Working Paper

Regulation of Control Systems under Inequality Contraints

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Foreword

A velocity controlled viability theory for control systems with inequality contraints is introduced. We do this by constructing a velocity controlled regulation map and change the control problem in that sense. By doing that the solutions to the differential inclusion are prevented from running into the border of the viability domain too fast. This leads to a Machaud system, which provides for an application of the viability theorem as well as the smooth control theorem.

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REGULATION OF CONTROL SYSTEMS UNDER INEQUALITY CONSTRAINTS

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1. Viability Theory for Control Systems.

Let X and Z be two finite dimensional vector spaces. We consider a control system denoted by (U, f) described by a set valued feedback map $U : X \rightsquigarrow Z$ and a single valued map $f : \text{Graph}(U) \to X$ denoting the dynamics of the system. The evolution of the system (U, f) is governed by the differential inclusion [2]:

$$\begin{cases} \text{for almost all } t: & x'(t) \in F(x(t)) := \{f(x(t), u(t))\}_{u(t) \in U(x(t))} \\ \text{with the initial state:} & x(0) = x_0 \end{cases} \tag{1}$$

$f(x, u)$ denotes the velocity of the state x controled by $u \in U(x)$. Hence $F(x)$ is the set of velocities available to the system at state x. An alternative approach to feedback control problems has been given by G. Leitmann and his coauthors [7-9] and A. B. Kurzhanski and T. F. Fillipowa [6]. Here we will follow the viability theoretical approach introduced by J. P. Aubin [1].

Consider the case when the set $K \subseteq \operatorname{dom} U \subseteq X$ of viable states is described by inequality and equality constraints. Hence we have

$$\begin{cases} K := \{x \in L \subseteq X \,|\, g_i(x) \geq 0 \quad \forall i = 1, \ldots, p \text{ and } h_j(x) = 0 \quad \forall j = 1, \ldots, q\} \\ \qquad \text{with } L \text{ closed in the finite dimensional vector space } X \text{ and } g_i \text{ and} \\ \qquad h_j \text{ twice continuously differentiable functions from } X \text{ to } \mathbb{R}. \end{cases} \tag{2}$$

In many cases $L = X$.

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We denote by

$$I(x) := \{i \in \{1, \dots, p\} \,|\, g_i(x) = 0\} \tag{3}$$

the subsets of active constraints. Assume once and for all the transversality condition:

$$\begin{cases} \forall x_0 \in K \quad \exists v_0 \in C_L(x_0) := \{v \in X| \lim\limits_{\substack{h \to 0^+ \\ K \ni x' \to x_0}} d_K(x' + hv)/h = 0\} \\ \text{with } d_K(x) := \inf\limits_{z \in K} \|x - z\| \\ \text{such that } \langle g_i'(x_0), v_0 \rangle > 0 \qquad \forall i \in I(x) \\ \text{and } h'(x) C_L(X) = \mathbb{R}^q \end{cases} \tag{4}$$

We require that for every solution $x(\cdot)$ to (1) with the initial state $x_0 \in K$, $x(t)$ stays in K. We say that a function $x : \mathbb{R}_+ \to X$ is viable in K if $x(t) \in K$ for all $t \in \mathbb{R}_+$. Our aim is to find out whether there are controls $u(t) \in U(x(t))$ such that the solution of the differential equation $x'(t) = f(x(t), u(t))$ *with* $x(0) = x_0 \in K$, is viable in K. We say K satifies the viability property for $F : K \subseteq X \rightsquigarrow X$ if for all initial states $x_0 \in K$ there exists a solution $x : \mathbb{R}_+ \to X$ to the differential inclusion (1) which is viable in K. K satisfies the invariance condition if all solutions are viable in K. The **contingent cone** of K at x is according to viability theory [1,4] given by

$$\begin{cases} T_K(x) := \{v \in X \,|\, \liminf\limits_{h \to 0^+} \dfrac{d_K(x + hv)}{h} = 0 \qquad \text{with } d_K(x) := \inf\limits_{z \in K} \|x - z\|\} \\ \qquad = \{v \in T_L(x) \,|\, \langle h_j'(x), v \rangle = 0 \quad \forall j \text{ and } \langle g_i'(x), v \rangle \geq 0 \quad i \in I(x)\}. \end{cases} \tag{5}$$

If $L = X$ we have $T_L(x) = T_X(x) = X$. We associate with any subset $K \subseteq \text{dom}(U)$ the **regulation map**

$$\begin{cases} R_K : X \rightsquigarrow Z \qquad \text{defined by} \\ R_K(x) : = \{u \in U(x) | f(x, u) \in T_K(x)\} \\ \qquad = \{u \in U(x) | \langle h_j'(x), f(x, u) \rangle = 0 \; \forall j \;\; \wedge \;\; \langle g_i'(x), f(x, u) \rangle \geq 0 \; \forall i \in I(x)\} \end{cases} \tag{6}$$

Furthermore the subset $K \subseteq \text{dom}(U)$ is called a **viability domain** of the control system (U, f) if the **viability condition**

$$R_K(x) \neq \emptyset \qquad \forall x \in K \tag{7}$$

holds, which is the case if and only if

$$T_K(x) \cap F(x) \neq \emptyset \qquad \forall x \in K.$$

Analogously we formulate the **invariance condition** by

$$F(x) \subseteq T_K(x) \qquad \forall x \in K. \tag{8}$$

We may now formulate both theorems, namely the viability theorem for control systems and the smooth control theorem, which we use later on.

Theorem 1. (Viability Theorem).

Let us assume that the control system (U, f) is a Marchaud System, i. e. that it satisfies the following conditions:

(1) Graph(U) *is closed*
(2) *f is continuous*
(3) *the velocity subsets $F(x)$ are convex*
(4) *f and U have linear growth*
i. e. $\|f(x,u)\| \leq c(\|x\| + \|u\| + 1)$ *for some $c > 0$*
and $\|u(x)\| \leq d(\|x\| + 1)$ *for some $d > 0$*

Then a closed subset $K \subseteq$ dom(U) *satisfies the viability property if and only if it is a viability domain.*

Furthermore any open loop control $u(\cdot)$ regulating a viable solution $x(\cdot)$ in the sense that

$$\text{for almost all } t: \quad x'(t) = f(x(t)), u(t)) \quad \text{with } x_0 = x(0)$$

obeys the **regulation law***:*

$$\text{for almost all } t: \quad u(t) \in R_K(x(t))$$

Proof.
We refer to theorem 3.5.5 in [1], which is based on Haddad's Theorem [5]. □

Theorem 2. (Smooth Control Solutions).

Let K be closed in the finite dimensional vector space X. Consider a control system (f, U) such that $K \subseteq$ dom$\,U$*. Assume that the set valued feedback map $U(\cdot)$ is closed and that the dynamic f is continuous with linear growth. Then the following statements are equivalent:*

(1) *For any initial state $x_0 \in K$ and any initial control $u_0 \in U(x_0)$ there exists a smooth control solution $(x(\cdot), u(\cdot))$ to the control system starting at (x_0, u_0). This means that $x(\cdot)$ and $u(\cdot)$ are both absolutely continuous.*
(2) $\forall (x, u) \in \text{Graph}\, U: \quad DU(x,u)f(x,u) \neq \emptyset$
with D the contingent derivative of U at $(x, u) \in$ Graph$\,U$
defined by Graph$(DU(x,u)) := T_{\text{Graph}\, U}(x,u)$

Proof. We apply theorem 7.2.2 in [1], which provides the conditions for the existence of smooth state-control solutions. □

The above theorems provide conditions for the existence of viable or smooth control solutions, of the system. It seems likely that these viability conditions can be changed into invariance conditions, guaranteeing that all solutions are viable, by simply changing the differential inclusion given in (1) into

$$\begin{cases} \text{for almost all } t: & x'(t) \in \{f(x(t), u(t))\}_{u(t) \in R_K(x(t))} \\ \text{with the initial state: } x(0) = x_0. \end{cases}$$

$F(x)$ then satisfies the invariance condition given in (8) by construction.

But the reason why we often cannot use the above theorems for $U = R_K : X \rightsquigarrow Z$ is that T_K and therefore also R_K are both in general not closed. We solve this problem by introducing a velocity control. As we will see in the following section this is not only theoretically useful but has also an intuitive interpretation. An alternative way of dealing with the problem of non-closure of the regulation map is given by Aubin and Frankowska [3].

2. The Velocity Controlled Contingent Cone.

Although we have found appropriate conditions for the viability of the model, the situation is not satisfactory, mainly because of two reasons.

The first one is of technical kind: In order to apply theorem 1, we need the set valued feedback map $R_K(\cdot)$ to be closed. For simplicity let us assume that $L = X$ in K given in (2). Then the graph of T_K defined as in (5) and thus also the graph of R_K are in general not closed. This follows from the fact that, $T_K(x) = \ker h'(x)$ for any interior point but not necessarily on the boundary.

In order to visualize this let us consider $K = \mathbb{R}$ for an example. We thus have $T_K(x) = \{v \in \mathbb{R} | v \geq 0 \quad \text{if } x = 0\}$. Hence the sequence $(x_n, v_n) := (\frac{1}{n}, -1) \in \text{Graph}\, T_K$ converges to $(x, v) = (0, -1) \notin \text{Graph}\, T_K$.

The second problem arises from interpretation. It seems strange that a solution is allowed to run into the boundary of K with so to say full steem. In many cases we would say that we have to decrease velocity if we are near the boundary, in order to be able to stop or change direction before it is too late. For example if we consider a human controler we would rather expect him to decrease the velocity of the system smoothly when it approaches the border line than to let the system crash into the border with uncontrolled velocity.

It turns out that these two problems are closely related. In order to solve them we introduce a subset of the contingent cone and according to that also a subset to the regulation map. Intuitively we want to say that the system has to decrease velocity if it approaches a border line. Technically this leads to:

Definition 3. (The Velocity Controlled Contingent Set).
Let K be as in (2). We define the velocity controlled contingent set by

$$\begin{cases} T_K^{\diamond} : K \rightsquigarrow X \qquad \text{with} \\ T_K^{\diamond}(x) := \{u \in T_L(x) |\, g_i(x) + \langle g_i'(x), u \rangle \geq 0 \qquad \forall i = 1, \ldots, p \\ \qquad\qquad\qquad \text{and} \quad \langle h_j'(x), u \rangle = 0 \quad \forall j = 1, \ldots, q\} \end{cases} \tag{9}$$

Theorem 4.
Let K be as in (2). We impose the transversality condition (4) on K. Then $T_K^{\diamond}(x)$ is contained in the contingent cone $T_K(x)$. Its graph is closed whenever the graph of $x \rightsquigarrow T_L(x)$ is closed.

Conversely we obtain the following inclusion:

$$T_K(x) \cap \gamma_K(x)B \subseteq T_K^{\diamond}(X) \subseteq T_K(x)$$
$$\text{with } \gamma_K(x) := \min_{i \notin I(x)} \frac{g_i(x)}{\|g_i'(x)\|} \tag{10}$$
$$\text{and } B \text{ the unit ball in } X$$

In order to prove the theorem we need the following

Lemma 5.
The function $\gamma_K : K \to]0, \infty]$ defined by (10) is upper semicontinuous. Furthermore if there is a constant $c > 0$ such that

$$\|g_i'(x)\| \geq c\frac{g_i(x)}{\|x\| + 1} \qquad \forall i = 1, \dots, p$$

then γ_K has linear growth.

Proof of the Lemma.
Let $x_n \in K$ converge to x_0 and $a_n \leq \gamma_K(x_n)$ converge to a_0. $x_0 \in K$, since K is closed. $g_i(x_0) > 0$ whenever $i \notin I(x_0)$. From that we infer that $i \notin I(x_n)$ for n large enough. Hence the inequalities $a_n\|g_i'(x_n)\| \leq g_i(x_n)$ hold true for any $i \notin I(x_0)$ if n is large enough. They imply that $a_0\|g_i'(x_0)\| \leq g_i(x_0)$ for all $i \notin I(x_0)$. Hence $a_0 \leq \gamma_K(x_0)$.
The second statement follows directly from the definition of linear growth given as in theorem 1. By taking $d = \frac{1}{c}$ we achieve $|\gamma_K(x)| = \gamma_K(x) \leq d(\|x\| + 1) \quad \forall x$. □

Proof of the Theorem.
Let v belong to $T_K^{\diamond}(x)$. If $i \in I(x)$ it follows that $\langle g_i'(x), u\rangle = g_i(x) + \langle g_i'(x), u\rangle \geq 0$. Hence $u \in T_K(x)$.
Conversely let u belong to $T_K(x) \cap \gamma_K(x)B$. Then either $i \in I(x)$ and $g_i(x) + \langle g_i'(x), u\rangle = \langle g_i'(x), u\rangle \geq 0$, or $i \notin I(x)$. Then $g_i(x) > 0$ and since $\|u\| \leq \gamma_K(x) \leq \frac{g_i(x)}{\|g_i'(x)\|}$, we see that $g_i(x) + \langle g_i'(x), u\rangle \geq \|g_i'(x)\|\|u\| + \langle g_i'(x), u\rangle \geq 0$. It follows that u belongs to $T_K^{\diamond}(x)$.
It is left to show that the graph of $T_K^{\diamond}$ is closed whenever the graph of T_L is closed. Let $x_n \in K$ converge to x_0 and $u_n \in T_K^{\diamond}(x_n)$ converge to u_0. Since K is closed we only have to prove that $u_0 \in T_K^{\diamond}(x_0)$, in other words that $g_i(x_0) + \langle g_i'(x_0)\rangle \geq 0 \quad \forall i$. Assume that $g_i(x_0) + \langle g_i'(x_0), u_0\rangle := -\varepsilon < 0$ for one i. We know that there exists an n and $\delta_1, \delta_2 > 0$ such that $|g_i(x_n) + \langle g_i'(x_n), u_n\rangle - g_i(x_0) + \langle g_i'(x_0), u_0\rangle| < \varepsilon$ provided $\|x_n - x_0\| < \delta_1$ and $\|u_n - u_0\| < \delta_2$. Hence $g_i(x_n) + \langle g_i'(x_n), u_n\rangle < g_i(x_0) + \langle g_i'(x_0), u_0\rangle + \varepsilon = -\varepsilon + \varepsilon = 0$. But this is a contradiction. □

3. The Velocity Controlled Differential Inclusion.

Naturally we can also redefine $R_K(\cdot)$ and $F(\cdot)$ in the new sense:

Definition 6. (The Velocity Controlled Regulation Map).

Consider a control system (U, f) described by a feedback map U and dynamics f. Let $K \subseteq \operatorname{dom} U$ be as in (2). We define the f velocity controlled regulation map

$$\begin{cases} R_K^{\diamond} : K \rightsquigarrow Z & \text{with} \\ R_K^{\diamond}(x) := \{u \in U(x) | f(x,u) \in T_K^{\diamond}(x)\} \end{cases} \tag{11}$$

Lemma 7.

The set valued map $R_K^{\diamond}(\cdot)$, defined above in (11), is closed.

Proof.

This follows from theorem 4. □

Definition 8.

We consider the same situation as in definition 5. We define the set valued map

$$\begin{cases} F^{\diamond} : K \rightsquigarrow X & \text{by} \\ F^{\diamond}(x) := \{f(x,u)\}_{u \in R_K^{\diamond}(x)} \end{cases} \tag{12}$$

In the following we consider the differential inclusion:

$$\begin{cases} \text{for almost all } t: & x'(t) \in F^{\diamond}(x(t)) \\ \text{with the initial state} & x(0) = x_0 \in K \quad \text{given} \end{cases} \tag{13}$$

Theorem 9.

Let us assume that the control system (U, f) has an upper semicontinuous set valued feedback map $U : X \rightsquigarrow Z$ with closed values. Let the dynamic of the system $f : X \times Z \to X$ be continuous. Then the graph of $F^{\diamond}$ is closed.

Proof.

We use the fact that the graph of an upper semicontinuous set valued map with closed values is closed [4]. Thus we have to show that $F^{\diamond}$ is upper semicontinuous and that $F^{\diamond}(x)$ is closed for all x. The second statement follows directly from theorem 4, which says that $T_K^{\diamond}(x)$ is closed for all x and from the assumption that U has closed values. Hence it is left to show that $F^{\diamond}$ is upper semicontinuous.

$R_K^{\diamond} : X \rightsquigarrow X$ defined in (11) is upper semicontinuous since $R_{K,Z}^{\diamond} : X \rightsquigarrow X$ with $R_{K,Z}^{\diamond}(x) := \{u \in Z | \langle g_i'(x), f(x,u)\rangle + g_i(x) \geq 0 \quad \forall i\}$ and $U : X \rightsquigarrow X$ are both upper semicontinuous and $R_K^{\diamond}(x) = U(x) \cap R_{K,Z}^{\diamond}(x)$ for all x.

Choose $x \in K$ and $\varepsilon > 0$. We have to look for some $\mu > 0$ such that the velocity set $F^{\diamond}(x_1)$ is a subset of $B_X(F^{\diamond}(x), \varepsilon)$ whenever $\|x - x_1\| < \mu$. This means:

$$\forall x_1 \in K \quad \text{with} \quad \|x - x_1\| < \mu :$$
$$\forall u_1 \in R_K^{\diamond}(x_1) \qquad \exists\, u \in R_K^{\diamond}(x) \quad \text{such that} \quad \|f(x_1, u_1) - f(x,u)\| < \varepsilon.$$

Since f is continuous there exist $\delta_1, \delta_2 > 0$ such that for all $u, u_1 \in Z$ and for all $x, x_1 \in X$ with $\|x - x_1\| < \delta_1$ and $\|u - u_1\| < \delta_2$ one has $\|f(x,u) - f(x_1, u_1)\| < \varepsilon$.

We use that $R_K^{\diamond}(.)$ is upper semicontinuous. Thus there exists $\nu > 0$ such that $R_K^{\diamond}(x_1) \subseteq B_Z(R_K^{\diamond}(x), \delta_2)$ whenever $\|x - x_1\| < \nu$. In other words:

$$\forall u_1 \in R_K^{\diamond}(x_1) \qquad \exists u \in R_K^{\diamond}(x) \quad \text{such that} \quad \|u - u_1\| < \delta_2$$

We choose $\mu = \min\{\delta_1, \nu\}$ and conclude that whenever $\|x - x_1\| < \mu$, $F^{\diamond}(x_1)$ is a subset of $B_X(F^{\diamond}(x), \varepsilon)$. $\square$

We proved that if the control system (U, f) obeys the corresponding continuity conditions, $T_K^{\diamond}$ and hence also $R_K^{\diamond}$ as well as $F^{\diamond}$ are closed maps even if the original set valued map T_K was not closed. We constructed $F^{\diamond}$ by reducing the set of possible dynamics $F(x) = \{f(x,u)\}_{u \in R_K(x)}$ to those which are velocity controlled, i. e. to the set of velocities $f(x, u)$ with $u \in R_K^{\diamond}(x)$.

We can now apply these results. First we conclude in theorem 10 that $F^{\diamond}$ is a Marchaud map whenever its values are convex and it has linear growth.

Theorem 10. (Velocity Controlled Viability Theorem).

We make the same assumptions for the control system (U, f) as in theorem 9. If furthermore $F^{\diamond}$ has linear growth and convex values, then $F^{\diamond}$ is a Marchaud map and hence the system $(R_K^{\diamond}, f)$ is a Marchaud system. Thus whenever the stronger **velocity controled viability condition**

$$R_K^{\diamond}(x) \neq \emptyset \qquad \forall x \in K \tag{14}$$

holds, there exists for each initial point $x_0 \in K$ and each initial control $u_0 \in R_K^{\diamond}(x_0)$ a velocity controled viable solution, that is a solution to the differential inclusion (13). Moreover any solution to (13) is viable.

Furthermore we claim that the velocity controlled viability condition (14) holds whenever the usual viability condition (7) is replaced by

$$\forall x \in K: \qquad \exists u \in R_K(x) \qquad \textit{with} \quad \|f(x,u)\| \leq \gamma_K(x)$$

where $\gamma_K(\cdot)$ is defined as in (10).

Proof.

The first statement is true by theorem 9 and the definition of a Marchaud map [1]. Viability is then a direct deduction from theorem 1. Invariance holds since $F^{\diamond}(x)$ is a subset of $T_K(x)$ by definition.

To prove the last statement, we choose for a given $x \in K$ a control $u \in R_K(x)$ such that $\|f(x,u)\| \leq \gamma_K(x) = \min_{i \notin I(x)} \frac{g_i(x)}{\|g_i'(x)\|}$. If $i \in I(x)$ then $g_i(x) = 0$ and hence $g_i(c) + \langle g_i'(x), f(x,u)\rangle = \langle g_i'(x), f(x,u)\rangle \geq 0$. Let $i \notin I(x)$. We know that $\|f(x,u)\| \leq \gamma_K(x) \leq \frac{g_i(x)}{\|g_i'(x)\|}$. It follows that $|\langle g_i'(x), f(x,u)\rangle| \leq \|g_i'(x)\|\|f(x,u)\| \leq g_i(x)$. We conclude that $g_i(x) + \langle g_i'(x), f(x,u)\rangle \geq 0$ holds in both cases. $\square$

In the following we may not only fix an initial state x_0 but also an initial control u_0. Furthermore the theorem provides for the existence of smooth control solutions.

Theorem 11. (Smooth Control Solutions for $F^\diamond$).

Let X and Z be finite dimensional vector spaces. Consider a control system (U, f) described by an upper semi continuous feedback map $U : X \rightsquigarrow Z$ with closed values and a continuous dynamics $f : X \times Z \to X$, exhibiting linear growth. Let $K \subseteq \operatorname{dom} U$ be defined by inequality conditions as in (1.2). Assume that

$$\left\{ \begin{array}{l} \forall (x,u) \in \mathrm{Graph}(R_K^\diamond) \qquad \exists\, v \in DU(x,u)(y) \quad s.th: \\ \langle g_i'(x), y + f_x'(x,u)y - f_u'(x,u)v \rangle + g_i''(x)(f(x,u), y) \geq 0 \qquad \forall\, i \in I(x,u) \\ \text{where} \qquad I(x,u) := \{ i = 1, \ldots, p \,|\, g_i(x) - \langle g_i'(x), f(x,u) \rangle = 0 \} \end{array} \right. \tag{15}$$

Then if the velocity controlled viability condition (14) holds there exists for any initial state $x_0 \in K$ and any initial control $u_0 \in R_K^\diamond(x_0)$ a smooth control solution $(x(\cdot), u(\cdot))$ to the differential inclusion (13) with $F^\diamond$ as in (12). This means $x(\cdot)$ and $u(\cdot)$ are both absolutely continous.

Proof.

In order to apply theorem 2 we have to show that $DR_K^\diamond(x,u)(f(x,u)) \neq \emptyset$. But this is, in case where K is given by inequality conditions, equivalent to assumption (15). □

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