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Value Functions and Optimality Conditions for Semilinear Control Problems. II: Parabolic Case

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IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

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Value Function and Optimality Conditions for Semilinear Control Problems. II: Parabolic Case

P. Cannarsa *
Dipartimento di Matematica
II Università di Roma
Via O. Raimondo
00173 Roma, Italy

H. Frankowska
CNRS, CEREMADE,
Université Paris-Dauphine
75775 Paris Cedex 16, France

Foreword

In this paper the authors study properties of the value function and of optimal solutions of a semilinear Mayer problem in infinite dimensions. Applications concern systems governed by a state equation of parabolic type. In particular, the issues of the joint Lipschitz continuity and semiconcavity of the value function are treated in order to investigate the differentiability of the value function along optimal trajectories.

A.B.Kurzhanski, Chairman
Systems and Decision Sciences

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1 Introduction

In a previous paper [CF1], we studied the Mayer problem

$$\text{minimize } g(x(T)) \tag{1}$$

over all solutions of the semilinear control system

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t), u(t)), \quad u(t) \in U \\ x(t_0) = x_0 \end{cases} \tag{2}$$

where x_0 belongs to a Banach space $X, t_0 \in [0, T]$ and A is the infinitesimal generator of a *strongly continuous* semigroup on X. We have shown that useful information on optimal trajectories may be derived from properties of the value function defined as

$$V(t_0, x_0) = \inf\{g(x(T)) | x(\cdot) \text{ is a solution of } (2)\}$$

for all $(t_0, x_0) \in [0, T] \times X$.

Under suitable assumptions, V was proved to be Lipschitz *with respect to* x ([BDP]) and semiconcave *with respect to* x ([CF1]). These properties can be used to deduce the differentiability of V with respect to x along optimal trajectories ([CF1]).

The present paper is mainly devoted to the description of the richer set of properties that V possesses when $-A$ is a *sectorial* operator. Analogous properties were obtained in [CF2], [CF3] for a finite dimensional context.

For example, we show that, if A is the generator of an analytic semigroup, then V is *jointly* Lipschitz (Theorem 3.1) and semiconcave (Theorem 4.1) in (t, x) on $[0, T[\times X$ (elementary examples show that this fails to be true on the whole domain $[0, T] \times X$). This increase in smoothness of V in the analytic case, is due to the fact that the solution $x(t)$ of (2) belongs to the domain of the fractional power $(-A)^\alpha$ for all $\alpha \in [0, 1[$ and all $t \in]t_0, T]$.

It is known that V satisfies the Hamilton-Jacobi equation (in the viscosity sense)

$$-V_t(t, x) + H(t, x, -V_x(t, x)) - \langle V_x(t, x), Ax \rangle = 0,$$

where $H(t, x, p) = \sup_{u \in U} \langle p, f(t, x, u) \rangle$ ([CL1], [CL2]). In particular, this fact implies that for all $(t, x) \in]0, T[\times D(A)$ and all $(p_t, p_x) \in D^+V(t, x)$ we have

$$-p_t + H(t, x, -p_x) - \langle p_x, Ax \rangle \leq 0. \tag{3}$$

In this paper we show that equality holds in (3) along any optimal trajectory $\bar{x}(\cdot)$ in the following form: for all $(p_t, p_x) \in D^+V(t, \bar{x}(t))$ and $\alpha \in]0,1[$

$$-p_t + H(t, \bar{x}(t), -p_x) + \langle (A^*)^\alpha p_x, (-A)^{1-\alpha} \bar{x}(t) \rangle = 0 \qquad (4)$$

for any $t \in]t_0, T[$, see Theorem 5.2. To justify equation (4) we note that, as a consequence of the Lipschitz properties of V, $D_x^+ V(t,x)$ is contained in $D((-A^*)^\alpha)$ for all $\alpha \in [0,1[$ and all $(t,x) \in [0,T[\times X$ (Corollary 3.4). If the hamiltonian H is strictly convex with respect to p, then (4) yields the differentiability of V along optimal trajectories except for end points (Corollary 5.4).

For a finite dimensional space X, equality (4) was derived in [Zh] for *almost every* $t \in [t_0, T]$. Therefore, the result of Theorem 5.2 improves the analogous result for finite dimensions.

Let us consider the subset $D^*V(t,x)$ of $D^+V(t,x)$, which consists of all weak$-*$ limits of $\nabla V(t_i, x_i)$ where $(t_i, x_i) \to (t,x)$. We recall that $D^*V(t,x)$ is a set of generators for the convex set $D^+V(t,x)$, due to the semiconcavity of V (see Section 2). In Section 5 we shall prove that equality holds in (3) at all points of $D^*V(t,x)$ i.e.

$$p_t + \langle (-A^*)^{1-\alpha} p_x, (-A)^\alpha x \rangle = H(t, x, -p_x) \qquad (5)$$

for all $(t,x) \in [0,T[\times D(-A)^\alpha, (p_t,p_x) \in D^*V(t,x), \alpha \in]0,1[$. In particular, (5) implies that V is (t,x)–differentiable at all points $(t,x) \in [0,T[\times D(-A)^\alpha$ at which $D_x^+V(t,x)$ is a singleton.

This property has in turn several applications. Suppose that V is differentiable with respect to x at a point (t_0, x_0) and let $\bar{x}(\cdot)$ be any optimal trajectory for problems (1), (2). Then V is differentiable with respect to (t,x) at $(t, \bar{x}(t))$ for all $t \in]t_0, T]$ and $D^*V(t, \bar{x}(t)) = \{\nabla V(t, \bar{x}(t))\}$ (Theorem 5.6).

Moreover, if (1) has a unique optimal solution $\bar{x}(\cdot)$, then, for all $t \in]t_0, T]$, V is differentiable at $(t, \bar{x}(t))$ (Corollary 5.11).

Furthermore, given any optimal trajectory $\bar{x}(\cdot)$ of problem (1), (2), the corresponding co-state $\bar{p}(\cdot)$, obtained in [CF1], satisfies the inclusion

$$(\langle (-A^*)^\alpha \bar{p}(t), (-A)^{1-\alpha} \bar{x}(t) \rangle + H(t, \bar{x}(t), \bar{p}(t)), -\bar{p}(t)) \in D^+V(t, \bar{x}(t))$$

for all $t \in]t_0, T]$ (Theorem 5.2).

We conclude this introduction with the outline of the paper. In Section 2 we collect preliminary material on evolution equations and generalized differentials. The Lipschitz regularity of V is derived in Section 3 and the semiconcavity in Section 4. Section 5 contains the applications mentioned above. Finally, in Section 6, we investigate the closedness properties of the feedback map.

2 Preliminaries

Let X be a Banach space with norm $|\cdot|$. In this paper we assume that $|\cdot|$ is differentiable away from 0. For any $r > 0$ and $x_0 \in X$ set

$$B_r(x_0) = \{x \in X \mid |x - x_0| < r\}$$

We denote by X^* the dual of X and by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X.

Let $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup, $e^{tA}(t \geq 0)$, in X. Then it is well known that there exist constants $M_0, M_1 > 0$ and $\omega \in \mathbf{R}$ such that

$$\begin{cases} (i) & |e^{tA}x| \leq M_0 e^{\omega t}|x| \\ (ii) & |Ae^{tA}x| \leq (\omega M_0 + \frac{M_1}{t})e^{\omega t}|x| \end{cases} \tag{6}$$

for all $x \in X$ and $t > 0$ (see e.g. [Pa, p. 60]).

Suppose now that $\omega < 0$, so that 0 belongs to the resolvent set of $A, \rho(A)$. We denote by $(-A)^\alpha, \alpha \in \mathbf{R}$, the fractional powers of $-A$ with domain $D(-A)^\alpha$ (see [Pa, p. 69]) and set

$$|x|_\alpha = |(-A)^\alpha x|$$

for all $x \in D(-A)^\alpha$. Estimate (6)(ii) has the following version for fractional powers

$$|(-A)^\alpha e^{tA}x| \leq \frac{M_\alpha}{t^\alpha}|x| \quad \text{for all } t > 0 \tag{7}$$

for all $x \in X, t > 0$ and some constant $M_\alpha > 0$ (see [Pa, p. 74]).

Let $T > 0, x_0 \in X, f \in L^p(0,T;X), p > 1$. Then the Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t), & 0 \leq t \leq T \\ x(0) = x_0 \end{cases}$$

has a unique mild solution

$$x \in C([0,T];X) \cap C^\theta(]0,T];X), \theta = \frac{p-1}{p} \tag{8}$$

given by the formula

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A} f(\tau) d\tau, \quad t \in [0,T]$$

(see e.g [Pa]). Assume further that $f \in L^\infty(0,T;X)$. Then it is well known that $x(t) \in D(-A)^\alpha$ for any $\alpha \in]0,1[$ and $t > 0$. In fact, estimates (6) and (7) yield

$$|x(t)|_\alpha \leq M_\alpha \left(t^{-\alpha}|x_0| + \frac{t^{1-\alpha}}{1-\alpha} \|f\|_{L^\infty(0,T;X)} \right) \tag{9}$$

for all $t > 0$ and $\alpha \in]0,1[$. A slightly longer – yet standard – computation shows that

$$x(\cdot) \in C^{1-\theta}(]0,T]; D(-A)^\theta), \forall\, \theta \in]0,1[. \tag{10}$$

Let Ω be an open subset of X and $\varphi : \Omega \to \mathbf{R}$. For any fixed $x_0 \in \Omega$, the semi-differentials of φ at x_0 are defined as

$$D^+\varphi(x_0) = \left\{ p \in X^* | \limsup_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0 \rangle}{|\alpha - x_0|} \leq 0 \right\}$$

$$D^-\varphi(x_0) = \left\{ p \in X^* | \liminf_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle p, x - x_0 \rangle}{|x - x_0|} \geq 0 \right\}$$

and called super and subdifferential of φ at x_0, respectively (see [CEL]). The semi-differentials $D^+\varphi(x_0)$ and $D^-\varphi(x_0)$ are both non-empty if and only if φ is Fréchet differentiable at x_0. In this case we have

$$D^+\varphi(x_0) = D^-\varphi(x_0) = \{\nabla\varphi(x_0)\}$$

where $\nabla\varphi$ denotes the gradient of φ.

We denote by $D^*\varphi(x_0)$ the set of all points $p \in X^*$ for which there exists a sequence $\{x_n\}_{n \in N}$ in X with the following properties

$$\begin{cases} (i) & x_n \text{ converges to } x_0 \text{ as } n \to \infty \\ (ii) & \varphi \text{ is Fréchet differentiable at } x_n, \forall n \in N \\ (iii) & \nabla\varphi(x_n) \text{ weakly} - * \text{ converges to } p \text{ as } n \to \infty \end{cases} \tag{11}$$

If φ is Lipschitz in a neighborhood of x_0, then φ is Fréchet diffentiable on a dense subset of Ω (see [Pr]). Consequently, $D^*\varphi(x_0) \neq \phi$.

Let now Ω be convex. We say that φ is semi- concave if there exists a function

$$w : [0, +\infty[\times [0, +\infty[\to [0, +\infty[$$

satisfying

$$\begin{cases} (i) & w(r,s) \le w(R,S), \quad \forall 0 \le r \le R, \quad \forall 0 \le s \le S \\ (ii) & \lim_{s \downarrow 0} w(r,s) = 0, \quad \forall r > 0 \end{cases}$$

and such that

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) - \varphi(\lambda x + (1-\lambda)y) \le \lambda(1-\lambda)|x-y|w(r,|x-y|) \quad (12)$$

for every $r > 0, \lambda \in [0,1]$ and $x, y \in \Omega \cap B_r(0)$.

The superdifferential of a semiconcave function has several useful properties, some of which are recalled in the following

Proposition 2.1 *If φ is Lipschitz and semiconcave in $B_r(x_0)$ for some $r > 0$, then*

$$D^+\varphi(x_0) = \overline{co} D^*\varphi(x_0) \quad (13)$$

where $\overline{co}$ denotes the closed convex hull. In particular $D^+\varphi(x_0) \neq \phi$. Moreover,

$$\varphi(x) - \varphi(x_0) - \langle p, x - x_0 \rangle \le |x - x_0|\omega(r, |x - x_0|) \quad (14)$$

for all $p \in D^+\varphi(x_0)$ and all $x \in B_r(x_0)$. Furthermore, if $D^+\varphi(x_0)$ is a singleton, then φ is Gâteaux differentiable at x_0. If, in addition, $D^+\varphi(x)$ is contained in some compact subset of X^* for all $x \in B_r(x_0)$, then φ is Fréchet differentiable at x_0.

The proof of the first two statements (13) and (14) is given in [C1] (Corollary 4.7). The third statement follows from the fact that $D^+\varphi(x)$ coincides with the generalized gradient if φ is semiconcave (see Proposition 4.8 in [C1] and [C]). Finally, the last statement can be obtained adapting the proof of Corollary 4.12 in [CS1].

We next give a result which relates functions satisfying estimate (12) for any $\lambda \in [0,1]$ with functions satisfying estimate (12) for $\lambda = \frac{1}{2}$.

Proposition 2.2 *Let $\varphi : \Omega \to \mathbf{R}$ be locally Lipschitz and $\alpha \in]0,1]$. Suppose that for all $R > 0$ there exists $C_R > 0$ such that*

$$\varphi(x) + \varphi(y) - 2\varphi(\frac{x+y}{2}) \le C_R|x-y|^{1+\alpha} \quad (15)$$

for all $x, y \in \Omega$ *with* $|x|, |y| \leq R$. *Then for all* $R > 0$ *there exists* $C'_R > 0$ *such that*

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) - \varphi(\lambda x + (1-\lambda)y) \leq C'_R \lambda(1-\lambda)|x-y|^{1+\alpha} \quad (16)$$

for all $\lambda \in [0,1]$ *and* $x, y \in \Omega$ *satisying* $|x|, |y| \leq R$. *In particular,* φ *is semiconcave.*

Proof. Let $B_r(x) \subset \Omega$ and $h \in X, 0 \neq |h| < r$. Define

$$f(t) = \frac{\varphi(x+th)}{|h|^{1+\alpha}}, \quad t \in [-1,1].$$

From (15) it follows that

$$f(t) + f(s) - 2f(\frac{t+s}{2}) \leq C_R|t-s|^{1+\alpha}$$

for all $t, s \in [-1,1]$. Moreover, f is Lipschitz continuous. Therefore, from Lemma 4.2 in [CS1] we conclude that

$$f(t) - f(s) - f'(s)(t-s) \leq C'_R|t-s|^{1+\alpha}$$

for a.e. $t, s \in [-1,1]$. Also,

$$f(s) - f(t) - f'(t)(s-t) \leq C'_R|t-s|^{1+\alpha}$$

so that

$$[f'(t) - f'(s)](t-s) \leq 2C'_R|t-s|^{1+\alpha} \quad (17)$$

for a.e. $t, s \in [-1,1]$. Now, for all $\lambda \in [0,1]$, we have

$$\begin{aligned} \lambda f(1-\lambda) &+ (1-\lambda)f(-\lambda) - f(0) \\ &= \lambda \int_0^1 \tfrac{d}{dt} f((1-\lambda)t)dt + (1-\lambda)\int_0^1 \tfrac{d}{dt} f(-\lambda t)dt \\ &= \lambda(1-\lambda)\int_0^1 [f'((1-\lambda)t) - f'(-\lambda t)]\tfrac{t}{t}dt \\ &\leq 2C'_R \lambda(1-\lambda)\tfrac{1}{1+\alpha} \end{aligned}$$

recalling (17). The above inequality reads as follows

$$\lambda\varphi(x + (1-\lambda)h) + (1-\lambda)\varphi(x - \lambda h) - \varphi(x) \leq \frac{2C'_R}{1+\alpha}\lambda(1-\lambda)|h|^{1+\alpha},$$

which is equivalent to (16).

QED

Remark. When $\alpha = 1$, (16) can also be recovered from (15) by an induction procedure and a density argument.

Proposition 2.3 *Let $\varphi : X \to \mathbf{R}$ be locally Lipschitz and semiconcave. Then for all $x \in X$ and $\theta \in X$*

$$\frac{\partial \varphi}{\partial \theta}(x) := \lim_{h \to 0+} \frac{\varphi(x + h\theta) - \varphi(x)}{h} = \liminf_{\substack{x' \to x \\ h \to 0+}} \frac{\varphi(x' + h\theta) - \varphi(x')}{h} =: \varphi^0_-(x)(\theta)$$

and the set-valued map $Q : X \to X$ defined by $Q(x) = \{\theta \in X : \frac{\partial \varphi}{\partial \theta}(x) \leq 0\}$ has closed graph and nonempty images.

Proof. It is enough to adapt proofs of Theorem 2.9 and Proposition 2.5 from [CF2] to the infinite dimensional case.

We conclude this section with the following lemma, which is a simple consequence of Gronwall's inequality.

QED

Lemma 2.4 *Let $\varphi : [a, b] \to \mathbf{R}$ be an integrable function such that*

$$\varphi(t) \leq A + \frac{B}{(t-a)^\alpha} + L \int_a^t \varphi(s)ds \tag{18}$$

for a.e. $t \in [a, b]$ and some constants $L, A, B \geq 0, \alpha \in]0, 1[$. Then, for a.e. $t \in [a, b]$

$$\varphi(t) \leq [1 + L(b-a)e^{L(b-a)}]A + \left[\frac{Le^{L(b-a)}}{1-\alpha} + \frac{1}{(t-a)^\alpha}\right] B. \tag{19}$$

Proof. Let $\psi(r) = \int_a^r \varphi(t)dt$. Then, integrating (19) with respect to t yields

$$\psi(r) \leq A(b-a) + \frac{B}{1-\alpha}(b-a)^{1-\alpha} + L \int_a^r \psi(t)dt.$$

Thus, the Gronwall lemma implies that

$$\psi(r) \leq e^{L(r-a)}[A(b-a) + \frac{B}{1-\alpha}(b-a)^{1-\alpha}]$$

Inserting this estimate in (18), we get (19).

QED

3 The optimal control problem: Lipschitz regularity of the value function

Let X be a Banach space and U a complete separable metric space. Fix $T > 0$ and let $(t_0, x_0) \in [0,T] \times X$. Consider a system $x(\cdot)$ governed by the semilinear state equation

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t), u(t)), & t \in [t_0, T] \\ x(t_0) = x_0. \end{cases} \tag{20}$$

Let $g : X \to \mathbf{R}$ be a given continuous function. We are interested in the Mayer optimal control problem below:

$$\text{minimize } g(x(T)) \text{ over all solutions to (20) with measurable } u. \tag{21}$$

In this section we impose the following assumptions on the data of our problem:

$$\begin{cases} (i) & A : D(A) \subset X \to X \text{ is the infinitesimal generator of} \\ & \text{an analytic semigroup, } e^{tA}, t \geq 0, \text{ satisfying (6) for some } \omega < 0; \\ (ii) & f : [0,T] \times X \times U \to X \text{ is continuous and such that} \\ & |f(t,x,u)| \leq C_0(1 + |x|),\ |f(t,x,u) - f(t,y,u)| \leq C_0|x - y| \\ & \text{for some } C_0 > 0 \text{ and all } t \in [0,T], x, y \in X, u \in U; \\ (iii) & g \text{ is Lipschitz on all bounded subsets of } X. \end{cases} \tag{22}$$

It is well known that, under assumptions (22), for every measurable $u : [t_0, T] \to U$ problem (20) has a unique mild solution $x(\cdot) \in C([t_0, T]; X)$ satisfying

$$x(t) = e^{(t-t_0)A} x_0 + \int_0^t e^{(t-s)A} f(s, x(s), u(s)) ds \tag{23}$$

for all $t \in [t_0, T]$. We denote this solution by

$$x(\cdot; t_0, x_0, u).$$

Moreover, (22), (23) and the Gronwall Lemma yield

$$|x(t)| \leq M_0 e^{M_0 C_0 T}(|x_0| + C_0 T), \quad \forall t \in [t_0, T]. \tag{24}$$

Remark. As is well known, the assumption that e^{tA} is of negative type ($\omega < 0$ in (22)) implies no loss of generality. Indeed, let (6) be fulfilled for some $\omega_0 \in \mathbf{R}$ and $x(\cdot)$ be a solution of (20). Then $y(t) = e^{-(\omega_0+1)(t-t_0)}x(t)$ satisfies

$$\begin{cases} y'(t) &= A_0 y(t) + f_0(t, y(t), u(t)), \quad t_0 \le t \le T \\ y(t_0) &= x_0 \end{cases} \tag{25}$$

where

$$\begin{cases} A_0 = A - (\omega_0 + 1)I \\ f_0(t,x,u) = e^{-\omega_0(t-t_0)} f(t, e^{\omega_0(t-t_0)}x, u). \end{cases}$$

Notice that e^{tA_0} is of negative type, while f_0 satisfies (22) (ii) with the same constants as f. Therefore, problem (20) is equivalent to minimizing $g(e^{(\omega_0+1)(T-t_0)}y(T))$ over all trajectories of (25).

The value function of problem (20), (21), defined as

$$V(t_0, x_0) = \inf\{g(x(T; t_0, x_0, u)| \; u : [t_0, T] \to U \text{ is measurable }\},$$

has many properties which are relevant for the original optimal control problem. Among these, let us recall the Optimality Principle: for all $t \in [t_0, T]$

$$V(t_0, x_0) = \inf\{V(t; x(t; t_0, x_0, u)) \,|u : [t_0, T] \to U \text{ is measurable }\}. \tag{26}$$

Theorem 3.1 *Assume (22) and let $R > \frac{1}{T}$. Then there exists a constant $C_R > 0$ such that*

$$|V(t_1, x_1) - V(t_0, x_0)| \le C_R(|t_1 - t_0| + |x_1 - x_0|) \tag{27}$$

for all $t_1, t_0 \in [0, T - \frac{1}{R}]$ and all $x_1, x_0 \in X$ satisfying $|x_1|, |x_0| \le R$.

Proof.

Step 1: reduction to $D(-A)^\alpha$.

Fix $t_1, t_0 \in [0, T - \frac{1}{R}], x_1, x_0 \in X, |x_i| \le R$. Define $s_i = t_i + \frac{1}{2R}, i = 0, 1$, and let $u_0(\cdot)$ be such that

$$V(t_0, x_0) + |t_1 - t_0| + |x_1 - x_0| > V(s_0, x(s_0; t_0, x_0, u_0))$$

(if $|t_1 - t_0| + |x_1 - x_0| = 0$, then (27) is trivial). Fix $\bar{u} \in U$ and define

$$u_1(t) = \begin{cases} u(t - t_1 + t_0), & t \in [t_1, s_1] \\ \bar{u}, & t \in [s_1, T] \end{cases}$$

Set also

$$\bar{x}_i(t) = x(t; t_i, x_i, u_i), \quad y_i = \bar{x}_i(s_i), \quad i = 0, 1.$$

Then, recalling (24), we have that

$$|f(t, \bar{x}_i(t), u_i(t))| \leq C(R), \quad t \in [t_i, T]$$

where $C(R) = C_0[1 + e^{M_0 C_0 T} M_0(R + C_0 T)]$. Therefore, by (9) we conclude that $y_i \in D(-A)^\alpha$ for $i = 0, 1$ and all $\alpha \in]0, 1[$ and

$$|y_i|_\alpha \leq M_\alpha \left[\frac{R}{(2R)^\alpha} + \frac{C(R)}{(1-\alpha)(2R)^{1-\alpha}}\right] =: M_\alpha(R). \tag{28}$$

Moreover, setting $\bar{x}(t) = \bar{x}_1(t + t_1 - t_0) - \bar{x}_0(t), t_0 \leq t \leq s_0$, we have

$$|\bar{x}(t)| \leq M_0|x_1 - x_0| + C_0 M_0 \int_{t_0}^{t} |\bar{x}(s)| ds$$

so that $|\bar{x}(t)| \leq M_0|x_1 - x_0|e^{C_0 M_0/R}$. In particular

$$|y_1 - y_0| \leq M_0 e^{C_0 M_0/R}|x_1 - x_0| \tag{29}$$

Now, we have by (26)

$$V(t_1, x_1) - V(t_0, x_0) < V(s_1, y_1) - V(s_0, y_0) + |t_1 - t_0| + |x_1 - x_0|.$$

Therefore, interchanging (t_1, x_1) and (t_0, x_0),

$$|V(t_1, x_1) - V(t_0, x_0)| \leq |V(s_1, y_1) - V(s_0, y_0)| + |t_1 - t_0| + |x_1 - x_0|. \tag{30}$$

Step 2. Estimate on $V(s_1, y_1) - V(s_0, y_0)$.

In this step we denote by $C_{\alpha,R}$ any positive constant depending on α and R. Let $\bar{u}_0 : [s_0, T] \to U$ be measurable and such that

$$V(s_0, y_0) + |t_1 - t_0| + |x_1 - x_0| > g(x(T; s_0, y_0, \bar{u}_0)).$$

Suppose $s_1 \leq s_0$ (or, equivalently, $t_1 \leq t_0$) and define

$$\bar{u}_1(t) = \begin{cases} \bar{u}, & s_1 \leq t \leq s_0 \\ \bar{u}_0(t), & s_0 \leq t \leq T \end{cases} \tag{31}$$

$$\tilde{x}_i(t) = x(t; s_i, y_i, \bar{u}_i), \quad i = 0, 1.$$

Then,

$$\begin{aligned} V(s_1,y_1) - V(s_0,y_0) &\leq |t_1 - t_0| + |x_1 - x_0| + g((\tilde{x}_1(T)) - g(\tilde{x}_0(T)) \\ &\leq |t_1 - t_0| + |x_1 - x_0| + C_R|\tilde{x}_1(T) - \tilde{x}_0(T)| \end{aligned} \tag{32}$$

for some constant $C_R > 0$. Now, for all $t \geq s_0$

$$\begin{aligned} |\tilde{x}_1(t) - \tilde{x}_0(t)| &\leq |e^{(t-s_1)A}y_1 - e^{(t-s_0)A}y_0| \\ &+ |\int_{s_1}^{s_0} e^{(t-s)A} f(s,\tilde{x}_1(s),\bar{u}_1(s))ds| \\ &+ |\int_{s_0}^{t} e^{(t-s)A}[f(s,\tilde{x}_1(s),\bar{u}_0(s)) - f(s,\tilde{x}_0(s),\bar{u}_1(s))]ds| \end{aligned}$$

From (6), (7), (28) and (29) it follows that, for all $t > s_0$,

$$\begin{aligned} |e^{(t-s_1)A}y_1 - e^{(t-s_0)A}y_0| &\leq |(e^{(s_0-s_1)A} - I)e^{(t-s_0)A}y_1| + |e^{(t-s_0)A}(y_1 - y_0)| \\ &\leq |s_0 - s_1||Ae^{(t-s_0)A}y_1| + M_0|y_1 - y_0| \\ &\leq \frac{M_{1-\alpha}}{(t-s_0)^{1-\alpha}}|y_1|_\alpha|t_0 - t_1| + M_0^2 e^{C_0M_0/R}|x_1 - x_0| \\ &\leq C_{\alpha,R}\left[\frac{|t_0 - t_1|}{|t-s_0|^{1-\alpha}} + |x_1 - x_0|\right] \end{aligned}$$

for all $\alpha \in]0,1[$. Hence,

$$\begin{aligned} |\tilde{x}_1(t) - \tilde{x}_0(t)| &\leq C_{\alpha,R}\left[\left(1 + \frac{1}{|t-s_0|^{1-\alpha}}\right)|t_1 - t_0| + |x_1 - x_0|\right] \\ &+ M_0C_0\int_{s_0}^{t}|\tilde{x}_1(s) - \tilde{x}_0(s)|ds \end{aligned}$$

for all $t > s_0$. Thus Lemma 2.4 yields

$$|\tilde{x}_1(t) - \tilde{x}_0(t)| \leq C_{\alpha,R}\left[|x_1 - x_0| + |t_1 - t_0|(1 + \frac{1}{|t-s_0|^{1-\alpha}})\right]$$

for all $t > s_0$. Therefore

$$|\tilde{x}_1(T) - \tilde{x}_0(T)| \leq C_{\alpha,R}(|x_1 - x_0| + |t_1 - t_0|)$$

for some $C_{\alpha,R} > 0$. The above estimate and (32) yield

$$V(s_1,y_1) - V(s_0,y_0) \leq C_{\alpha,R}(|t_1 - t_0| + |x_1 - x_0|) \tag{33}$$

under the extra assumption $s_1 \le s_0$.

On the other hand, if $s_1 > s_0$, then instead of (31) we define $\bar{u}_1(s) = \bar{u}_0(s), t \in [s,T]$ and repeating the above argument we obtain (33) once again. Therefore, interchanging (s_1, y_1) and (s_0, y_0), we have

$$|V(s_1,y_1) - V(s_0,y_0)| \le C_R(|t_1 - t_0| + |x_1 - x_0|) \qquad (34)$$

for some $C_R > 0$ (fixing, for instance, $\alpha = \frac{1}{2}$). This estimate and (30) imply the conclusion (27).

QED

We note that the interest of the above result is due to the fact that it provides the joint Lipschitz continuity of V with respect to (t,x). The Lipschitz continuity of $V(t,\cdot)$ with respect to x for all $t \in [0,T]$ is a known result (see [BDP]), even when the semigroup e^{tA} is just strongly continuous. Indeed, when e^{tA} is analytic, a stronger Lipschitz property holds true for V, as we show below.

Theorem 3.2 *Assume (22) and let $R > \frac{1}{T}, \alpha \in [0,1[$. Then there exists a constant $C = C(\alpha, R, T)$ such that*

$$|V(t,x_1) - V(t,x_0)| \le C|(-A)^{-\alpha}(x_1 - x_0)| \qquad (35)$$

for all $t \in [0, T - \frac{1}{R}]$ and all $x_1, x_0 \in X$ satisfying $|x_1|, |x_0| \le R$.

Proof. Assuming $|(-A)^{-\alpha}(x_1 - x_0)| > 0$ (otherwise there is nothing to prove), let $u_0 : [t,T] \to U$ be such that

$$V(t,x_0) + |(-A)^{-\alpha}(x_1 - x_0)| > g(x(T;t,x_0,u_0))$$

and set $\bar{x}_1(\cdot) = x(\cdot;t,x_1x_1,u_0), \bar{x}_0(\cdot) = x(\cdot;t,x_0,u_0)$. Then, recalling (26),

$$V(t,x_1) - V(t,x_0) \le C_R|\bar{x}_1(T) - \bar{x}_0(T)| + |(-A)^{-\alpha}(x_1 - x_0)| \qquad (36)$$

for some constant $C_R > 0$. On the other hand, in view of (7),

$$\begin{aligned} |\bar{x}_1(s) - \bar{x}_0(s)| &\le |(-A)^{\alpha} e^{(s-t)A}(-A)^{-\alpha}(x_1 - x_0)| \\ &+ |\int_t^s e^{(t-\sigma)A}[f(\sigma,\bar{x}_1(\sigma),u(\sigma)) - f(\sigma,\bar{x}_0(\sigma),u(\sigma))]d\sigma| \\ &\le \frac{M_\alpha}{(s-t)^\alpha}|(-A)^{-\alpha}(x_1 - x_0)| + C_0 M_0 \int_s^t |\bar{x}_1(\sigma) - \bar{x}_0(\sigma)| d\sigma \end{aligned}$$

for all $s \in]t,T]$. Hence, applying Lemma 2.4 we obtain

$$|\bar{x}_1(s) - \bar{x}_0(s)| \leq C_{\alpha,T}\frac{1}{(s-t)^\alpha}|(-A)^{-\alpha}(x_1 - x_0)|.$$

The conclusion (35) follows from the above estimate and (36), since the argument is symmetric with respect to x_1, x_0.

QED

From estimates (27) and (35) we immediately obtain the following

Corollary 3.3 *Assume (22) and let $R > \frac{1}{T}, \alpha \in [0,1[$. Then there exists a constant $C = C(\alpha, R, T)$ such that*

$$|V(t_1,x_1) - V(t_0,x_0)| \leq C[|t_1 - t_0| + |(-A)^{-\alpha}(x_1 - x_0)|] \tag{37}$$

for all $t_1, t_0 \in [0, T - \frac{1}{R}]$ and all $x_1, x_0 \in X$ satisfying $|x_1|, |x_0| \leq R$.

The result below is useful for the applications in Section 5. For linear state equations, it was proved in [CDP1]. We denote by $D_x^+V(t,x)$ the superdifferential of $V(t,\cdot)$ at x.

Corollary 3.4 *Assume (22) and let $(t_0, x_0) \in [0,T[\times X$. Then, for all $\alpha \in [0,1[$,*

$$D_x^+V(t_0,x_0) \subset D((-A^*)^\alpha) \;\;\&\;\; D_x^-V(t_0,x_0) \subset D((-A^*)^\alpha)$$

Moreover, for each $R > 0$ there exists a constant $C_R = C_R(\alpha, T) > 0$ such that, if $|x_0| \leq R$, then

$$|(-A^*)^\alpha p| \leq C_R \tag{38}$$

for all $p \in D_x^+V(t_0,x_0) \cup D_x^-V(t_0,x_0)$.

Proof. We provide the proof for the superdifferential only, because the argument below applies to the subdifferential as well.

For all $x \in X, p \in D_x^+V(t_0,x_0)$ and $\lambda > 0$

$$-\langle p, x\rangle \leq \frac{V(t_0, x_0 + \lambda x) - V(t_0,x_0) - \lambda\langle p, x\rangle}{\lambda} + \frac{|V(t_0, x_0 + \lambda x) - V(t_0,x_0)|}{\lambda}$$

Hence, taking $\limsup_{\lambda \to 0}$ of both sides, estimate (35) yields

$$-\langle p, x\rangle \leq C_\alpha|(-A)^{-\alpha}x|$$

Hence

$$|\langle p, x\rangle| \leq C_\alpha |(-A)^{-\alpha} x|$$

for all $\alpha \in [0,1[$. Thus, for all $x \in D(-A)^\alpha$,

$$|\langle p, (-A)^\alpha x\rangle| \leq C_\alpha |x|$$

which in turn implies that $p \in D(((-A)^\alpha)^*) = D((-A^*)^\alpha)$ and (38).

QED

4 Semiconcavity of the value function

In this section we show that the value function of our optimal control problem (20), (21) is semiconcave in (t,x) on $[0,T[\times X$. For this purpose we have to strengthen assumptions (22) as follows:

$$\begin{cases} (i) & f(\cdot,\cdot,u) \text{ is differentiable and } \exists \alpha \in]0,1] \text{ such that} \\ & \left\| \frac{\partial f}{\partial(t,x)}(t,x,u) - \frac{\partial f}{\partial(t,x)}(s,y,u) \right\| \leq C_R(|x-y| + |t-s|)^\alpha \\ & \text{for all } s,t \in [0,T], x,y \in B_R(0), u \in U; \\ (ii) & \exists \alpha \in]0,1] \text{such that } g(x) + g(y) - 2g(\frac{x+y}{2}) \leq C_R |x-y|^{1+\alpha} \\ & \text{for all } x,y \in B_R(0). \end{cases} \tag{39}$$

In (i) we have denoted by $\|\cdot\|$ the standard norm of a bounded linear operator on X. Also, by Proposition 2.2, (ii) implies that g is semiconcave in X.

Remark. It can be easily seen that assumption (i) above implies that

$$\begin{aligned} &|\lambda f(t_1,x_1,u) + (1-\lambda) f(t_0,x_0,u) - f(\lambda(t_1,x_1) + (1-\lambda)(t_0,x_0),u)| \\ &\qquad \leq C_R \lambda(1-\lambda)(|t_1-t_0| + |x_1-x_0|)^{1+\alpha} \end{aligned}$$

for all $\lambda \in [0,1]$ and all $x_0, x_1 \in B_R(0)$.

Theorem 4.1 *Assume (22), (39) and let $R > 0$. Then there exists $C_R > 0$ such that*

$$V(t_1,x_1) + V(t_0,x_0) - 2V\left(\frac{t_1+t_0}{2}, \frac{x_1+x_0}{2}\right) \leq C_R(|t_1-t_0| + |x_1-x_0|)^{1+\alpha} \tag{40}$$

for all $t_1, t_0 \in [0, T - \frac{1}{R}]$ and all $x_1, x_0 \in B_R(0)$.

Proof. Without loss of generality, we may assume that f is independent of t.

Step 1: reduction to $D((-A)^\beta), 0<\beta<1$.
Fix $t_1, t_0 \in [0, T-\frac{1}{R}], x_1, x_0 \in B_R(0)$ and define

$$t_2 = \frac{t_1+t_0}{2}, \quad x_2 = \frac{x_1+x_0}{2}$$

$$s_i = t_i + \frac{1}{2R}, \quad i=0,1,2.$$

Let $u_2(\cdot)$ be such that

$$V(t_2,x_2) + (|t_1-t_0|+|x_1-x_0|)^{1+\alpha} > V(s_2, x(s_2; t_2, x_2, u_2)).$$

(obviously, we may assume that $|t_1-t_0|+|x_1-x_0|>0$). Fix also $\bar{u} \in U$ and set

$$\begin{aligned} u_i(t) &= \begin{cases} u_2(t-t_i+t_2), & t\in[t_i,s_i[\\ \bar{u}, & t\in[s_i,T] \end{cases} \quad i=0,1 \\ \bar{x}_i(t) &= x(t;t_i,x_i,u_i), \quad i=0,1,2 \\ \tilde{x}_i(t) &= \bar{x}_i(t-t_2+t_i), \quad t_2\le t\le s_2, \quad i=0,1. \end{aligned}$$

Recalling (26) and (27), we have

$$\begin{aligned} V(t_1,x_1) &+ V(t_0,x_0) - 2V(t_2,x_2) \le \qquad (41) \\ &\le V(s_1,\bar{x}_1(s_1)) + V(s_0,\bar{x}_0(s_0)) - 2V(s_2,\bar{x}_2(s_2)) + \\ &+ 2(|t_1-t_0|+|x_1-x_0|)^{1+\alpha} \\ &\le V(s_1,\bar{x}_1(s_1)) + V(s_0,\bar{x}_0(s_0)) - 2V\left(s_2, \frac{\bar{x}_1(s_1)+\bar{x}_0(s_0)}{2}\right) \\ &+ C_R|\bar{x}_1(s_1)+\bar{x}_0(s_0)-2\bar{x}_2(s_2)| + 2(|t_1-t_0|+|x_1-x_0|)^{1+\alpha}. \end{aligned}$$

Now,

$$\bar{x}_1(s_1)+\bar{x}_0(s_0)-2\bar{x}_2(s_2) = \tilde{x}_1(s_2)+\tilde{x}_0(s_2)-2\bar{x}_2(s_2). \qquad (42)$$

Moreover, for all $t\in[t_2,s_2]$

$$\begin{aligned} \tilde{x}_1(t) &+ \tilde{x}_0(t) - 2\bar{x}_2(t) = \\ &= \int_{t_2}^{t} e^{(t-s)A}[f(\tilde{x}_1(s),u_2(s)) + f(\tilde{x}_0(s),u_2(s)) - 2f(\bar{x}_2(s),u_2(s))]ds. \end{aligned}$$

Therefore, by assumption (39) (i) and the remark below it,

$$\begin{aligned}|\tilde{x}_1(t)+\tilde{x}_0(t)-2\tilde{x}_2(t)| &\le C_2M_0\int_{t_2}^t|\tilde{x}_1(s)-\tilde{x}_0(s)|^{1+\alpha}ds+\\ &+ C_0M_0\int_{t_2}^t|\tilde{x}_1(s)+\tilde{x}_0(s)-2\tilde{x}_2(s)|ds \quad (43)\end{aligned}$$

On the other hand, for all $t\in[t_2,s_2]$,

$$|\tilde{x}_1(t)-\tilde{x}_0(t)|\le M_0|x_1-x_0|+M_0C_0\int_{t_2}^t|\tilde{x}_1(s)-\tilde{x}_0(s)|ds$$

and so, by Gronwall's lemma,

$$|\tilde{x}_1(t)-\tilde{x}_0(t)|\le M_0'|x_1-x_0|,\quad \forall t\in[t_2,s_2] \tag{44}$$

for some constant $M_0'>0$. Thus, (43), (44) and again Gronwall's lemma yield

$$|\tilde{x}_1(t)+\tilde{x}_0(t)-2\tilde{x}_2(t)|\le M_0''|x_1-x_0|^{1+\alpha},\quad \forall t\in[t_2,s_2]$$

for some $M_0''>0$ depending only on R.

The above inequality, (42) and (41) imply in turn that, for some $C>0$,

$$\begin{aligned}&V(t_1,x_1)+V(t_0,x_0)-2V(t_2,x_2)\le\\ &\le C(|t_1-t_0|+|x_1-x_0|)^{1+\alpha}+V(s_1,y_1)+V(s_0,y_0)-2V(s_2,y_2)\end{aligned} \tag{45}$$

where

$$\begin{cases} y_i=\bar{x}_i(s_i), & i=0,1\\ y_2=\frac{y_1+y_0}{2}\end{cases}$$

Step 2: estimates on the fractional norms of y_1,y_0.

We will now proceed to estimate the rightmost term in (45). We will take advantage of the fact that $y_i\in D((-A)^\beta), i=0,1,2$, for all $\beta\in[0,1[$ and

$$\begin{cases} (i) & |y_i|_\beta\le M_\beta(R), i=0,1\\ (ii) & |y_1-y_0|\le M_0e^{C_0M_0T}|x_1-x_0|\\ (iii) & |y_1-y_0|_\beta\le M_\beta|x_1-x_0|\end{cases} \tag{46}$$

for some $M_0,M_\beta(R)>0$. Estimates (i), (ii) above have essentially been proved in Step 1 of the proof of Theorem 3.1 (see (28), (29)). To prove (46)(iii) we note that

$$|y_1-y_0|_\beta=|(-A)^\beta(\tilde{x}_1(s_2)-\tilde{x}_0(s_2))|. \tag{47}$$

On the other hand, for all $t \in]t_2, s_2]$, recalling (7) and (44) we obtain

$$
\begin{aligned}
|(-A)^\beta[\tilde{x}_1(t) - \tilde{x}_0(t)]| &\leq |(-A)^\beta e^{(t-t_2)A}(x_1 - x_0)| + \\
+ \int_{t_2}^{t} |(-A)^\beta e^{(t-s)A}&[f(\tilde{x}_1(s), u_2(s)) - f(\tilde{x}_0(s), u_2(s)]|ds \\
&\leq \frac{M_\beta}{(t-t_2)^\beta}|x_1 - x_0| + M'_\beta |x_1 - x_0| \int_{t_2}^{t} \frac{ds}{(t-s)^\beta}.
\end{aligned}
$$

From the above inequality and (47), estimate (46) (iii) easily follows.

Step 3: estimate on $V(s_1, y_1) + V(s_0, y_0) - 2V(s_2, y_2)$.
Let $\bar{u}_2 : [s_2, T] \to U$ be such that

$$V(s_2, y_2) + (|t_1 - t_0| + |x_1 - x_0|)^{1+\alpha} > g(x(T; s_2, y_2, \bar{u}_2)).$$

Suppose $s_1 \leq s_0$ (or, equivalently, $t_1 \leq t_0$) and define

$$\tau(s) = \begin{cases} \frac{s+s_0}{2}, & s_1 \leq s \leq s_0 \\ s, & s_0 \leq s \leq T \end{cases}$$

$$
\begin{aligned}
\bar{y}_1(\cdot) &= x(\cdot; s_1, y_1, \bar{u}_2 \circ \tau) \\
\bar{y}_0(\cdot) &= x(\cdot, ; s_0, y_0, \bar{u}_2|_{[s_0, T]}) \\
\bar{y}_2(\cdot) &= x(\cdot; s_2, y_2, \bar{u}_2).
\end{aligned}
$$

Then, by assumptions (39) (ii), (22) (iii)

$$
\begin{aligned}
&V(s_1, y_1) + V(s_0, y_0) - 2V(s_2, y_2) \leq \\
&\leq (g(\bar{y}_1(T)) + g(\bar{y}_0(T)) - 2g(\bar{y}_2(T)) + 2(|t_1 - t_0| + |x_1 - x_0|)^{1+\alpha} \\
&\leq C_R|\bar{y}_1(T) - \bar{y}_0(T)|^{1+\alpha} + C_R|\bar{y}_1(T) + \bar{y}_0(T) - 2\bar{y}_2(T)| + \\
&+2(|t_1 - t_0| + |x_1 - x_0|)^{1+\alpha}
\end{aligned}
\tag{48}
$$

for some constant $C_R > 0$. We will now estimate the first two terms in the right-hand side of (48) separately.

Step 4: estimate on $|\bar{y}_1(T) - \bar{y}_0(T)|$.
First, we note that, for all $t \in]s_0, T]$,

$$
\begin{aligned}
&|\bar{y}_1(t) - \bar{y}_0(t)| \leq |e^{(t-s_1)A}(y_1 - y_0)| + |(e^{(s_0-s_1)A} - 1)e^{(t-s_0)A}y_0| + \\
&+M_0 \textstyle\int_{s_1}^{s_0} |f(\bar{y}_1(s), u_2 \circ \tau(s))|ds + M_0 \int_{s_0}^{t} |f(\bar{y}_1(s), u_2(s)) - f(\bar{y}_0(s), u_2(s))|ds \\
&\leq M_0|y_1 - y_0| + \frac{M_{1-\beta}}{(t-s_0)^{1-\beta}}|s_0 - s_1||y_0|_\beta + C_R|s_0 - s_1| + \\
&+C_0 M_0 \textstyle\int_{s_0}^{t} |\bar{y}_1(s) - \bar{y}_0(s)|ds.
\end{aligned}
$$

Therefore, Lemma 2.4 and (46) (i), (ii) imply that

$$|\bar{y}_1(t) - \bar{y}_0(t)| \le M_{\beta,R}\left[|x_1 - x_0| + |t_1 - t_0|\left(1 + \frac{1}{(t-s_0)^{1-\beta}}\right)\right] \qquad (49)$$

for all $\beta \in [0,1[$ and $t \in]s_0, T]$.

Step 5: estimate on $|\bar{y}_1(T) + \bar{y}_0(T) - 2\bar{y}_2(T)|$.
For all $t \in]s_0, T]$ we have

$$\begin{aligned}
|\bar{y}_1(t) &+ \bar{y}_0(t) - 2\bar{y}_2(t)| \le \\
&\le |e^{(t-s_1)A}y_1 + e^{(t-s_0)A}y_0 - 2e^{(t-s_2)A}y_2| + \\
&+ |\int_{s_1}^{s_0} e^{(t-s)A} f(\bar{y}_1(s), \bar{u}_2 \circ \tau(s))ds - 2\int_{s_2}^{s_0} e^{(t-s)} f(\bar{y}_2(s), \bar{u}_2(s))ds| + \\
&+ |\int_{s_0}^{t} e^{(t-s)A}\left[f(\bar{y}_1(s), \bar{u}_2(s)) + f(\bar{y}_0(s), \bar{u}_2(s)) - 2f(\bar{y}_2(s), \bar{u}_2(s))\right]ds| \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}
\end{aligned}$$

Now, using (46) (iii) we obtain

$$\begin{aligned}
\mathrm{I} &\le |[e^{(t-s_1)A} - e^{(t-s_2)A}](y_1 - y_0)| + |[e^{(t-s_1)A} + e^{(t-s_0)A} - 2e^{(t-s_2)A}]y_0| \le \\
&\le |s_1 - s_2||Ae^{(t-s_2)A}(y_1 - y_0)| + |([e^{(s_0-s_1)A} - 1] - 2[e^{(s_0-s_2)A} - 1])e^{(t-s_0)A}y_0| \\
&\le \frac{M_{1-\beta}}{(t-s_2)^{1-\beta}}|y_1 - y_0|_\beta \frac{|s_1 - s_0|}{2} + \left|\left(\int_0^{s_0-s_1} Ae^{\sigma A}d\sigma - 2\int_0^{s_0-s_1}\frac{1}{2}Ae^{\frac{\sigma}{2}A}d\sigma\right)e^{(t-s_0)A}y_0\right| \\
&\le \frac{M_{1-\beta}}{(t-s_0)^{1-\beta}}|x_1 - x_0||t_1 - t_0| + \int_0^{s_0-s_1}|A(e^{\frac{\sigma}{2}A} - 1)e^{(t-s_0+\frac{\sigma}{2})A}y_0|d\sigma \qquad (50)
\end{aligned}$$

Moreover, changing the variable $\eta = s_0 - \sigma/2$, we get

$$\begin{aligned}
\int_0^{s_0-s_1}|A(e^{\frac{\sigma}{2}A} &- 1)e^{(t-s_0+\frac{\sigma}{2})A}y_0|d\sigma \le \\
&\le \frac{1}{2}\int_0^{s_0-s_1}\sigma|(-A)^{2-\beta}e^{(t-s_0+\frac{\sigma}{2})A}(-A)^\beta y_0|d\sigma \\
&\le M'_{2-\beta}\int_0^{s_0-s_1}\frac{\sigma}{(t-s_0+\frac{\sigma}{2})^{2-\beta}}|y_0|_\beta d\sigma \\
&\le M_{\beta,R}\int_{s_2}^{s_0}\frac{2(s_0-\eta)}{(t-\eta)^{2-\beta}}2d\eta \\
&\le 4M_{\beta,R}\int_{s_2}^{s_0}\frac{(s_0-\eta)^\alpha}{(t-\eta)^{1-\beta+\alpha}}d\eta \\
&\le \frac{4M_{\beta,R}}{(t-s_0)^{1-\beta+\alpha}}\frac{(s_0-s_2)^{1+\alpha}}{1+\alpha}.
\end{aligned}$$

Let us now fix $\beta \in]0,1[$ so that $\beta > \alpha$, for example $\beta = \frac{\alpha+1}{2}$. Then the above estimate and (50) yield

$$\mathrm{I} \leq \frac{M'_\alpha}{(t-s_0)^{1-\alpha}}|x_1 - x_0|\,|t_1 - t_0| + \frac{M_{\alpha,R}}{(t-s_0)^{(1+\alpha)/2}}|t_1 - t_0|^{1+\alpha}. \qquad (51)$$

Next, by a change of variable in the first integral of II,

$$\begin{aligned} \mathrm{II} &= |2\int_{s_2}^{s_0}[e^{(t+s_0-2s)A}f(\bar{y}_1(2s-s_0),\bar{u}_2(s)) - e^{(t-s)A}f(\bar{y}_2(s),\bar{u}_2(s))]ds| \\ &\leq 2|\int_{s_2}^{s_0}[e^{(t+s_0-2s)A} - e^{(t-s)A}]f(\bar{y}_1(2s-s_0),\bar{u}_2(s))ds| \qquad (52) \\ &+ 2|\int_{s_2}^{s_0} e^{(t-s)A}[f(\bar{y},(2s-s_0),\bar{u}_2(s)) - f(\bar{y}_2(s),\bar{u}_s(s))]ds| \\ &=: \mathrm{II}_1 + \mathrm{II}_2. \end{aligned}$$

Also,

$$\begin{aligned} \mathrm{II}_1 &\leq \int_{s_2}^{s_0} ds \int_0^{s_0-s} |Ae^{(t-s+\sigma)A}f(\bar{y}_1(2s-s_0),\bar{u}_2(s))|d\sigma \\ &\leq C_R\int_{s_2}^{s_0}\log(1+\frac{s_0-s}{t-s})ds \\ &\leq C_R\int_{s_2}^{s_0}\frac{s_0-s}{t-s}ds \leq C_R\int_{s_2}^{s_0}\frac{(s_0-s)^\alpha}{(t-s_0)^\alpha}ds \\ &= \frac{C_R}{(1+\alpha)(t-s_0)^\alpha}\left(\frac{|t_1-t_0|}{2}\right)^{1+\alpha}. \end{aligned}$$

To estimate II_2 we note that, for all $s \in [s_2, s_0]$,

$$\begin{aligned} |\bar{y}_1(2s-s_0) - \bar{y}_2(s)| &\leq |e^{(2s-s_0-s_1)A}x_1 - e^{(s-s_2)A}x_2| + \\ +|\int_{s_1}^{2s-s_0} & e^{(2s-s_0-\sigma)A}f(\bar{y}_1(\sigma),\bar{u}_2(\tau(\sigma)))d\sigma| + |\int_{s_2}^{s} e^{(s-\sigma)A}f(\bar{y}_2(s),\bar{u}_2(s))d\sigma| \\ \leq M_0|x_1 - x_2| &+ |(e^{(s-s_2)A}-1)e^{(s-s_2)A}x_1| + C_R|2s-s_0-s_1| + C_R|s-s_2| \\ &\leq M_0|x_1-x_0| + C_\alpha\frac{|t_1-t_0|}{(s-s_2)^{1-\alpha}}|x_1|_\alpha + C_R|t_1-t_0|. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{II}_2 &\leq 2C_0M_0\int_{s_2}^{s_0}|\bar{y}_1(2s-s_0),-\bar{y}_2(s)|ds \\ &\leq C_{R,\alpha}|t_1-t_0|[|x_1-x_0| + |t_1-t_0| + |t_1-t_0|^\alpha]. \end{aligned}$$

Thus, (52) and the estimates on $\mathrm{II}_1, \mathrm{II}_2$ yield

$$\mathrm{II} \leq C_R \left\{ |x_1 - x_0|^2 + |t_1 - t_0|^{1+\alpha}[1 + \frac{1}{(t-s_0)^\alpha}] \right\} \tag{53}$$

for all $t \in]s_0, T]$.

Finally, to bound III we use assumption (39) (ii) and estimate (49) as follows

$$\begin{aligned} \mathrm{III} \;&\leq\; C_2 M_0 \int_{s_0}^t |\bar{y}_1(s) - \bar{y}_0(s)|^{1+\alpha} ds + \\ &+\; C_1 M_0 \int_{s_0}^t |\bar{y}_1(s) + \bar{y}_0(s) - 2\bar{y}_2(s)| ds \leq \\ &\leq\; M_{\alpha,R}\{\, |x_1 - x_0|^{1+\alpha} + |t_1 - t_0|^{1+\alpha}[1 + \frac{1}{(t-s_0)^{(1-\alpha^2)/2}}] \,\} + \\ &+\; C_1 M_0 \int_{s_0}^t |\bar{y}_1(s) + \bar{y}_0(s) - 2\bar{y}_2(s)| ds. \end{aligned}$$

From the above inequality, (51) and (53) we conclude that

$$\begin{aligned} |\bar{y}_1(t) + \bar{y}_0(t) - 2\bar{y}_2(t)| \leq M_{\alpha,R} \Bigg\{ |x_1 - x_0|^{1+\alpha} + |t_1 - t_0|^{1+\alpha} \Bigg[1 + \frac{1}{(t-s_0)^{(1+\alpha)/2}} + \\ + \frac{1}{(t-s_0)^\alpha} + \frac{1}{(t-s_0)^{(1-\alpha^2)/2}} \Bigg] \Bigg\} + C_1 M_0 \int_{s_0}^t |\bar{y}_1(s) + \bar{y}_0(s) - 2\bar{y}_2(s)| ds \end{aligned}$$

for all $t \in]s_0, T]$. Therefore, Lemma 2.4 implies

$$|\bar{y}_1(T) + \bar{y}_0(T) - 2\bar{y}_2(T)| \leq M'_{\alpha,R}(|x_1 - x_0|^{1+\alpha} + |t_1 - t_0|^{1+\alpha}).$$

The conclusion follows from (45), (48), (49) and the estimate above.

QED

Corollary 4.2 *Under all assumptions of Theorema 4.1, suppose further that e^{tA} is compact for $t > 0$. Then V is Fréchet differentiable at all points (t, x) such that $D^+V(t, x)$ is a singleton.*

The proof follows from Theorems 4.1, 3.1, Corollary 3.4 and Proposition 2.1, recalling that, since the semigroup is compact and analytic, its fractional powers are compactly embedded in X.

5 Applications

We provide here some applications of the above results to Mayer optimal control problems. First we associate with the control system (2) its Hamiltonian $H : [0,T] \times X \times X^* \to \mathbf{R}$ defined by

$$H(t,x,p) = \sup_{u \in U} \langle p, f(t,x,u) \rangle.$$

Theorem 5.1 *Assume that (22) (ii), (iii) hold true and that A generates a strongly continuous semigroup on X. Let $\bar{x}(\cdot)$ be an optimal solution to problem (21). Then for almost every $t \in [t_0, T]$ such that $\bar{x}(t) \in D(A)$ we have*

$$\forall (p_t, p_x) \in D^+V(t, \bar{x}(t)), -p_t - \langle p_x, A\bar{x}(t) \rangle + H(t, \bar{x}(t), -p_x) = 0.$$

Proof. Let $\bar{u}(\cdot)$ be an optimal control corresponding to $\bar{x}(\cdot)$. Consider the set of Lebesgue points of the function $f(\cdot, \bar{x}(\cdot), \bar{u}(\cdot))$

$$\mathcal{T} := \left\{ t \in [t_0, T] | \lim_{h \to 0+} \frac{1}{h} \int_{t-h}^{t+h} |f(s, \bar{x}(s), \bar{u}(s)) - f(t, \bar{x}(t), \bar{u}(t))| ds = 0 \right\}.$$

We recall that $\mathcal{T}$ has a full measure in $[t_0, T]$. Let $t \in \mathcal{T}$ be such that $\bar{x}(t) \in D(A)$, then it is not difficult to check that

$$\bar{x}'(t) = A\bar{x}(t) + f(t, \bar{x}(t), \bar{u}(t)). \tag{54}$$

Fix $(p_t, p_x) \in D^+V(t, \bar{x}(t))$. Then

$$0 \geq \limsup_{s \to t+} \frac{V(s, \bar{x}(s)) - V(t, \bar{x}(t)) - p_t(s-t) - \langle p_x, \bar{x}(s) - -\bar{x}(t) \rangle}{|s-t| + |\bar{x}(s) - \bar{x}(t)|}.$$

Since $\bar{x}(\cdot)$ is optimal, $V(\cdot, \bar{x}(\cdot)) \equiv \text{const}$. Thus the above inequality and (54) yield

$$-p_t - \langle p_x, A\bar{x}(t) + f(t, \bar{x}(t), \bar{u}(t)) \rangle \leq 0.$$

By the same argument, taking $s \to t$ we obtain

$$p_t + \langle p_x, A\bar{x}(t) + f(t, \bar{x}(t), \bar{u}(t)) \rangle \leq 0.$$

Consequently,

$$\forall (p_t, p_x) \in D^+V(t, \bar{x}(t)), -p_t - \langle p_x, \; A\bar{x}(t) + f(t, \bar{x}(t), \bar{u}(t)) \rangle = 0. \tag{55}$$

We next claim that for every $(p_t, p_x) \in D^+V(t, \bar{x}(t))$ and every $u \in U$

$$- p_t - \langle p_x, A\bar{x}(t) + f(t, \bar{x}(t), u)\rangle \leq 0. \tag{56}$$

Indeed fix $(p_t, p_x) \in D^+V(t, \bar{x}(t)), u \in U$. Consider the Cauchy problem

$$\begin{cases} x'(s) &= Ax(s) + f(s, x(s), u), \quad s \in [t, T] \\ x(t) &= \bar{x}(t) \end{cases}$$

Since $\bar{x}(t) \in D(A)$, its (unique) solution $x(\cdot)$ satisfies $x'(t) = A\bar{x}(t) + f(t, \bar{x}(t), u)$. Using the fact that $t \to V(t, x(t))$ is nondecreasing, we obtain

$$\begin{aligned} 0 &\leq \limsup_{h\to 0+} \frac{V(t+h, x(t+h)) - V(t, x(t))}{h} \\ &\leq \limsup_{h\to 0+} \frac{V(t+h, x(t+h)) - V(t, x(t)) - hp_t - h\langle p_x, A\bar{x}(t) + f(t, \bar{x}(t), u)\rangle}{h} \\ &+ p_t + \langle p_x, A\bar{x}(t) + f(t, \bar{x}(t), u)\rangle \leq p_t + \langle p_x, A\bar{x}(t) + f(t, \bar{x}(t), u)\rangle \end{aligned}$$

and (56) follows. To end the proof it is enough to apply (55), (56) and the definition of H.

QED

Remark. We recall that, when X is a Hilbert space and A generates an analytic semigroup, then $\bar{x}(t) \in D(A)$ a.e..

Theorem 5.2 *Assume (22) and (39). Let $\overline{x}(\cdot)$ be an optimal trajectory of problem (21). Then, for any $\theta \in]0,1[$*

$$- p_t + \left\langle (-A^\star)^\theta p_x, (-A)^{1-\theta}\overline{x}(t)\right\rangle + H(t, \overline{x}(t), -p_x) = 0 \tag{57}$$

for all $t \in]t_0, T[$ and all $(p_t, p_x) \in D^+V(t, \overline{x}(t))$.

Notice that $p_x \in D(-A^\star)$ in light of Corollary 3.4. We first prove a lemma

Lemma 5.3 *Assume (22) and let $\theta \in]0,1[$. Then*

$$- p_t + \left\langle (-A^\star)^\theta p_x, (-A)^{1-\theta} x\right\rangle + H(t, x, -p_x) \leq 0 \tag{58}$$

for all $(t, x) \in [0, T[\times D(-A)^{1-\theta}$ and all $(p_t, p_x) \in D^+V(t, x)$.

Proof. We use the same argument as in the second part of the proof of Theorem 5.1. Fix $u \in U$ and let $\overline{x}(\cdot) = x(\cdot, t, x, u)$. Then, for all $h > 0$

$$\frac{1}{h}\langle p_x, \overline{x}(t+h) - x\rangle = \frac{1}{h}\left\langle p_x, (e^{hA} - 1)x\right\rangle + \frac{1}{h}\int_0^h \left\langle e^{(h-s)A^\star} p_x, f(t+s, \overline{x}(t+s), u)\right\rangle ds$$

Clearly,

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \left\langle e^{(h-s)A^\star} p_x, f(t+s, \overline{x}(t+s), u)\right\rangle ds = \langle p_x, f(t,x,u)\rangle$$

Moreover, in view of Corollary 3.4,

$$\begin{aligned}&\lim_{h\to 0^+} \tfrac{1}{h}\left\langle p_x, (e^{hA} - 1)x\right\rangle = \lim_{h\to 0^+} -\tfrac{1}{h}\textstyle\int_0^h \left\langle (-A^\star)^\theta p_x, e^{sA}(-A)^{1-\theta}x\right\rangle ds \\ &= -\left\langle (-A^\star)^\theta p_x, (-A)^{1-\theta}x\right\rangle\end{aligned}$$

Therefore,

$$\begin{aligned}0 \le \limsup_{h\to 0^+} \tfrac{V(t+h, \overline{x}(t+h)) - V(t,x)}{h} &\le p_t + \limsup_{h\to 0^+} \tfrac{\langle p_x, \overline{x}(t+h) - x\rangle}{h} \\ &\le p_t + \langle p_x, f(t,x,u)\rangle - \left\langle (-A^\star)^\theta p_x, (-A)^{1-\theta}x\right\rangle\end{aligned}$$

and (58) follows recalling the definition of H.

QED

Remark. Estimate (58) is equivalent to saying that V is a viscosity subsolution of the Hamilton-Jacobi-Bellman equation

$$-V_t(t,x) + H(t,x,-V_x(t,x)) - \langle V_x(t,x), Ax\rangle = 0.$$

In fact, modifying the argument above as one does in the finite dimensional case (see e.g. [PL]), one can show that V is also a viscosity supersolution of the above equation, i.e.

$$-p_t + \left\langle (-A^\star)^\theta p_x, (-A)^{1-\theta}x\right\rangle + H(t,x,-p_x) \ge 0$$

for all $(t,x) \in [0,T[\times D(-A)^{1-\theta}$ and all $(p_t, p_x) \in D^-V(t,x)$.

Proof of Theorem 5.2. From Lemma 5.3 and (9) we know that

$$-p_t + \left\langle (-A^\star)^\theta p_x, (-A)^{1-\theta}\overline{x}(t)\right\rangle + H(t, \overline{x}(t), -p_x) \le 0$$

for all $t \in]t_0, T[$ and all $(p_t, p_x) \in D^+V(t, \overline{x}(t))$. Hence, it suffices to derive the opposite inequality. Recalling Theorem 4.1, Proposition 2.2 and (14) we obtain, for all $s \in]t_0, T[$ and $(p_t, p_x) \in D^+V(t, \overline{x}(t))$,

$$\begin{aligned}
&-(s-t)\left[p_t + \tfrac{1}{s-t}\langle p_x, \overline{x}(s) - \overline{x}(t)\rangle\right] = \\
&= V(s, \overline{x}(s)) - V(t, \overline{x}(t)) - p_t(s-t) - \langle p_x, \overline{x}(s) - \overline{x}(t)\rangle \leq \\
&\leq C_\alpha(|t-s| + |\overline{x}(s) - \overline{x}(t)|)^{1+\alpha}
\end{aligned}$$

Now, by (8) it follows that $\overline{x}(\cdot) \in C^\theta([\frac{t_0+t}{2}, t]; X)$ for any $0 < \theta < 1$. Let us fix $\theta = \frac{2+\alpha}{2+2\alpha}$. So, the above inequality yields

$$-p_t - \frac{1}{s-t}\langle p_x, \overline{x}(s) - \overline{x}(t)\rangle \geq -C_{\alpha,\theta}|t-s|^{\alpha/2} \tag{59}$$

for any $\frac{t_0+t}{2} \leq s < t$. Next,

$$\begin{aligned}
-\tfrac{1}{s-t}\langle p_x, \overline{x}(s) - \overline{x}(t)\rangle &= \tfrac{1}{s-t}\left\langle p_x, (e^{(t-s)A} - 1)\overline{x}(s)\right\rangle + \\
&+ \tfrac{1}{s-t}\int_s^t \left\langle p_x, e^{(t-\sigma)A} f(\sigma, \overline{x}(\sigma), \overline{u}(\sigma)\right\rangle d\sigma.
\end{aligned}$$

Recalling (10), we have

$$\begin{aligned}
&\lim_{s\to t^-} \tfrac{1}{s-t}\left\langle p_x, (e^{(t-s)A} - 1)\overline{x}(s)\right\rangle = \\
&= \lim_{s\to t^-} \tfrac{1}{t-s}\int_0^{t-s} \left\langle (-A^\star)^\theta p_x, e^{\sigma A}(-A)^{1-\theta}\overline{x}(t)\right\rangle d\sigma = \\
&= \left\langle (-A^\star)^\theta p_x, (-A)^{1-\theta}\overline{x}(t)\right\rangle.
\end{aligned}$$

Therefore, taking $\liminf_{s\to t^-}$ in (59), we obtain

$$\begin{aligned}
&-p_t + \left\langle (-A^\star)^\theta p_x, (-A)^{1-\theta}\overline{x}(t)\right\rangle + \\
&+ \liminf_{s\to t^-} \tfrac{1}{t-s}\int_0^{t-s} \left\langle -p_x, e^{(t-\sigma)A} f(\sigma, \overline{x}(\sigma), \overline{u}(\sigma))\right\rangle d\sigma \geq 0
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\liminf_{s\to t^-} \tfrac{1}{t-s}\int_0^{t-s} \left\langle -p_x, e^{(t-\sigma)A} f(\sigma, \overline{x}(\sigma), \overline{u}(\sigma)\right\rangle d\sigma = \\
&= \liminf_{s\to t^-} \tfrac{1}{t-s}\int_0^{t-s} \langle -p_x, f(t, \overline{x}(t), \overline{u}(\sigma))\rangle d\sigma \in \overline{co}\mathcal{F}
\end{aligned}$$

where $\mathcal{F} = \{\langle -p_x, f(t, \overline{x}(t), u)\rangle \,|u \in U\}$. Since $H(t, \overline{x}(t), -p_x) = \sup \mathcal{F}$, the conclusion follows.

QED

An interesting consequence of Theorem 5.2 is the smoothness of the value function along any optimal trajectory in case H is strictly convex in p, a classical result for finite dimensional problems in calculus of vaiations (see [Fl]).

Corollary 5.4 *Assume (22), (39) and let $\overline{x}(\cdot)$ be an optimal trajectory of problem (21). If $H(t,\overline{x}(t),\cdot)$ is strictly convex for some $t \in]t_0,T[$, the V is Fréchet differentiable at $(t,\overline{x}(t))$.*

Proof. The strict convexity of $H(t,\bar{x}(t),\cdot)$ and (57) yield that $D^+V(t,\bar{x}(t))$ is a singleton. Then Corollary 4.2 concludes the proof.

QED

To provide further applications we need to recall necessary conditions satisfied by optimal solutions to problem (21). Let $(\bar{x},\bar{u})$ be a trajectory-control pair for system (20). Denote by $G(s,t)$ the solution operator of the linear problem

$$\begin{cases} \frac{\partial G}{\partial s}(s,t) = (A + \frac{\partial f}{\partial x}(s,\bar{x}(s),\bar{u}(s)))G(s,t) \\ G(t,t) = Id. \end{cases}$$

Let $G^*(s,t)$ denote the adjoint of $G(s,t)$.

Theorem 5.5 *Assume that X is a Hilbert space, that f is Fréchet differentiable with respect to x and that (22), (39) hold true. Consider an optimal trajectory-control pair $(\bar{x},\bar{u})$ of problem (21). Then for every $p \in D^+g(\bar{x}(T))$, the function $\bar{p}(t) = -G^*(T,t)p$ satisfies the maximum principle*

$$\langle \bar{p}(t), f(t,\bar{x}(t),\bar{u}(t))\rangle = H(t,\bar{x}(t),\bar{p}(t)) \text{ a.e. in } [t_0,T]$$

and the co-state inclusion

$$-\bar{p}(t) \in D_x^+V(t,\bar{x}(t)) \quad \text{for all } t \in [t_0,T].$$

Furthermore, for every $0 < \alpha < 1$ and all $t \in]t_0,T]$

$$(\langle(-A^*)^{\alpha}\bar{p}(t),(-A)^{1-\alpha}\bar{x}(t)\rangle + H(t,\bar{x}(t),\bar{p}(t)),-\bar{p}(t)) \in D^+V(t,\bar{x}(t)).$$

Proof. The first two statements result from [CF1, Theorem 3.1] as well as the inclusion

$$(\langle\bar{p}(t),A\bar{x}(t)\rangle + H(t,\bar{x}(t),\bar{p}(t)),-\bar{p}(t)) \in D^+V(t,\bar{x}(t))$$

for almost all $t \in [t_0, T]$ such that $\bar{x}(t) \in D(A)$. By the maximal regularity result $[LM]$, we get $\bar{x}(t) \in D(A)$ almost everywhere in $[t_0, T]$. Fix $0 < \alpha < 1$. From (9) we already know that $\bar{x}(t) \in D((-A)^{1-\alpha})$ for all $t \in]t_0, T]$. Consequently, using Corollary 3.4, we deduce that for almost all $t \in [t_0, T]$

$$(\langle (-A^*)^{\alpha}\bar{p}(t), (-A)^{1-\alpha}\bar{x}(t)\rangle + H(t, \bar{x}(t), \bar{p}(t)), -\bar{p}(t)) \in D^+V(t, \bar{x}(t)).$$

Fix $\bar{t} \in]t_0, T]$ and let $t_n \to \bar{t}$ be such that the above inclusion holds true at every t_n. Taking the limit we obtain

$$\langle (-A^*)^{\alpha}\bar{p}(\bar{t}), (-A)^{1-\alpha}\bar{x}(\bar{t})\rangle + H(\bar{t}, \bar{x}(\bar{t}), \bar{p}(\bar{t})), -\bar{p}(\bar{t})) \in \operatorname*{Lim\,sup}_{n\to\infty} D^+V(t_n, \bar{x}(t_n))$$

whenever $\bar{p}(\bar{t}) \in D((-A^*)^{\alpha})$. Thanks to Corollary 3.4 it remains to show that

$$\operatorname*{Lim\,sup}_{n\to\infty} D^+V(t_n, \bar{x}(t_n)) \subset D^+V(t, \bar{x}(t)).$$

By Theorem 4.1 we know that V is locally Lipschitz and semiconcave at $(\bar{t}, \bar{x}(\bar{t}))$. Proposition 2.1 ends the proof.

QED

Theorem 5.6 *Let X be a separable Hilbert space and assume that g is continuously differentiable. Suppose further that (22), (39) hold true and*

i) $f(t, x, U)$ *is closed and convex for all* $(t, x) \in [0, T] \times X$
ii) e^{tA} *is compact for all* $t > 0$
iii) H *is differentiable with respect to* x *and* $\forall R > 0, \exists \ell_R \in L^1(0, T)$ *such that for all* $x, y \in B_R, p, q \in X^*$ *with* $|p|, |q| \le R$ $|\frac{\partial H}{\partial x}(t, x, p) - \frac{\partial H}{\partial x}(t, y, q)| \le \ell_R(t)(|x - y| + |p - q|)$.

Then for every $(t_0, x_0) \in [0, T] \times X$

$$D^*V(t_0, x_0) = \{\lim_{k\to\infty} \nabla V(t_k, x_k) : (t_k, x_k) \to (t_0, x_0), \exists\, \nabla V(t_k, x_k)\}.$$

Remark. From the proof given below, it is easy to realize that the same result holds true if A generates a strongly continuous (not necessarily analytic) semigroup on X.

Proof. Fix $(t_0, x_0) \in [0, T] \times X$ and let $(t_k, x_k) \to (t_0, x_0)$ be such that $\nabla V(t_k, x_k)$ is weakly-$*$ converging. It is enough to show that $\{\nabla V(t_k, x_k)\}$

has a strongly converging subsequence. From Theorem 5.2 and [CF1, Corollary 5.6 and Remark 3.3] there exist optimal trajectories $\bar{x}_k(\cdot)$ for problem (21) with (t_0, x_0) replaced by (t_k, x_k) and solutions $p_k(\cdot)$ to the backward Cauchy problem

$$\begin{cases} -p' &= A^*p + \frac{\partial H}{\partial x}(t, \bar{x}_k(t), p) \\ -p(T) &= \nabla g(\bar{x}_k(T)) \end{cases} \tag{60}$$

satisfying

$$-p_k(t_k) = V_x'(t_k, x_k). \tag{61}$$

Using [CF1, Lemma 5.4] we deduce that there exist a mild solution $\bar{x}(\cdot)$ to control system (20) and a subsequence $\bar{x}_{k_j}(\cdot)$ such that

$$\lim_{j\to\infty} \sup_{t\in[t_0,T]\cap[t_{k_j},T]} |\bar{x}(t) - \bar{x}_{k_j}(t)| = 0.$$

Thus $\lim_{j\to\infty} \nabla g(\bar{x}_{k_j}(T)) = \nabla g(\bar{x}(T))$. By the continuous depedence of solutions to (60) on data we deduce that

$$\lim_{j\to\infty} \sup_{t\in[t_0,T]\cap[t_{k_j},T]} |p(t) - p_{k_j}(t)| = 0$$

where p is the mild solution to

$$\begin{cases} -p' &= A^*p + \frac{\partial H}{\partial x}(t, \bar{x}(t), p) \\ -p(T) &= \nabla g(\bar{x}(T)) \end{cases}$$

In particular this yields that

$$\lim_{j\to\infty} p_{k_j}(t_{k_j}) = p(t_0).$$

This and (61) imply that $V_x'(t_{k_j}, x_{k_j})$ converge strongly. Since $V_t'(t_k, x_k) \in \mathbf{R}$ it is strongly convergent. Thus we deduce that $\nabla V(t_{k_j}, x_{k_j})$ is strongly convergent.

QED

Theorem 5.7 *Assume (22), (39) and let e^{tA} be compact for $t > 0$. Then, for every $(t_0, x_0) \in [0, T[\times X$,*

$$D^\star V(t_0, x_0) = \{\lim_{k\to\infty} \nabla V(t_k, x_k) : (t_k, x_k) \to (t_0, x_0),\ \ \exists \nabla V(t_k, x_k)\}.$$

Proof. First notice that, since $-A$ is sectorial and e^{tA} is compact for $t > 0$, then $(-A)^{-\theta}$ and $(-A^{\star})^{-\theta}$ are compact opetators on X and $X^{\star}$ respectively (see e.g. [He]). Hence, $D((-A^{\star})^{\theta})$ is compactly embedded in $X^{\star}$ and the conclusion follows from Corollary 3.4.

QED

Corollary 5.8 *Under all assumptions of Theorem 5.6 for every* $(t_0, x_0) \in [0,T] \times D((-A)^{1-\alpha})$ *and* $0 < \alpha < 1$ *we have*

$$\forall (p_t, p_x) \in D^{*}V(t_0, x_0), p_t + \langle (-A^{*})^{\alpha} p_x, (-A)^{1-\alpha} x_0 \rangle = H(t_0, x_0, -p_x).$$

Proof. Fix $0 < \alpha < 1, (t_0, x_0) \in [0,T] \times D((-A)^{1-\alpha})$ and $(p_t, p_x) \in D^{*}V(t_0, x_0)$. By Theorem 5.6 there exist $(t_k, x_k) \to (t_0, x_0)$ such that $\nabla V(t_k, x_k)$ converge strongly to (p_t, p_x). Furthermore, Theorems 4.1, 5.6 and Proposition 2.1 imply that for all $k \geq 1$

$$\lim_{x \to x_k} \sup_{p \in D^{+}V(t_k, x)} \|\nabla V(t_k, x) - p\| = 0.$$

Thus there exist $y_k \to x_0, y_k \in D((-A)^{1-\alpha})$ such that

$$\lim_{k \to \infty} \sup_{p \in D^{+}V(t_k, y_k)} \|(p_t, p_x) - p\| = 0. \tag{62}$$

It is known (see e. g. [CF1, Lemma 5.4]) that there exist optimal trajectories $\bar{x}_k(\cdot)$ of problem (21) with (t_0, x_0) replaced by (t_k, y_k). Let $\bar{p}_k(\cdot)$ denote the corresponding co-states given by Theorem 5.5.

Since $y_k \in D((-A)^{1-\alpha})$ we know that $\forall t \in [t_k, T], \bar{x}_k(t) \in D((-A)^{1-\alpha}))$. This and the last statement of Theorem 5.5 yield that

$$\langle (-A^{*})^{\alpha} \bar{p}_k(t_k), (-A)^{1-\alpha} y_k \rangle + H(t_k, y_k, \bar{p}_k(t_k)), -\bar{p}_k(t_k)) \in D^{+}V(t_k, y_k)$$

Set

$$\begin{aligned} p_t^k &= \langle (-A^{*})^{\alpha} \bar{p}_k(t_k), (-A)^{1-\alpha} y_k \rangle + H(t_k, y_k, \bar{p}_k(t_k)) \\ p_x^k &= -\bar{p}_k(t_k). \end{aligned}$$

Then

$$p_t^k + \langle (-A^{*})^{\alpha} p_x^k, (-A)^{1-\alpha} y_k \rangle = H(t_k, y_k, -p_x^k). \tag{63}$$

Since y_k converge to $x_0 \in D((-A)^{1-\alpha})$ and, by (62) $\lim_{k \to \infty} (p_t^k, p_x^k) = (p_t, p_x)$, taking the limit in (63) we end the proof.

QED

Corollary 5.9 *Under all assumptions of Theorem 5.6 suppose that for some* $(t_0, x_0) \in [0, T[\times D((-A)^{1-\alpha})$, $D_x^+ V(t_0, x_0)$ *is a singleton. Then* V *is Fréchet differentiable at* (t_0, x_0) *and* $D^* V(t_0, x_0) = \{\nabla V(t_0, x_0)\}$.

Proof. Let Π_x denote the projection of $\mathbf{R} \times X$ onto X. Since

$$\Pi_x D^+ V(t_0, x_0) \subset D_x^+ V(t_0, x_0) =: \{p_0\}$$

we deduce from Corollary 5.8 and the equality

$$D^+ V(t_0, x_0) = \overline{co} D^* V(t_0, x_0)$$

that for all $(p_t, p_x) \in D^* V(t_0, x_0), p_x = p_0$ and

$$p_t = H(t_0, x_0, -p_0) - \langle (-A^*)^\alpha p_0, (-A)^{1-\alpha} x_0 \rangle.$$

Thus $D^* V(t_0, x_0)$ is a singleton. Since V is locally Lipschitz and semiconcave at (t_0, x_0), Proposition 2.1 ends the proof.

QED

Theorem 5.10 *Under all assumptions of Theorem 5.6 suppose that* A *is self-adjoint and that the Gâteaux derivative* $V_x'(t_0, x_0)$ *does exist. Let* $\bar{x}(\cdot)$ *be an optimal solution to problem (21). Then for all* $t \in]t_0, T[, V$ *is Fréchet differentiable at* $(t, \bar{x}(t))$ *and*

$$D^* V(t, \bar{x}(t)) = \{\nabla V(t, \bar{x}(t))\}.$$

Proof. Let $\bar{p}(\cdot)$ denote the co-state corresponding to $\bar{x}(\cdot)$ and given by Theorem 5.5. From [CF1, Theorem 5.1] and [CG] we deduce that for all $t \in [t_0, T]$

$$D_x^+ V(t, \bar{x}(t)) = \{-p(t)\}.$$

The proof follows by the application of Corollary 5.9 and using the fact that $\bar{x}(t) \in D((-A)^{1-\alpha})$ for all $t \in]t_0, T]$.

QED

Corollary 5.11 *Under all assumptions of Theorem 5.6 suppose that problem (21) has a unique optimal solution* $\bar{x}(\cdot)$. *Then for every* $t \in]t_0, T[$, V *is Fréchet differentiable at* $(t, \bar{x}(t))$.

Proof. From [CF1, Theorem 5.3] we know that $V(t,\cdot)$ is Fréchet differentiable at $\bar{x}(t)$ for all $t \in]t_0, T]$. Applying Theorem 5.10 we end the proof.

QED

Theorem 5.12 *Under all assumptions of Theorem 5.6 suppose that g is convex and for all $t \in [0,T]$*

$$Graph\ (f(t,\cdot,U)) \text{ is convex.}$$

Then for every $t \in [0,T], V(t,\cdot)$ is convex and continuously differentiable on X.

Proof. Fix $t_0 \in [0,T[$. From [CF1, Corollary 5.6] we know that for all $x_0^i \in X, i = 1,2$ there exist optimal trajectories $\bar{x}^i(\cdot)$ to problem (21) with (t_0, x_0) replaced by (t_0, x_0^i). Fix $\lambda \in [0,1]$. Since $z(\cdot) = \lambda\bar{x}^1 + (1-\lambda)\bar{x}^2$ is a trajectory of the control system (20) with x_0 replaced by $\lambda x_0^1 + (1-\lambda)x_0^2$ we deduce that

$$\begin{aligned} V(t_0, \lambda x_0^1 + (1-\lambda)x_0^2) &\leq g(z(T)) \\ &\leq \lambda g(\bar{x}^1(T)) + (1-\lambda)g(\bar{x}^2(T)) \\ &= \lambda V(t_0, x_0^1) + (1-\lambda)V(t_0, x_0^2). \end{aligned}$$

Consequently, $V(t_0,\cdot)$ is continuous and convex. So, its subgradient at x_0 is nonempty. Since, by Theorem 4.1, $V(t_0,\cdot)$ is also semiconcave, we deduce from Proposition 2.1 that its superdifferential at x_0 is nonempty. Thus $V(t_0,\cdot)$ is differentiable at x_0. This and Proposition 2.1 end the proof.

QED

6 Optimal feedback

We provide here a result concerning the optimal synthesis for problem (21). With any $(t,x) \in [0,T] \times D(A)$ we associate the feedback set

$$F(t,x) = \left\{ v \in f(t,x,U) : \lim_{h\to 0+} \frac{V(t+h, x+h[Ax+v]) - V(t,x)}{h} = 0 \right\}.$$

Clearly $F(t,x) = \emptyset$ if the above limits do not exist for any $v \in f(t,x,U)$. We proved in [CF1] the following result.

Theorem 6.1 *Assume (22) and let* $u : [t_0, T] \to U$ *be measurable and* $\bar{x}$ *be a solution to (20) such that* $\bar{x}(t) \in D(A)$ *almost everywhere in* $[t_0, T]$*. Then* $\bar{x}'(t)$ *exists for almost all* t *and the following two statements are equivalent*

i) $\bar{x}$ *is optimal for problem (21)*
ii) $\bar{x}'(t) - A\bar{x}(t) \in F(t, \bar{x}(t))$ *a.e. in* $[t_0, T]$

Theorem 6.2 *Assume that (22) and (39) hold true and that the sets* $f(t, x, U)$ *are closed. Then the graph of the set-valued* F *is closed in* $[0, T] \times D(A) \times X$*.*

Proof. Consider the set-valued map $\hat{F} : [0, T] \times D(A) \longrightarrow X$ defined by

$$\hat{F}(t, x) = \{v \in X : V_-^{\circ}(t, x)(1, Ax + v) \leq 0\}.$$

From Proposition 2.3 we know that graph($\hat{F}$) is closed in $[0, T] \times D(A) \times X$. On the other hand, by (55), for all $x \in D(A)$ and $v \in f(t, x, U)$

$$\langle \nabla V(t, x), (1, Ax + v) \rangle \geq 0$$

This and Proposition 2.1 yield that for all $(t, x) \in [0, T] \times D(A)$

$$F(t, x) = \hat{F}(t, x) \cap f(t, x, U)$$

and the result follows.

QED

Corollary 6.3 *Assume that (22), (39) hold true, that the sets* $f(t, x, U)$ *are closed and that the set-valued map* F *defined above is single-valued. Then it is continuous.*

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