Working Paper

Observability of Parabolic Systems under Scanning Sensors

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Foreword

This paper continues the investigations in SDS on observability issues motivated by environmental monitoring and related problems. Here the author introduces a specific class of scanning sensors that ensure solvability of the problem and can further lead to numerically robust techniques.

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A.Yu. Khapalov

1. Introduction , Statement of Problem.

Let Ω be an open bounded domain of an n-dimensional Euclidean space R^n with sufficiently smooth boundary $\partial\Omega$. Consider the following homogeneous problem for the parabolic equation

$$\frac{\partial u(x,t)}{\partial t} = Au(\cdot,t)\,, \tag{1.1}$$

$$t \in T = (0,\theta)\,,\ x \in \Omega \subset R^n\,,\ Q = \Omega \times T\,,\ \Sigma = \partial\Omega \times T,$$

$$u(\xi,t) = 0, \xi \in \partial\Omega, t \in T,$$

$$u(x,0) = u_0(x),\ u_0(\cdot) \in L_2(\Omega)$$

with an unknown initial condition $u_0(x)$.

Here A is the infinitesimal generator of a strongly continuous semigroup $S(t)(t > 0)$ in the Hilbert space $L_2(\Omega)$:

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) - a(x),$$

$$v\sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}\,(x)\xi_i\xi_j,\ \xi_i \in R,\ v = \text{const} \ > 0,$$

$$a_{ij}(x) = a_{ji}(x),\ a_{ij}(\cdot), a(\cdot) \in L_\infty(\Omega),\ i,j = 1,2,\ldots n,$$

$$a(x) \geq 0.$$

We will treat the solution of the initial-boundary value problem (1.1) as a generalized solution [15, 16] from the Banach space $\overset{o}{V}{}_2^{1,0}(Q)$ consisting of all elements of the Sobolev space $H_0^{1,0}(Q)$

that are continuous in t in the norm of $L_2(\Omega)$, although most results of this paper require further smoothness for solutions to the system (1.1), as it will be specified below.

Below we will use the standard notation for the Sobolev spaces [22 , 15 , 16].

Let $\hat{x}(t)$, $t \in T$ denote a spatial trajectory in the domain Ω ($\hat{x}(t) \in \Omega$) , $\hat{x}(t)$ is measurable on T.

Assume that the available *measurement data* are defined by a scanning sensor (in general, it might be fixed, so as $\hat{x}(t) \equiv \hat{x}$)

$$y(t) = u(\hat{x}(t), t) + \zeta(t), \; t \in T, \tag{1.2}$$

where $y(t)$, $t \in T$ is a scalar observation data, $\zeta(t)$ is an unknown but bounded *measurement error*,

$$\zeta(\cdot) \in \Xi \;\; , \;\; \Xi \subset \mathbf{B}, \tag{1.3}$$

with the set Ξ given and a Banach space $\mathbf{B}$ to be suitably selected below.

In the case of the singleton set Ξ, the equation (1.2) is written as

$$y(t) = u(\hat{x}(t), t), \; t \in T. \tag{1.4}$$

The traditional definition of observability for the parabolic system (1.1) (which derives from the observability theory for ordinary differential equations) says [20, 2, 4] that the system (1.1), (1.4) is *observable* if an initial (or final, which is the same) state of the system can be uniquely determined from the observation $y(t)$ over the time interval $(0, \theta)$.

The infinite - dimensional nature of distributed - parameter systems generates various definitions of observability that are determined by the topological structure of the problem (1.1) - (1.3).

The system described by (1.1) and (1.4) is said [20, 2, 4] to be *exactly observable (or continuously observable) at final time* θ if

$$\exists \gamma > 0 \;\text{ such that }\; \| \mathbf{G}(\cdot)\mathbf{S}(\cdot)u(\cdot, 0) \|_B \geq \; \gamma \, \| u(\cdot, \theta) \|_{L_2(\Omega)}$$

for any solution $u(x, t)$ of the system (1.1).

The latter definition can also be formulated in an equivalent form in terms of *informational domains* [11]:

Definition 1.1 [10, 12]. The informational domain $U(\theta, y(\cdot))$ of states $u(x, \theta)$ of the system (1.1), (1.2) is the set of all those functions $u(x, \theta)$ for each of which there exists a pair

$$\{u_0^*(\cdot), \; \zeta^*(\cdot)\}$$

with the second component satisfying (1.3) and $u_0^*(\cdot) \in L_2(\Omega)$ that, in turn, generates due to (1.1), (1.2) a pair $\{u^*(x,t), y^*(t)\}$ satisfying the equalities

$$u^*(x,\theta) \equiv u(x,\theta), y^*(t) \equiv y(t), t \in T.$$

Definition 1.2 [13 , 14] .The system (1.1) - (1.3) is said to be *strongly observable* if the set $U(\theta, y(\cdot))$ for this problem under

$$\Xi = \{\zeta(\cdot) \mid \| \zeta(\cdot) \|_B \leq 1\} \tag{1.5}$$

is a bounded subset of $L_2(\Omega)$, regardless of the measurement $y(\cdot)$.

Let λ_i, $\omega_i(\cdot)$ $(i = 1, 2, \ldots)$ denote sequences of eigenvalues and respective orthonormalized (in the norm of $L_2(\Omega)$) eigenfunctions for the spectral problem

$$A\omega_i(\cdot) = -\lambda_i \omega_i(\cdot), \ \omega_i(\cdot) \in H_0^1(\Omega),$$

$$< \omega_i(\cdot), \omega_j(\cdot) >= \delta_{ij},$$

so that

$$\lambda_{i+1} \geq \lambda_i; \ \lambda_i \to +\infty, \ \ i \to +\infty \, ; \ \ \delta_{ij} = \begin{cases} 1, \ i = j, \\ 0, \ i \neq j. \end{cases}$$

Denote by $L_2^{(k)}(\Omega)$ the subspace of $L_2(\Omega)$ spanned by the functions

$$\omega_i(\cdot), \quad i = 1, 2, \ldots, k.$$

Definition 1.3 [13, 14].The system (1.1) - (1.3) is said to be *weakly observable* if all the projections $U^{(k)}(\theta, y(\cdot)), k = 1, 2, \ldots$ of the set $U(\theta, y(\cdot))$ on the sequence of finite-dimensional subspaces $L_2^{(k)}(\Omega)$ are bounded, regardless of the measurement $y(\cdot)$.

We should stress that Definition 1.3 is given in the relation to the basis in $L_2(\Omega)$ consisting of eigenfunctions associated with the observed system in question. However, in general, the specified property of sets $U(\theta, y(\cdot))$ may not occur with respect to another basis.

Consider a sequence of functions

$$\psi_i(t) = e^{-\lambda_i t} \omega_i(\hat{x}(t)) \, , \ t \in T, \ i = 1, 2, \ldots$$

and denote by B_i the closed subspace of B spanned by the functions

$$\psi_j(\cdot) \, , \ j = 1, \ldots \, ; j \neq i.$$

Finally let

$$d_i = \inf_{\psi(\cdot)\in B_i} \| \psi_i(\cdot) - \psi(\cdot) \|_B, \quad i = 1, 2, \ldots. \tag{1.6}$$

Then one can observe that the system (1.1) - (1.3) is weakly observable if and only if all of the values d_i $(i = 1, 2, \ldots)$ are positive. Furthermore, if the latter are such that

$$\sum_{i=1}^{\infty} \frac{\exp(-2\lambda_i\theta)}{d_i^2} < \infty,$$

then the system (1.1) - (1.3) is strongly observable.

In case of stationary sensors ($\hat{x}(t) \equiv \hat{x}, t \in T$) the problem (1.6) turns out to be the one of exponential sequences which was investigated in [17, 6] and results were applied to the controllabilty and observability theory by many authors [5, 6, 20, 2, 4, 9].

It is known that if dimensionality of spatial variable x is higher than 1, then

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty,$$

in which case all the values d_i turn to be 0 .The latter means that there does not exist any stationary sensor that can make the system (1.1) - (1.3) weakly or strongly observable under $n \geq 2$.

The main objective of present paper is to present a method (which does not depend upon dimensionality of x) of constructing measurement trajectories $\hat{x}(t)$ in (1.2) that can provide both weak and strong observability for the system (1.1) - (1.3). We assume that disturbances $\zeta(\cdot)$ in the observation equation (1.2) are restricted in the norm of the spaces $C[\varepsilon, \theta]$ or $L_\infty(\varepsilon, \theta)$ with an arbitrary (but preassigned) value $\varepsilon > 0$.

Some results on existence of such a type of scanning sensors under $B = C[\varepsilon, \theta]$ have been presented in [13, 14].

Another approach to the solution of observability problems with scanning sensors was considered in [18] for lumped - parameter systems.

In the last section of the paper we also discuss the case of disturbances from $L_2(\varepsilon, \theta)$ for the one-dimensional parabolic equation.

Section 3 deals with the application of the above method to discrete - time observability problem with scanning sensor. A similar problem in case of stationary sensor and $\theta = \infty$ has been studied in [9].

Remark 1.1. To ensure the correctness of the value $u(\hat{x}(t), t)$ we will require a proper smoothness of solutions to the mixed problem (1.1).

If the dimension n of the spatial variable x is equal to 1, due to embedding theorems [22, 15, 16] we have

$$H^1(\Omega) \subset C(\bar{\Omega}).$$

In case of $n = 2,3$ we will assume in addition that the coefficients of A and the boundary $\partial\Omega$ are sufficiently regular and

$$u(\cdot,\cdot) \in H^{2,1}(Q).$$

Then again due to embedding theorems [22, 15, 16]

$$H^{2,1}(Q) \subset C(\bar{\Omega})$$

In both cases the superposition $u(\hat{x}(t),t)\,, t \in (\varepsilon,\theta)$ will be measurable and bounded [21, 16].

Finally , assuming that the coefficients of the elliptic operator A and the boundary $\partial\Omega$ are such that the system (1.1) admits a unique classical solution, one can find (taking into account the asymptotics of eigenvalues λ_i) a series of systems of type (1.1) such that an arbitrary generalized solution to those satisfies

$$u(\cdot,\cdot) \in C(\bar{\Omega} \times [\varepsilon,\theta]) \tag{1.7}$$

for any $\varepsilon \in (0,\theta)$. The latters give us a class of parabolic systems with an arbitrary dimension of the spatial variable x, which admits scanning sensors (1.2).

For the sake of simplicity we will assume below that all the solutions $u(x,t)$ to the system (1.1) satisfy (1.7) (for general situation see Remark 4.3), in particular

$$\omega_i(\cdot) \in C(\bar{\Omega})\,, \quad i = 1,2,\dots .$$

Remark 1.2 . Due to the generalized maximum principle for solutions of the initial - boundary value problem (1.1) [15] we have an estimate

$$\operatorname*{vrai\,max}_{x\in\Omega} \mid u(x,t') \mid \geq M \operatorname*{vrai\,max}_{x\in\Omega} \mid u(x,t'') \mid, \; t'' \geq t' \geq 0,$$

$$M = \text{const.}$$

According to Remark 1.1 this estimate will be used below in the form (without loss of generality we can put $M = 1$) [7] :

$$\max_{x\in\bar{\Omega}} \mid u(x,t') \mid \geq \max_{x\in\bar{\Omega}} \mid u(x,t'') \mid, \; t'' \geq t' \geq \varepsilon > 0. \tag{1.8}$$

2. Observability under Disturbances from $C[\varepsilon, \theta]$ and $L_\infty(\varepsilon, \theta)$.

Let ε be an arbitrary positive number from T. Consider the initial - boundary value problem (1.1) under the measurement data $y(t)$ taken over subinterval $T_\varepsilon = (\varepsilon, \theta)$ of the time interval T :

$$y(t) = u(\hat{x}(t), t) + \zeta(t), \; t \in T_\varepsilon. \tag{2.1}$$

The main results of this section are derived under assumption that disturbances $\zeta(\cdot)$ are subjected to the restriction

$$\max_{t \in [\varepsilon, \theta]} | \zeta(\cdot) | \leq 1. \tag{2.2}$$

In the sequel we will extend them to the case of $L_\infty(T_\varepsilon)$.

It is clear that if some trajectory $\hat{x}(t), t \in T$ makes the system (1.1), (2.1), (2.2) be observable then it does the same for the system (1.1) - (1.3).

It is well-known that any solution to problem (1.1) admits a unique representation as

$$u(x,t) = \mathbf{S}(t) u_0(\cdot) = \sum_{i=1}^{\infty} e^{-\lambda_i t} < u_0(\cdot), \, \omega_i(\cdot) > \omega_i(x), \tag{2.3}$$

where

$$< u_0(\cdot), \, \omega_i(\cdot) >= \int_\Omega u_0(x) \omega_i(x) dx.$$

Due to Remark 1.1 we assume that the series (2.3) converges in the norm $C(\bar{\Omega} \times [\varepsilon, \theta])$ for all $\varepsilon > 0$.

Let us select in the interval T_ε an arbitrary monotone sequence of points $\{t_i\}_{i=0}^{\infty}$ such that

$$\varepsilon = t_0 < t_1 < t_2 < \ldots < t_i < t_{i+1} < \ldots < \theta.$$

Denote

$$\tau_i = (t_{i-1}, t_i), \;\; i = 1, \ldots .$$

Assume first that the unknown solution $u(x,t)$ is generated by initial condition $u_0(x)$ from $L_2^{(1)}(\Omega)$, e.g

$$u(x,t) = e^{-\lambda_1 t} u_{01} \omega_1(x),$$

where

$$u_{01} = \int_\Omega u(x,0) \, \omega_1(x) \, dx.$$

Take t_1^1 from τ_1 and denote by $\hat{x}^1_{(1)}$ the solution of the optimization problem

$$\hat{x}^1_{(1)} = \arg\{\max \mid \omega_1(x) \mid \mid x \in \bar{\Omega}\}.$$

Consider an arbitrary measurement trajectory $\hat{x}(t)$ such that at time t_1^1 it passes through a point $\hat{x}^1_{(1)}$:

$$\hat{x}(t_1^1) = \hat{x}^1_{(1)}.$$

Then , if any element $u^*(\cdot, \theta)$ belongs to $U(\theta, y(\cdot)) \bigcap L_2^{(1)}(\Omega)$ so that

$$u^*(x,t) = e^{-\lambda_1 t} u^*_{01} \omega_1(x),$$

one can obtain

$$y(t_1^1) = u(\hat{x}(t_1^1), t_1^1) + \zeta(t_1^1)\ . \tag{2.4}$$

We note next that due to linearity of equations (1.1), (2.1) with respect to $u(\cdot,\cdot)$ to prove observability (weak or strong) of the system (1.1), (2.1), (2.2) it is sufficient to consider the case when the information domain $U(\theta, y(\cdot))$ for the latter is "largest" possible , so as

$$y(t) \equiv 0,\ t \in T_\varepsilon.$$

Thus , for any element $u^*(\cdot, \theta)$ of the set $U(\theta, \{0\}) \bigcap L_2^{(1)}(\Omega)$ from (2.2) - (2.4) the following estimate is fulfilled:

$$\mid u^*_{01} \mid \leq e^{+\lambda_1 t_1^1} \mid \omega_1^{-1}(\hat{x}^1) \mid .$$

This gives

$$\| u^*(\cdot, \theta) \|_{L_2(\Omega)} \leq e^{+\lambda_1 (t_1^1 - \theta)} (\text{meas } \Omega)^{1/2},$$

since

$$\| \omega_1(\cdot) \|^2_{L_2(\Omega)} \leq \max\{\mid \omega_1(x) \mid^2 \mid x \in \bar{\Omega}\} \text{ meas } \Omega.$$

Let us proceed now to the general case. Let $u(\cdot, \theta)$ be an element of $U(\theta, y(\cdot)) \bigcap L_2^{(k)}(\Omega)$ (if the latter is non-empty) and hence

$$u(x,t) = \sum_{i=1}^{k} e^{-\lambda_i t} < u_0(\cdot),\ \omega_i(\cdot) > \omega_i(x). \tag{2.5}$$

Denote by Φ_k the set

$$\Phi_k = \{v(\cdot) \mid \| v(\cdot) \|_{L_2(\Omega)} = 1, \quad v(\cdot) \in L_2^{(k)}(\Omega)\}.$$

We note next that Φ_k is also a bounded finite - dimensional subset of $C(\bar{\Omega})$. This enables us to associate with it for any positive γ (whose dependence upon k will be specified below) the finite γ-net

$$\Phi_k^\gamma = \{v_j(\cdot)\}_{j=1}^{J_k}.$$

Thus for any element $v(\cdot) \in \Phi_k$ there exists an integer $j = j_* \leq J_k$ such that

$$\| v(\cdot) - v_{j_*}(\cdot) \|_{C(\bar{\Omega})} \leq \gamma. \tag{2.6}$$

The maximum principle (1.8) (applied for the set Φ_k) turns the γ-net Φ_k^γ (due to finite dimensionality of the latter (1.8) can be extended for $[0, \theta]$) into the (also finite) γ-net

$$\Phi_k^\gamma(\cdot) = \{u_k^j(x,t), \ x \in \bar{\Omega}, t \in [0, \theta] \ \}_{j=1}^{J_k}$$

in the space $C(\bar{\Omega} \times [0, \theta])$ for the set all of the possible solutions $u(x,t)$ generated at instant $t = 0$ by the set $\Phi_k \subset C(\bar{\Omega})$.

Choosing in τ_k an arbitrary sequence of instants of time t_k^j, $j = 1, 2, \ldots J_k$, so as

$$t_{k-1} < t_k^1 < t_k^2 < \ \ldots \ < t_k^{J_k} < t_k \ \leq \theta, \tag{2.7}$$

we introduce a series of optimization problems

$$\max \{| u_k^j(x, t_k^j) | \mid x \in \bar{\Omega}\} , \ j = 1, \ldots, J_k. \tag{2.8}$$

Let $\hat{x}_{(k)}^j, j = 1, \ldots, J_k$ denote a sequence of solutions to (2.8) .

Let $\hat{x}(t) \equiv \hat{x}^*(t) \, (t \in T_\epsilon)$ be an arbitrary spatial curve in Ω that passes at instants t_k^j through spatial points $\hat{x}_{(k)}^j, j = 1, \ldots, J_k$.

Now, if any element $u^*(\cdot, \theta)$ belongs to $U(\theta, y(\cdot)) \bigcap L_2^{(k)}(\Omega)$ (so as (2.5) is valid) , then from (2.1) it follows that

$$y(t_k^j) = u(\hat{x}^*(t_k^j), t_k^j) + \zeta(t_k^j), \ j = 1, \ldots, J_k. \tag{2.9}$$

Let α denote the norm in the space $L_2(\Omega)$ of function $u^*(\cdot, t)$ taken at $t = 0$:

$$\alpha = \| u^*(\cdot, 0) \|_{L_2(\Omega)} \ .$$

Select an element $u_k^{j_*}(\cdot, \cdot) \in \Phi_k^\gamma(\cdot)$ so that

$$| \alpha^{-1} u^*(x,t) - u_k^{j_*}(x,t) | \leq \gamma \quad \text{for all } x \in \Omega \, , \ t \in [0, \theta]. \tag{2.10}$$

Thus we have

$$\| u^*(\cdot, t_k^{j_*}) \|_{C(\bar\Omega)} \leq \alpha \| u_k^{j_*}(\cdot, t_k^{j_*}) \|_{C(\bar\Omega)} + \alpha\gamma. \tag{2.11}$$

On the other hand (using (2.8) and again (2.10)),

$$\alpha \| u_k^{j_*}(\cdot, t_k^{j_*}) \|_{C(\bar\Omega)} = | \alpha u_k^{j_*}(\hat{x}_{(k)}^{j_*}, t_k^{j_*}) | \leq | u^*(\hat{x}^*(t_k^{j_*}), t_k^{j_*}) | + \alpha\gamma. \tag{2.12}$$

Assuming ("the worst case ") $y(t) \equiv 0, t \in T_e$ and combining (2.2) , (2.9) - (2.12) we obtain

$$\| u^*(\cdot, t_k^{j_*}) \|_{C(\bar\Omega)} \leq 1 + 2\alpha\gamma. \tag{2.13}$$

Now note that

$$\alpha^2 e^{-2\lambda_k t_k^{j_*}} = e^{-2\lambda_k t_k^{j_*}} \sum_{i=1}^{k} < u^*(\cdot, 0), \omega_i(\cdot) >^2 \leq \sum_{i=1}^{k} e^{-2\lambda_i t_k^{j_*}} < u^*(\cdot, 0), \omega_i(\cdot) >^2 =$$

$$= \int_\Omega u^{*2}(x, t_k^{j_*}) dx \leq \| u^*(\cdot, t_k^{j_*}) \|^2_{C(\bar\Omega)} (\text{meas}\, \Omega). \tag{2.14}$$

Combining (2.13) and (2.14) we obtain

$$(\text{meas}\, \Omega)^{-1/2} (\sum_{i=1}^{k} e^{-2\lambda_i t_k^{j_*}} < u^*(\cdot, 0), \omega_i(\cdot) >^2)^{1/2} - 2\gamma (\sum_{i=1}^{k} < u^*(\cdot, 0), \omega_i(\cdot) >^2)^{1/2} \leq 1. \tag{2.15}$$

Assume now that the value of γ depends upon k. The latter affects only the number of elements in the γ-net $\Phi_k^\gamma(\cdot)$. We will choose $\gamma = \gamma_k$ from the following condition

$$0 < \gamma_k < \frac{1}{2} (\text{meas}\, \Omega)^{-1/2} e^{-\lambda_k t_k},$$

which in its turn yields

$$\gamma_k = \beta_k \frac{1}{2} (\text{meas}\Omega)^{-1/2} e^{-\lambda_k t_k} \quad \text{with} \quad 0 < \beta_k < 1. \tag{2.16}$$

Taking into account the inequality

$$e^{-\lambda_k t_k} (\sum_{i=1}^{k} < u^*(\cdot, 0), \omega_i(\cdot) >^2)^{1/2} \leq (\sum_{i=1}^{k} e^{-2\lambda_i t_k^{j_*}} < u^*(\cdot, 0), \omega_i(\cdot) >^2)^{1/2}$$

and combining (2.15) , (2.16) we finally obtain

$$\| u^*(\cdot, \theta) \|_{L_2(\Omega)} \leq$$

$$\| u^*(\cdot, t_k) \|_{L_2(\Omega)} \leq (\sum_{i=1}^{k} e^{-2\lambda_i t_k^{j_*}} < u^*(\cdot, 0), \omega_i(\cdot) >^2)^{1/2} \leq (\text{meas}\, \Omega)^{1/2} \frac{1}{1 - \beta_k}, \tag{2.17}$$

which is valid for all $k = 1, \ldots$.

The estimate (2.17) allows us to construct measurement trajectories making the system (1.1), (2.1), (2.2) be both weakly and strongly observable.

Indeed , let us introduce in the domain Ω a class of spatial curves $\hat{X}(\beta)$ to be continuous in the time interval $[0,\theta]$, excluding perhaps a single instant of time, as follows.

Procedure 2.1. Let β be an arbitrary number from the interval $(0,1)$. For a chosen monotone sequence $\{t_k\}_{k=1}^{\infty} \subset T_\varepsilon$, the relation (2.16) defines the values

$$\gamma_k = \gamma_k(\beta),\ J_k = J_k(\beta),\ k = 1,2\dots .$$

Then, selecting due to (2.7) an arbitrary monotone sequence of instants of time

$$t_k^j\ ,\ \ k = 1,\dots; j = 1,\dots J_k(\beta)\ ,$$

we can obtain from (2.8) the respective sequence of spatial points

$$\hat{x}_{(k)}^j,\ \ k = 1,\dots; j = 1,2\dots J_k(\beta).$$

It is clear that there exists a limit

$$\lim_{k\to\infty} t_k = a \leq \theta.$$

Definition 2.1. We will say that an arbitrary spatial curve $\hat{x}^*(t), t \in [\varepsilon,\theta]$ in the domain Ω *belongs to the class* $\hat{X}(\beta)$ if it is continuous everywhere in $[\varepsilon,\theta]$ except possibly at the instant $t = a$ and such that

$$\hat{x}^*(t_k^j) = \hat{x}_{(k)}^j,\ k = 1,2,\dots,; j = 1,\dots,J_k(\beta), \tag{2.18}$$

where the sequence of pairs $\{t_k^j, \hat{x}_{(k)}^j\}$ is selected according to Procedure 2.1.

Thus we have proved

Lemma 2.1. Let $\hat{x}^*(t)$ be an arbitrary measurement trajectory from the class $\hat{X}(\beta)$ $(0 < \beta < 1$) and let the set $U(\theta, y(\cdot)) \bigcap L_2^{(k)}(\Omega)$ for the system (1.1),(2.1),(2.2) be non-empty. Then to assure that the estimate (2.17) is uniform (over $k = 1,\dots$) it is sufficient for each $k = 1,\dots$ to take into account measurement data (2.1) taken only over the time interval τ_k.

We may *reformulate Lemma 2.1* in the following way :

to solve observability problem (in weak or strong sence) in an arbitrary subspace $L_2^{(k)}(\Omega)$ with the estimate (2.17) it is sufficient if a measurement curve would only pass through a finite number of spatial points specified according to Procedure 2.1 in advance.

This result immediately leads to

Theorem 2.1. An arbitrary measurement trajectory $\hat{x}^*(t)$ from the class $\hat{X}(\beta)$ with $0 < \beta < 1$ makes the system (1.1), (2.1), (2.2) strongly observable.

Proof. Let $u^*(\cdot,\theta)$ be an element of $U(\theta, y(\cdot))$, so as

$$u^*(x,t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} < u_0^*(\cdot),\ \omega_i(\cdot) > \omega_i(x). \tag{2.19}$$

We assume again

$$y(t) \equiv 0 \ \ , \ \ t \in T_\varepsilon.$$

Take any positive constant δ and divide the sum on the right-hand side of (2.19) into two parts

$$u^*(x,t) = u^*_{N^*}(x,t) + v^*_{N^*}(x,t),$$

where

$$u^*_{N^*}(x,t) = \sum_{i=1}^{N^*} e^{-\lambda_i t} < u_0^*(\cdot),\ \omega_i(\cdot) > \omega_i(x),$$

$$v^*_{N^*}(x,t) = \sum_{i=N^*+1}^{\infty} e^{-\lambda_i t} < u_0(\cdot),\ \omega_i(\cdot) > \omega_i(x)$$

in such a way that

$$\| v^*_{N^*}(\cdot,\theta) \|_{L_2(\Omega)} \le \delta, \tag{2.20}$$

and ,besides, due to (2.3)

$$\| v^*_{N^*}(\hat{x}^*(\cdot),\cdot) \|_{C[\varepsilon,\theta]} \le \delta. \tag{2.21}$$

We note next that (2.21) implies

$$u^*_{N^*}(\cdot,\theta) \in U^\delta(\theta,\{0\}) \bigcap L_2^{(N^*)}(\Omega),$$

where $U^\delta(\theta,\{0\})$ stands for the informational domain for problem (1.1), (2.1) under the constraint

$$\max_{t\in[\varepsilon,\theta]} | \zeta(\cdot) | \le \ 1 + \delta. \tag{2.22}$$

Consequently lemma 2.1 gives us the estimate

$$\| u^*_{N^*}(\cdot,\theta) \|_{L_2(\Omega)} \le \ (\text{meas}\,\Omega)^{1/2} \frac{1+\delta}{1-\beta}.$$

Finally , combining estimate (2.20) with the latter , one can obtain

$$\| u^*(\cdot,\theta) \|_{L_2(\Omega)} \le \ (\text{meas}\,\Omega)^{1/2} \frac{1+\delta}{1-\beta} + \delta, \forall \delta > 0. \tag{2.23}$$

The following assertion allows us to make a conclusion about *weak observability* for the system (1.1), (2.1), (2.2) at the *initial instant of time.*

Corollary 2.1. Let $\hat{x}^*(t)$ be an arbitrary measurement trajectory from the class $\hat{X}(\beta)$, $0 < \beta < 1$. If $u^*(\cdot,\theta)$ belongs to $U(\theta,\{0\})$, then

$$< u^*(\cdot,0), \omega_i(\cdot) > \leq (\text{meas}\,\Omega)^{1/2}\frac{e^{\lambda_i\theta}}{1-\beta},\ i = 1,\ldots \tag{2.24}$$

Taking into account the continuity of the curves from the class $\hat{X}(\beta)$, we obtain

Theorem 2.2. An arbitrary measurement trajectory $\hat{x}^*(t)$ from the class $\hat{X}(\beta)$ with $0 < \beta < 1$ makes the system (1.1), (2.1) under the constraint

$$\| \zeta(\cdot) \|_{L_\infty(T_\varepsilon)} \leq 1$$

strongly observable.

Indeed, with disturbances from $L_\infty(T_\varepsilon)$, at each instant t_k^j $(k = 1,2\ldots ; j = 1,\ldots,J_k)$ one can find such a neighborhood of the latter that the estimate (2.17) (or its approximation) is valid for the entire corresponding neighborhood.

Corollary 2.2. An arbitrary measurement trajectory $\hat{x}(t), t \in T$ such that its restriction on T_ε belongs the class $\hat{X}(\beta)$ $(0 < \beta < 0$) makes the system (1.1), (1.4) observable .

Remark 2.1. From Corollary 2.1 it follows that we can obtain estimates for values d_i $(i = 1,2,\ldots$,see(1.6)) when $B = C[\varepsilon,\theta]$ or $L_\infty(T_\varepsilon)$ and

$$\hat{x}(t)) \in \hat{X}(\beta).$$

Indeed , combining inequalities (2.24) with formula (2.5) we obtain

$$d_i = \inf_{\psi(\cdot)\in B_i} \| e^{-\lambda_i t}\omega_i(\hat{x}(t)) - \psi(\cdot) \|_B \geq (\text{meas}\,\Omega)^{-1/2}(1-\beta)e^{-\lambda_i\theta}, i = 1,2,\ldots .$$

3. Discrete – Time Observability under Scanning Sensors.

Consider now the initial-boundary value problem (1.1) under discrete - time measurement data $\{y(t_i)\}_{i=1}^\infty$ that are available through the observation equation

$$y(t_i) = u(x^i,t_i) + \zeta_i\ ,\ i = 1,2,\ \ldots . \tag{3.1}$$

Here $\{t_i\}_{i=1}^\infty(t_i \in T_\varepsilon)$ is a monotone sequence of instants of time ; the spatial points x^i $(x^i \in \Omega)$ describe the location of scanning sensor at the instants $t_i(i = 1,\ldots)$.

Assume that disturbances ζ_i are subjected to the constraint

$$|\ \zeta_i\ | \leq 1,\ i = 1,\ldots. \tag{3.2}$$

Definitions of informational domain and observability can be easily adjusted for this class of observed systems.

Lemma 3.1 . Let the sequence of pairs $\{t_i, x^i\}_{i=1}^{\infty}$ in (3.1) be selected according to Procedure 2.1. Then the system (1.1), (3.1), (3.2) will be strongly observable and the estimate

$$\| u^*(\cdot, \theta) \|_{L_2(\Omega)} \leq (\text{meas}\, \Omega)^{1/2} \frac{1}{1-\beta} \tag{3.3}$$

will be valid for all those solutions $u^*(x,t)$ to the system (1.1) that satisfy (3.1), (3.2).

The proof of this lemma immediately follows from the theorem 2.1.

The following assertion demonstrates a principal application of Procedure 2.1.

Theorem 3.1 . Among all sequences of pairs that satisfy Procedure 2.1 there is a subclass of those such that any element of the latter, as well as its restriction on an arbitrary time interval $(a, b) \subset T_\varepsilon$ (regardless of the duration), makes the system (1.1), (3.1), (3.2) strongly observable and the estimate (3.3) holds.

*Proof.*The idea of the proof is based on the fact that results of the previous section exploit measurement data taken only at the countable set of instants of time. Furthermore,these instants can be allocated arbitrarily in the interval T_ε.

Indeed, let $\{t_i\}_{i=1}^{\infty}$ be an arbitrary subset of the time interval T_ε that is dense in the latter and let $\{\delta_j\}_{j=1}^{\infty}, \delta_j > 0$ denote an arbitrary sequence such that

$$\lim_{j \to \infty} \delta_j = 0.$$

Now for each interval $(t_i - \delta_j, t_i) \bigcap T_\varepsilon$ we can select a sequence of pairs according to Procedure 2.1

$$\{t_{ij}^k, x_{ij}^k\}_{k=1}^{\infty} \tag{3.4}$$

with $\beta > 0$ given.

In addition we assume that the sequence $\{t_{ij}^k\}$ is monotone and

$$\lim_{k \to \infty} t_{ij}^k = t_i \ ; \ i, j = 1, \ldots .$$

We note next that countability of the set of indices $i, j, k = 1, 2 \ \ldots$ allows one to select all of the instants t_{ij}^k to be distinct. This means that by an appropriate relabelling, we can obtain an increasing sequence of instants of time (dense in T_ε)

$$\varepsilon < \ \ldots \ < t^{m-1} < t^m < t^{m+1} < \ \ldots \ < \theta \,,$$

and a corresponding (due to (3.4)) sequence of points $\{x^m\}_{-\infty}^{+\infty}$.

The sequence of pairs

$$\{t^m, x^m\}_{-\infty}^{\infty} \tag{3.5}$$

ensures strong observability for the system (1.1), (3.1), (3.2) , as well as its restriction on an arbitrary time interval $(a,b) \in T_\varepsilon$.

*Remark 3.1.*Theorem 3.1 ensures a priori some guaranteed result for solution of the state estimation problem defined by (1.1), (3.1), (3.2).

Indeed , the sequence (3.5) allows one to start the process of observation at any instant of time t^* from T_ε and to end it at any time $t^{**} > t^*$. The measurement data obtained from the time interval (t^*, t^{**}) ensures the estimate (3.3).

4. One - Dimensional Case.

Disturbances from $\boldsymbol{L_2(T_\varepsilon)}$. In this section let us consider the initial - boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) - a(x)u(x,t), \quad 0 < x < 1, \quad t \in T, \tag{4.1}$$

$$a(x) \geq 0$$

$$u(t,0) = u(t,1) = 0, \quad u(x,0) = u_0(x).$$

The observability problem for the system (4.1) under stationary pointwise observations

$$y(t) = u(\bar{x},t) + \zeta(t), \quad t \in T_\varepsilon \tag{4.2}$$

has been studied by many authors [20, 2, 4].

In particular, it is well-known [20, 2, 4] that (4.1) and (4.2), with $a(x) = 0$, is observable if the point $\bar{x}$ is taken to be irrational, and it is strongly observable if $\bar{x}$ is an irrational number of a special type.

Let us now consider the dynamic measurement equation

$$y(t) = y(\hat{x}(t),t) + \zeta(t), \quad t \in T_\varepsilon, \tag{4.3}$$

assuming at the begining that disturbances $\zeta(t)$ are subjected to the restriction (2.2).

Suppose that the trajectory $\hat{x}(t), t \in T_\varepsilon$ is a monotone (increasing) smooth function of time connecting ends of the interval [0,1], so as

$$\hat{x}(\varepsilon) = 0\ ,\ \hat{x}(\theta) = 1. \tag{4.4}$$

Then, applying the maximum principle (see Remark 1.2) [7, 15] to the domain

$$D = \{(x,t) \mid 0 \le x \le \hat{x}(t), t \in T_\varepsilon\},$$

one can obtain the estimate

$$\| u(x,\theta) \|_{C[0,1]} \le \| u(\hat{x}(t),t) \|_{C[\varepsilon,\theta]}, \tag{4.5}$$

which ensures strong observability for the system (4.1), (4.3), (2.2) with $B = C[\varepsilon,\theta]$.

The estimate (4.5) is the same for an arbitrary monotone trajectory $\hat{x}(t), t \in T_\varepsilon$ satisfying (4.4). The latter makes such a class of trajectories stable with respect to those possible perturbations that force perturbed measurement curve to stay in the mentioned class.

Now let us multiply both sides of equation (4.1) by $u(x,t)$ and apply Green's formula for the domain D :

$$\begin{aligned}\int_D \frac{\partial u(x,t)}{\partial t} u(x,t)dxdt - \int_D (\Delta u(x,t) - a(x)u(x,t))u(x,t)dxdt = \\ = \frac{1}{2}\int_0^1 u^2(x,t)dx \,|_{t(x)}^{\theta} - \int_\varepsilon^\theta \frac{\partial u(x,t)}{\partial x} u(x,t)dt \,|_0^{\hat{x}(t)} + \\ + \int_D a(x)u^2(x,t)dxdt + \int_D (\frac{\partial u(x,t)}{\partial x})^2 dxdt = 0,\end{aligned} \tag{4.6}$$

where $t(x)$ is inverse to $x(t)$.

Combining (4.6) with the boundary condition in (4.1), we obtain

$$\frac{1}{2}\int_0^1 u^2(x,\theta)dx = \frac{1}{2}\int_0^1 u^2(x,t(x))dx +$$

$$+\int_\varepsilon^\theta \frac{\partial u(\hat{x}(t),t)}{\partial x} u(\hat{x}(t),t)dt - \int_D a(x)u^2(x,t)dxdt - \int_D (\frac{\partial u(x,t)}{\partial x})^2 dxdt.$$

Note next that

$$\int_0^1 u^2(x,t(x))dx = \int_\varepsilon^\theta u^2(\hat{x}(t),t)\dot{\hat{x}}(t)dt,$$

with $\dot{\hat{x}}(t) \ge 0$.

Finally, if we introduce the additional constraint for the derivative of $\hat{x}(t)$:

$$| \dot{\hat{x}}(t) | \le c, \quad \forall t \in T_\varepsilon, c = \text{const}, \tag{4.7}$$

for any trajectory that satisfies (4.7) we obtain

$$\| u(\cdot,\theta) \|_{L_2(\Omega)} \le C(\| u(\hat{x}(\cdot),\cdot) \|_{L_2(T_\varepsilon)} + \| \frac{\partial u(\hat{x}(\cdot),\cdot)}{\partial x} \|_{L_2(T_\varepsilon)}), \tag{4.8}$$

where the constant C is the same for an arbitrary monotone smooth curve $\hat{x}(t)$ such that conditions (4.4), (4.7) are fulfilled.

Denote the latter class of trajectories by $X(C)$.

Thus we have proved

Theorem 4.1 . An arbitrary trajectory $\hat{x}(t)$, $t \in T_\varepsilon$ from the class $X(C)$ ensures strong observability with the estimate (4.8) for the system (4.1) , (4.2) complemented by one more observation equation

$$z(t) = \frac{\partial u(\hat{x}(t), t)}{\partial x} + \xi(t), \quad t \in T_\varepsilon,$$

under constraints

$$\| \zeta(\cdot) \|_{L_2(T_\varepsilon)} \leq 1, \quad \| \xi(\cdot) \|_{L_2(T_\varepsilon)} \leq 1.$$

Remark 4.1. As an advantage of scanning observations itroduced above in this section (with respect to stationary sensors) one may consider a principal possibility to construct simple classes of them that are stable with respect to some reasonable perturbations.

Remark 4.2. We note next that stationary property of elliptic operator in (4.1) was not used in this section.

Remark 4.3. The results of this paper can be extended to the special class of spatially-averaged scanning observations [13, 14]:

$$\mathbf{G}(t)u(\cdot, t) = \int_\Omega \chi(x, \hat{x}(t))u(x, t)dx,$$

where

$$\chi(x, \hat{x}(t)) = \beta(t) \times \begin{cases} 1, & \text{if } x \in S_{h(t)}(\hat{x}(t)) \cap \Omega, \\ 0, & \text{if } x \bar{\in} S_{h(t)}(\hat{x}(t)) \cap \Omega, \end{cases}$$

$S_{h(t)}(\hat{x}(t))$ is the Euclidean neighborhood (in R^n) of radius $h(t)$ of point $\hat{x}(t)$; $\hat{x}(t), t \in T$ is a trajectory in the domain Ω; the function $\beta(t)$ is given.

This type of observations does not require continuity of solutions to the system (1.1). Instead of this, propeties of Lebesque points play the crucial role in the construction of measurement trajectories.

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