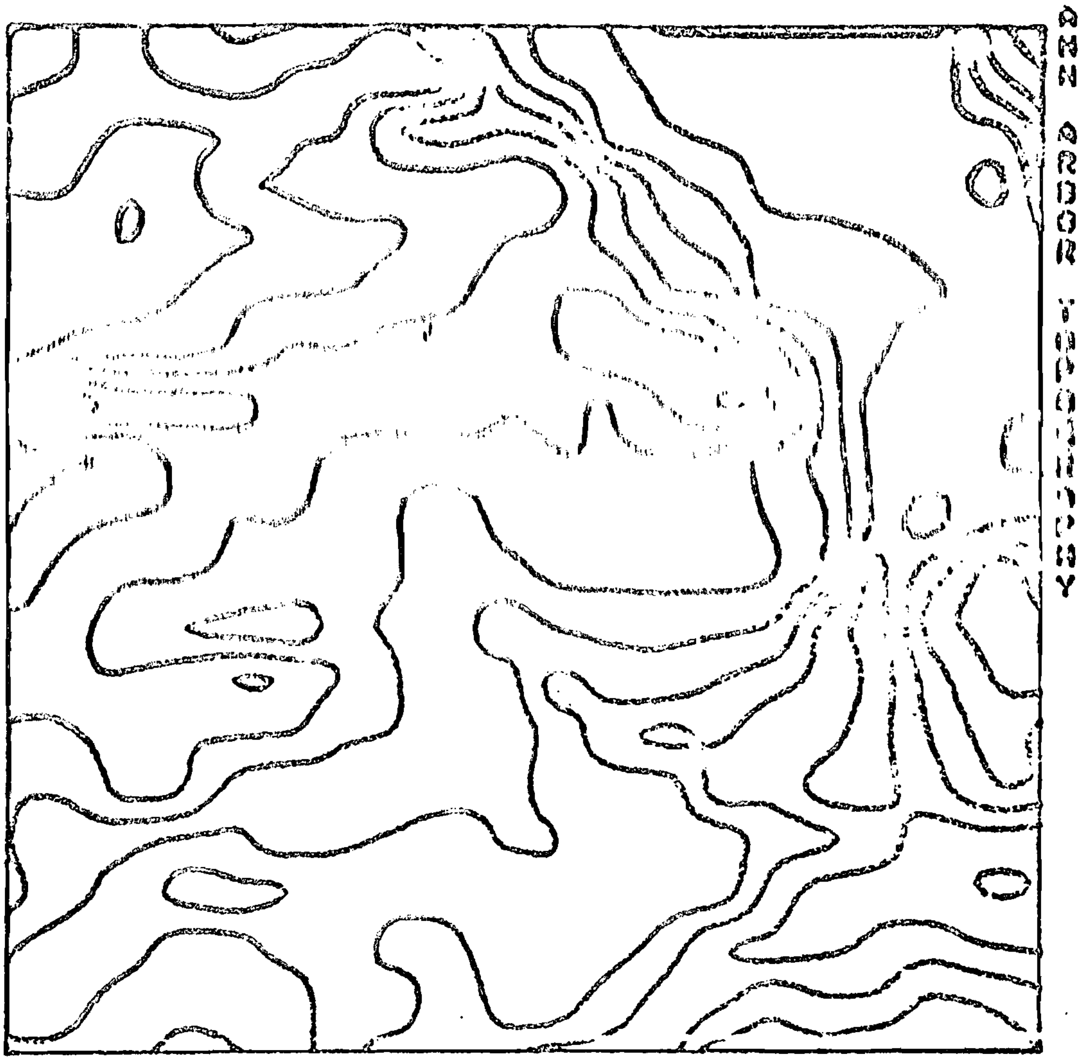
(only eight more points could be deleted before destroying the polygon topology)

FIGURE 5a



Example: a 20 by 20 lattice of data contoured.

FIGURE 5b



Example: Same data splined to an 77 by 77 array, then contoured.

FIGURE 6

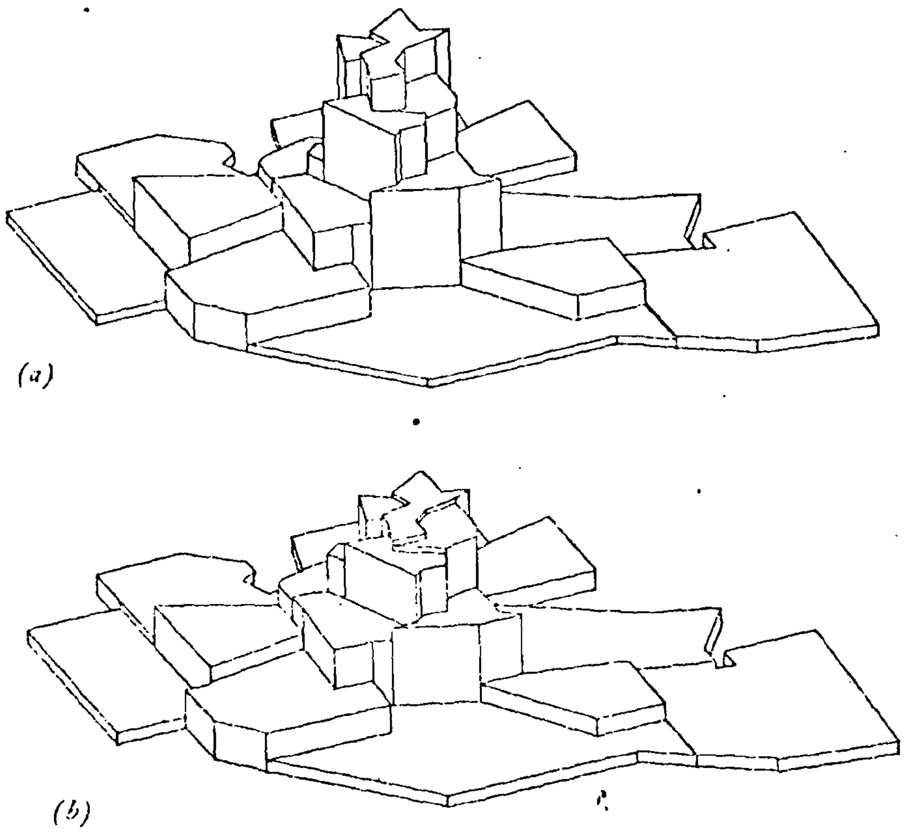


Figure 6 Ann Arbor 1970 population density by census tracts shown in perspective as a piecewise continuous function. Original and linearly modified values.

FIGURE 7a

Example: Michigan Soils
(Atlas of the United States)

FIGURE 7b

Example: Michigan Geology
(Atlas of the United States)

FIGURE 7c

Example: Intersection of Michigan Soils and Geology

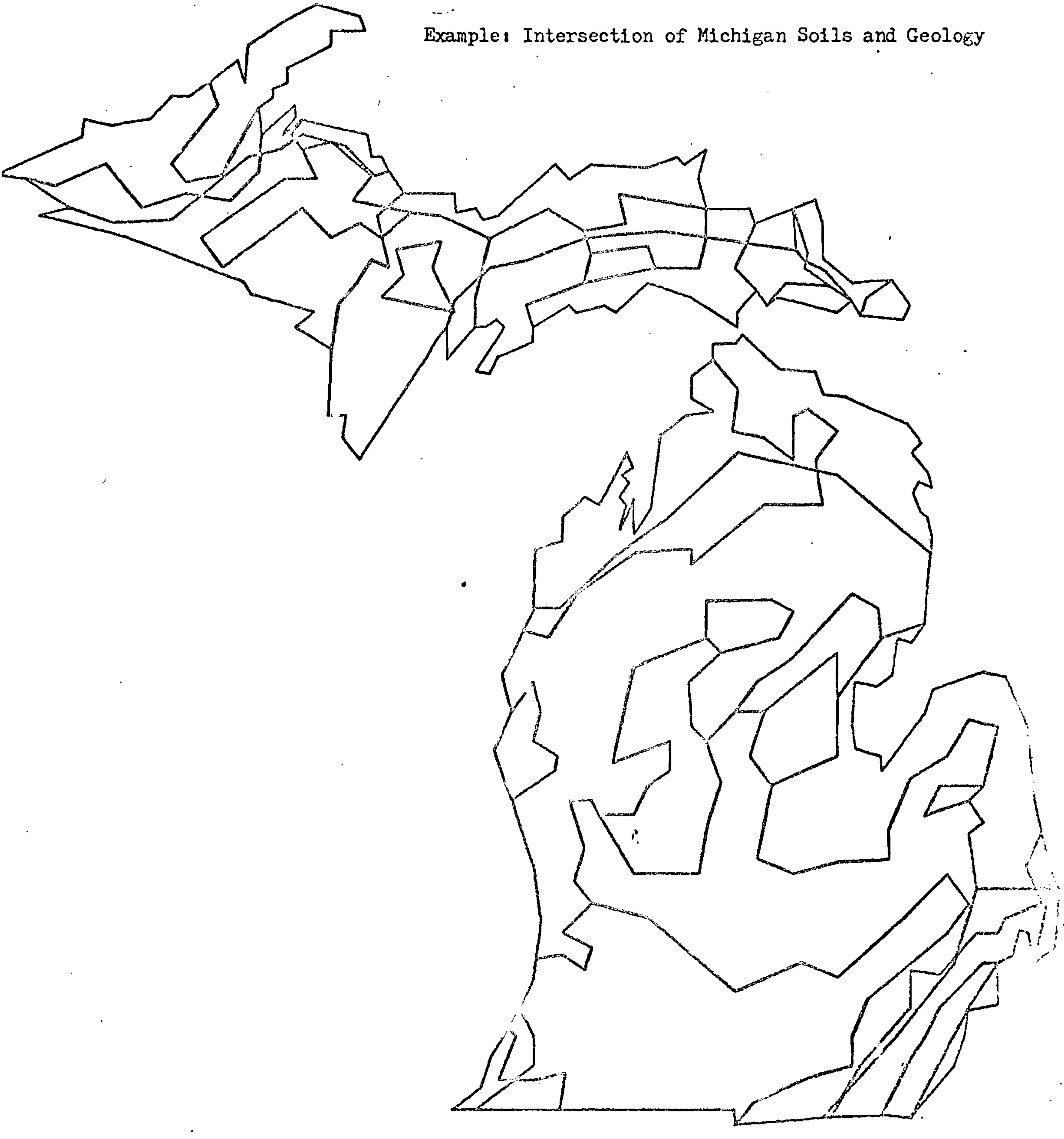


Figure 8

Tobler: Continuous Transformation

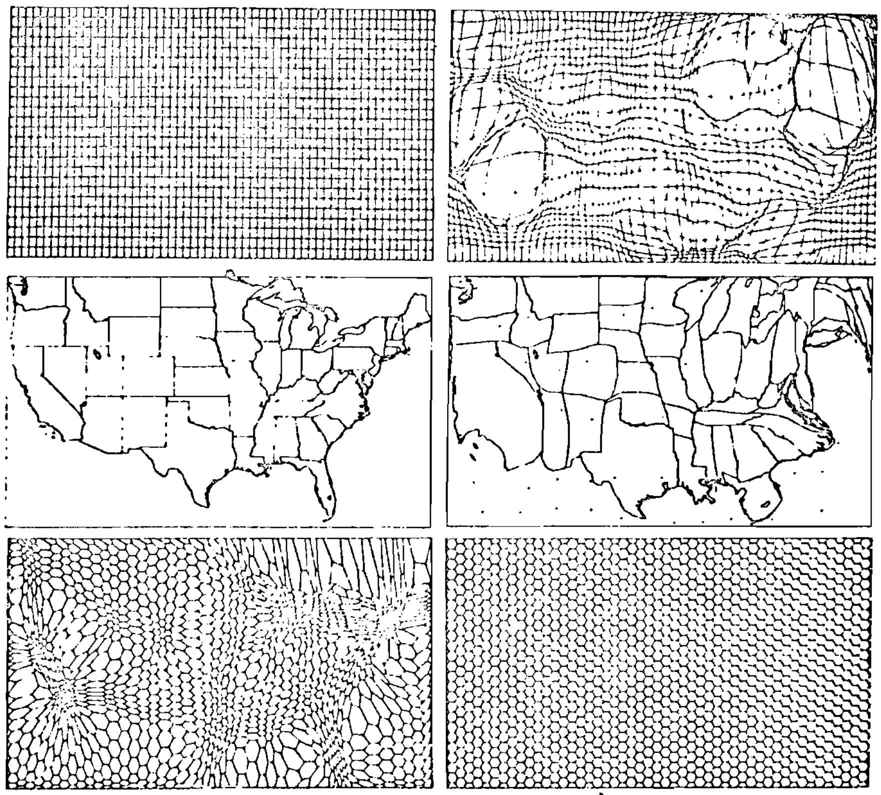


FIGURE 8. The left column pertains to the usual type of map; the right column, to the population cartogram. The cartogram has converged to 68% of its desired accuracy after 99 iterations. Top row: one degree latitude and longitude graticule for the secant plate carée map projection and for the cartogram, respectively Middle row: maps corresponding to the above. Bottom row: compact regions of equal population for each of the two maps.