# Working Paper

## Ellipsoidal Techniques: Control Synthesis for Uncertain Systems

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# Foreword

This is the second of a series of papers giving an early account of the application of ellipsoidal techniques to problems of modeling dynamical systems.The paper deals with the problem of *control synthesis* for a linear system with unknown but bounded disturbances which ends up in a synthesized nonlinear differential inclusion. The third paper deals with guaranteed state estimation — also to be interpreted as a tracking problem.

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# Ellipsoidal Techniques: Control Synthesis for Uncertain Systems

*A.B. Kurzhanski, I. Vályi*

## Introduction

This paper deals with a technique of solving the problem of control synthesis under unknown but bounded disturbances that allows an algrithmization with an appropriate graphic simulation. The original theoretical solution scheme taken here comes from the theory introduced by N. N. Krasovski [1], from the notion of the "alternated integral" of L. S. Pontriagin [2] and the "funnel equation" in the form given in [3]. For alternative treatment of related problems see also [5], [6] and [7]. The theory is used as a point of application of constructive schemes generated through ellipsoidal techniques developed by the authors. A concise exposition of the latter is the objective of this paper. A particular feature is that the ellipsoidal techniques introduced here do indicate an *exact* approximation of the original solutions based on set-valued calculus by solutions formulated in terms of ellipsoidal valued functions only.

## 1 The Problem of Control Synthesis

Consider a controlled system

$$\dot{x}(t) = A(t)x(t) + u(t) - v(t), \quad x(t), u(t), v(t) \in \mathcal{R}^n, \qquad t_0 \le t \le t_1 \tag{1}$$

with control parameters $u(t)$ subjected to a constraint

$$u(t) \in \mathcal{P}(t)$$

and disturbance $v(t)$ which is unknown but bounded, subjected to a constraint

$$v(t) \in \mathcal{Q}(t).$$

Here $\mathcal{P}(t), \mathcal{Q}(t)$ are *multivalued* maps with values in conv $\mathcal{R}^n$ — the set of all convex compact subsets of $\mathcal{R}^n$. The $(n \times n)$-matrix $A(t)$ is assumed to be continuous.

The system (1) under discussion is an *uncertain system* since its *input* $v = v(t)$, or $v = v(t, x)$, is taken to be *unknown* in advance. The complete information on the state space vector $x$ is

assumed to be given at each instant of time $t$ with no bias. Therefore we presume that for each $t \in [t_0, t_1]$ the available information is the *position* $\{t, x_t\}$, ($t \in [t_0, t_1]$, $x_t = x(t)$) of the system and also the functions $A(t), \mathcal{P}(t), \mathcal{Q}(t)$ of which the last two are multivalued.

Let $\mathcal{M} \in \text{conv } \mathcal{R}^n$ be a given set. The problem of control synthesis under the informational conditions of the above will consist in specifying a set-valued function $\mathcal{U} = \mathcal{U}(t, x)$, ($\mathcal{U}(t, x) \subset \mathcal{P}(t)$) — "the synthesizing control strategy" which *for any admissible realization* $v(t)$ of the (unknown) parameter $v$, $v(t) \in \mathcal{Q}(t)$ would ensure that all the solutions $x(t, \tau, x_\tau) = x[t]$ to the equation

$$\dot{x}(t) = A(t)x(t) + \mathcal{U}(t, x(t)) - v(t), \qquad t_0 \leq t \leq t_1 \tag{2}$$

that start at a given position $\{\tau, x_\tau\}$, would reach the terminal set $\mathcal{M}$ at the prescribed instant of time $t = t_1$ — provided $x_\tau \in \mathcal{W}(\tau, \mathcal{M})$. Here $\mathcal{W}(\tau, \mathcal{M})$ is the *solvability set* for the problem, namely the set of all those states $x_\tau$ from which the solution to the problem does exist in a given class $\mathbf{U}$ of strategies $\mathcal{U}(t, x)$.

The set $\mathcal{W}(\tau, \mathcal{M})$ is the "largest" set (with respect to inclusion) from which the problem is solvable.

We further presume

$$\mathcal{W}[\tau] = \mathcal{W}(\tau, \mathcal{M}) \neq \emptyset, \qquad t_0 \leq \tau \leq t_1.$$

The strategy $\mathcal{U}(t, x)$ will then be selected in a class $\mathbf{U}$ of *feasible feedback strategies* which would ensure that the synthesized system — a *differential inclusion*

$$\dot{x}(t) \in A(t)x(t) + \mathcal{U}(t, x(t)) - \mathcal{Q}(t), \qquad t_0 \leq t \leq t_1 \tag{3}$$

- does have a solution that starts at any point $x(t_0) = x_{t_0} \in \mathcal{R}^n$ and is defined throughout the interval $[t_0, t_1]$.

The aim of the solution to the problem of control synthesis will now be to find a solution strategy $\mathcal{U}(t, x)$ such that *all of the trajectories* $x[t] = x(t, t_0, x_{t_0})$ of the system (3) that start at an initial point $x_{t_0} \in \mathcal{W}[t_0]$, would satisfy the inclusion

$$x(t) \in \mathcal{W}[t], \qquad t_0 < t \leq t_1, \tag{4}$$

whatever is the point $x_{t_0} \in \mathcal{W}[t_0]$.

As we shall see in the sequel, the strategy $\mathcal{U}(t, x)$ can be constructed on the basis of $\mathcal{W}[t]$ provided the latter is calculated in advance. The calculation of the set-valued function $\mathcal{W}[t]$, (*the solvability tube*) is therefore a crucial point in finding the overall solution $\mathcal{U}(t, x)$.

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Without any lost of generality, by substituting $z = S(t, t_1)x$, where $S(t, t_1)$ is the $(n \times n)$-matrix solution to the equation

$$\dot{S}(t, t_1) = -S(t, t_1)A(t), \quad t_0 \le t \le t_1, \qquad S(t_1, t_1) = I,$$

and by returning to the old notation we can transform system (1) into

$$\dot{x}(t) = u(t) - v(t), \qquad u(t) \in \mathcal{P}(t), \quad v(t) \in \mathcal{Q}(t), \qquad t_0 \le t \le t_1.$$

## 2 The Solvability Tube

The *solvability set* $\mathcal{W}(\tau, \mathcal{M}) = \mathcal{W}[\tau]$ for a prescribed instant of time $\tau \in [t_0, t_1]$ could be defined as the set of all those vectors $x_\tau$ for each of which there exists an admissible feedback strategy $\mathcal{U}(t, x[t]) \subset \mathcal{P}(t)$ such that any solution $x[t] = x(t, \tau, x_\tau)$ to the equation

$$\dot{x}[t] \in \mathcal{U}(t, x[t]) - \mathcal{Q}(t), \quad \tau \le t \le t_1, \qquad x(\tau) = x_\tau \tag{5}$$

would satisfy the terminal condition

$$x(t_1) \in \mathcal{M}. \tag{6}$$

It suffices that the class **U** of admissible strategies would consist of multivalued maps $\mathcal{U}(t, x) \in \text{conv } \mathcal{R}^n$, continuous in $t$ and upper semicontinuous in $x$. This ensures the existence of (absolutely continuous) solutions to the equation (5).

Denote $h(\mathcal{W}', \mathcal{W}'')$ to be the *Hausdorff distance* between $\mathcal{W}', \mathcal{W}''$ namely,

$$h(\mathcal{W}', \mathcal{W}'') = \max\{h_+(\mathcal{W}', \mathcal{W}''), h_-(\mathcal{W}', \mathcal{W}'')\},$$

where

$$h_+(\mathcal{W}', \mathcal{W}'') = \min\{r | \mathcal{W}' \subset \mathcal{W}'' + r\mathcal{S}\},$$

and $h_-(\mathcal{W}'', \mathcal{W}') = h_+(\mathcal{W}', \mathcal{W}'')$ are the *Hausdorff semidistances*, $\mathcal{S}$ is the unit ball in $\mathcal{R}^n$.

Consider the "funnel equation"

$$\lim_{\sigma \to +0} \sigma^{-1} h_+(\mathcal{Z}(t - \sigma) - \sigma \mathcal{Q}(t), \mathcal{Z}(t) - \sigma \mathcal{P}(t)) = 0, \quad t_0 \le t \le t_1, \qquad \mathcal{Z}(t_1) \subset \mathcal{M}. \tag{7}$$

A multivalued map $\mathcal{Z}(t)$ is understood to be a *solution* of (7) if it satisfies equation (7) almost everywhere.

A solution $\mathcal{Z}_*(t)$ is said to be a *"maximal solution"* of (7) if there exists no other solution $\mathcal{Z}(t)$ of (7) such that

$$\mathcal{Z}_*(t) \subset \mathcal{Z}(t) \qquad \text{and} \qquad \mathcal{Z}(t) \not\equiv \mathcal{Z}_*(t), \qquad t_0 \le t \le t_1.$$

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**Lemma 2.1** *The "funnel equation" (7) has a unique "maximal solution" $\mathcal{Z}_*(t)$ with $\mathcal{Z}_*(t_1) = \mathcal{M}$.*

**Lemma 2.2** *The solvability tube $\mathcal{W}[t]$ coincides with the unique maximal solution $\mathcal{Z}_*(t)$ to the equation (5), so that*

$$\mathcal{W}[t] = \mathcal{Z}_*[t], \quad t_0 \le t \le t_1, \qquad \mathcal{W}[t_1] = \mathcal{M}.$$

The "funnel equation" (7) with boundary condition $\mathcal{Z}(t_1) = \mathcal{M}$ can be "integrated". Its maximal solution turns to be a multivalued integral known as the "alternated integral" of L. S. Pontryagin, [2]. We recall that the latter is defined as follows:

(i) divide the interval $[\tau, t_1]$ for all $N$ into $N + 1$ subintervals $[\tau^i, \tau^{i+1})$; $\quad i = 0, \ldots N$; due to the grid

$$H_N = \{\tau_i | i = 1, \ldots N, \ \tau = \tau_0 < \tau_1 < \ldots < \tau_N = t_1\}$$

so that $\lim_{N \to \infty} \Delta(H_N) = 0$, where

$$\Delta(H_N) = \max\{| \ \tau_{i-1} - \tau_i \ |, \ | \ \tau_i \in H_N, \ i = 1, \ldots N\}.$$

(ii) construct the integral sums for $i = 0, \ldots N$;

$$X_N^{(N)} = \mathcal{M}, \qquad X_{i-1}^{(N)} = \left( X_i^{(N)} \overset{\cdot}{-} \int_{\tau_{i-1}}^{\tau_i} \mathcal{Q}(t)dt \right) + \int_{\tau_{i-1}}^{\tau_i} \mathcal{P}(t)dt$$

and denote

$$X_0^{(N)} = I^{(N)}[\tau, H_N, \mathcal{M}].$$

Here and above the symbol $\mathcal{P} \overset{\cdot}{-} \mathcal{Q}$ stands for the geometric difference (the "Minkowski" difference) of sets $\mathcal{P}$ and $\mathcal{Q}$, namely

$$\mathcal{P} \overset{\cdot}{-} \mathcal{Q} = \{c | \mathcal{Q} + c \subset \mathcal{P}\}.$$

(iii) The "alternated integral" $I[\tau, \mathcal{M}]$ is then considered to be the Hausdorff limit

$$\lim_{N \to \infty} I^{(N)}[\tau, H_N, \mathcal{M}] = I[\tau, \mathcal{M}]. \tag{8}$$

This limit exists and does not depend on the sequence of subdivisions $H_N$ if, for example, there exists an $\varepsilon > 0$ such that

$$\mathcal{P}(t) \overset{\cdot}{-} (\mathcal{Q}(t) + \varepsilon\mathcal{S}) \ne \emptyset, \qquad \tau \le t \le t_1.$$

The integral

$$I[\tau, \mathcal{M}] = \int_{\tau, \mathcal{M}}^{t_1} (\mathcal{P}(t)dt \overset{\cdot}{-} \mathcal{Q}(t)dt)$$

is then correctly defined.

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**Lemma 2.3** *The set* $\mathcal{W}(\tau, \mathcal{M})$ *can be expressed as*

$$\mathcal{W}(\tau, \mathcal{M}) = I(\tau, \mathcal{M}), \qquad t_0 \le \tau \le t_1.$$

Therefore the tube $\mathcal{W}[\tau] = \mathcal{W}(\tau, \mathcal{M})$ could be calculated as the multivalued "alternated" integral $I(\tau, \mathcal{M})$ with a variable lower limit $\tau$. From here it follows:

**Lemma 2.4** *The set-valued function* $\mathcal{W}[t]$ *is convex compact valued, continuous in* $t$.

Once the solvability tube is calculated, the solution, i.e. the control strategy $\mathcal{U}(t, x)$ can be defined.

# 3    The Synthesizing Control Strategy

According to N.N. Krasovski [1] the synthesizing strategy $\mathcal{U}(t, x)$ can be defined as

$$\mathcal{U}(t, x) = \partial_\ell \rho(-\ell^0(x) \mid \mathcal{P}(t)), \tag{9}$$

where

$$\ell^0(x) = \arg\min\{\|\ell\| \mid \ell \in \partial_x d(x, \mathcal{W}[t])\}. \tag{10}$$

Here $\partial_x f(x^*, t)$ stands for the subdifferential of function $f(x, t)$ in the variable $x$ at point $x^*$ and $d(x, \mathcal{W}[t]) = \min\{\|x - w\| \mid w \in \mathcal{W}[t]\}$ is the Euclidean distance from $x$ to $\mathcal{W}[t]$.

Strategy (9), (10) is therefore such that

$$\ell^0(x) = \begin{cases} 0 & \text{if } x \in \mathcal{W}[t] \\ \partial_x d(x, \mathcal{W}[t]) & \text{if } x \notin \mathcal{W}[t] \end{cases}.$$

Hence

$$\mathcal{U}(t, x) = \begin{cases} \mathcal{P}(t) & \text{if } x \in \mathcal{W}[t] \\ \arg\max\{(-\ell^0(x), u) \mid u \in \mathcal{P}(t)\} & \text{if } x \notin \mathcal{W}[t] \end{cases}$$

where $\ell^0(x)$ is the unique maximizer for the problem

$$(\ell^0(x), x) - \rho(\ell^0(x) \mid \mathcal{W}[t]) = \max\{(\ell, x) - \rho(\ell \mid \mathcal{W}[t]) \mid \|\ell\| \le 1\}.$$

From Lemma 2.3 and from the definition of $\mathcal{U}(t, x)$ it follows:

**Lemma 3.1** *The multivalued map* $\mathcal{U}(t, x)$ *is convex compact valued, continuous in* $t$ *and upper semincontinuous in* $x$.

The latter property ensures the existence of solutions to the inclusion (5) and indicates that $\mathcal{U}(t, x) \in \mathcal{P}(t)$. By [2] then we have

**Lemma 3.2** *Once* $x_\tau \in \mathcal{W}[\tau]$, *the following inclusion is true*

$$x(t) \in \mathcal{W}[t], \quad \tau < t \le t_1,$$

*so that* $x(t_1) \in \mathcal{M}$.

Strategy $\mathcal{U}(t, x)$ therefore solves the target problem under uncertainty.

The final aim is to define a constructive scheme for the solution that would yield an appropriate algorithmic procedure. This will be done by way of approximating the tube $\mathcal{W}[t]$ through *ellipsoidal-valued* functions. We will also indicate a procedure that allows an *exact approximation* of $\mathcal{W}[t]$ by a variety of such functions.

# 4   The Discrete-time Scheme

As it was observed earlier the function $\mathcal{W}[t]$ could be represented either through an "alternated" integral (8) or through a "funnel" equation (7). The latter equation yields a *discrete-time scheme*

$$\mathcal{W}^{(\sigma)}[\tau_k^{(\sigma)}] = \left( \mathcal{W}^{(\sigma)}[\tau_{k-1}^{(\sigma)}] - \sigma \mathcal{P}(\tau_{k-1}^{(\sigma)}) \right) \dot{-} \left( -\sigma \mathcal{Q}(\tau_{k-1}^{(\sigma)}) \right) \tag{11}$$

$$\tau_k^{(\sigma)} = t_1 - \sigma k, \quad , k = 0, \dots N, \dots,$$

$$\mathcal{W}[\tau_0^{(\sigma)}] = \mathcal{M}.$$

**Lemma 4.1** *Let a* $t \in [t_0, t_1]$ *be fixed and suppose that* $\mathrm{int}(\mathcal{W}[t]) \ne \emptyset$, *then the discrete-time scheme (10) yields the relation*

$$\lim_{\sigma \to 0} h(\mathcal{W}^{(\sigma)}[\tau_{k_\sigma}^{(\sigma)}], \mathcal{W}[t]) = 0$$

*where* $k_\sigma$ *is chosen in such a way that*

$$| t - \tau_{k_\sigma}^{(\sigma)} | \le \sigma$$

*holds.*

From (11) it is clear that this scheme requires the addition and the geometrical subtraction of convex compact sets. Therefore the issue is how to organize a scheme of ellipsoidal approximations for these types of operations.

# 5 Ellipsoidal Techniques: Discrete Time

In this paper we do not elaborate on the ellipsoidal calculus in whole but do indicate the necessary amount of techniques for the specific problem of control synthesis.

The further notations are such that the support function $\rho(\ell \mid \mathcal{E}) = \sup\{(\ell, x) \mid x \in \mathcal{E}\}$ for an ellipsoid $\mathcal{E} = \mathcal{E}(a, Q)$ is

$$\rho(\ell \mid \mathcal{E}(a, Q)) = (\ell, a) + (\ell, Q\ell)^{1/2}.$$

With $\det Q \neq 0$, this is equivalent to the inequality

$$\mathcal{E}(a, Q) = \{x \in \mathcal{R}^n \mid (x - a)'Q^{-1}(x - a) \leq 1\}.$$

Therefore $a$ stands for the center of the ellipsoid and $Q > 0$ for the symmetric matrix that determines its configuration.

Suppose that two ellipsoids

$$\mathcal{E}_1 = \mathcal{E}(a_1, Q_1), \qquad \mathcal{E}_2 = \mathcal{E}(a_2, Q_2)$$

are given. The sum $\mathcal{E}(a_1, Q_1) + \mathcal{E}(a_2, Q_2)$ of these in general is not an ellipsoid, and the same is true for the geometrical difference $\mathcal{E}(a_1, Q_1) \dot{-} \mathcal{E}(a_2, Q_2)$.

We will indicate some parametrized varieties of ellipsoids that allow an *exact approximation* (both external and internal) for $\mathcal{E}_1 + \mathcal{E}_2$ and $\mathcal{E}_1 \dot{-} \mathcal{E}_2$. Consider an ellipsoid $\mathcal{E}(a_1 + a_2, Q^{1,2}(\pi))$,

$$Q^{1,2}(\pi) = (\pi_1 + \pi_2)(\pi_1^{-1}Q_1 + \pi_2^{-1}Q_2)$$

where $\pi \in \Pi_+$,

$$\Pi_+ = \{(\pi_1, \pi_2) \in \mathcal{R}^2 \mid \pi_1/\pi_2 \in [\lambda_{\min}^{1/2}, \lambda_{\max}^{1/2}]\}, \tag{12}$$

and $\lambda_{\min}, \lambda_{\max}$ are respectively the smallest and the largest solutions to the equation $\det(Q_1 - \lambda Q_2) = 0$. Consider in addition an ellipsoid

$$\mathcal{E}(a_1 + a_2, Q_+^{1,2}[S]),$$

with

$$Q_+^{1,2}[S] = S^{-1}[(SQ_1S)^{1/2} + (SQ_2S)^{1/2}]^2 S^{-1},$$

where $S \in \Sigma$, $\Sigma$ being the class of invertible (symmetric) matrices.

**Lemma 5.1** *The following inclusions are true*

$$\mathcal{E}(a_1 + a_2, Q_+^{1,2}[S]) \subset \mathcal{E}(a_1, Q_1) + \mathcal{E}(a_2 + Q_2) \subset \mathcal{E}(a_1 + a_2, Q^{1,2}(\pi))$$

*whatever is the parameter $\pi \in \Pi_+$ and the matrix $S \in \Sigma$.*

The given lemma allows to be amplified into

**Theorem 5.1** *The following equalities are true*

$$\overline{\bigcup\{\mathcal{E}((a_1 + a_2), Q_+^{1,2}[S]) | S \in \Sigma\}} = \mathcal{E}((a_1, Q_1) + \mathcal{E}((a_2, Q_2) = \bigcap\{\mathcal{E}(a_1 + a_2, Q^{1,2}(\pi)) | \pi \in \Pi_+\}$$

*where $\overline{\mathcal{K}}$ stands for the closure of set $\mathcal{K}$.*

The next step is to approximate the geometric differences. The important point is that for this sake we may again use the formulae of the above but with some changes in the signs of the parameters

Denote

$$\Pi_- = \{(\pi_1, -\pi_2) | (\pi_1, \pi_2) \in \Pi_+, \pi_1/\pi_2 < \lambda_{\min})\} \tag{13}$$

and

$$Q_-^{1,2}[S] = S^{-1}[(SQ_1 S)^{1/2} - (SQ_2 S)^{1/2}]^2 S^{-1}.$$

**Lemma 5.2** *Provided $\mathcal{E}(a_1, Q_1) \overset{\cdot}{-} \mathcal{E}(a_2, Q_2) \neq \emptyset$, the following inclusions are true:*

$$\mathcal{E}((a_1 - a_2, Q^{1,2}(\pi)) \subset \mathcal{E}(a_1, Q_1) \overset{\cdot}{-} \mathcal{E}(a_2, Q_2) \subset \mathcal{E}(a_1 - a_2, Q_-^{1,2}[S]),$$

*whatever are $\pi \in \Pi_-$, $S \in \Sigma$.*

Again, the latter proposition allows a stronger version, namely

**Theorem 5.2** *Under the condition $\mathcal{E}(a_1, Q_1) \overset{\cdot}{-} \mathcal{E}(a_2, Q_2) \neq \emptyset$ the following equalities are true*

$$\overline{\bigcup\{\mathcal{E}((a_1 - a_2, Q^{1,2}(\pi)) | \pi \in \Pi_-\}} \subset \mathcal{E}(a_1, Q_1) \overset{\cdot}{-} \mathcal{E}(a_2, Q_2) \subset \bigcap\{\mathcal{E}(a_1 - a_2, Q_+^{1,2}[S] | S \in \Sigma\}.$$

Theorems 5.1, 5.2 reflect a duality in the approximation of the (nonellipsoidal) sums and geometric differences of ellipsoids by intersections and unions of parametrized sets of type $\mathcal{E}(a, Q(\pi))$ and $\mathcal{E}(a, Q[S])$.

A further operation that follows from the discrete-time scheme (11) is to approximate the set

$$\mathcal{E} = [\mathcal{E}(a_1, Q_1) + \mathcal{E}(a_2, Q_2)] \overset{\cdot}{-} \mathcal{E}(a_3, Q_3) \tag{14}$$

for given three ellipsoids $\mathcal{E}(a_i, Q_i), i = 1, 2, 3$.

This can be done by combining the results of Theorems 5.1 and 5.2:

$$\mathcal{E}(a_1 + a_2, Q_+^{1,2}[S]) \overset{\cdot}{-} \mathcal{E}(a_3, Q_3) \subset \mathcal{E} \subset \mathcal{E}(a_1 + a_2, Q^{1,2}(\pi)) \overset{\cdot}{-} \mathcal{E}(a_3, Q_3)$$

$$\pi \in \Pi_+, S \in \Sigma,$$

and then, once more, to obtain

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$$\mathcal{E}(a_1 + a_2 - a_3, Q_{\text{int}}(S, q)) \subset \mathcal{E} \subset \mathcal{E}(a_1 + a_2 - a_3, Q_{\text{ext}}(\pi, S)), \tag{15}$$

where

$$Q_{\text{int}}(S, \nu) = (\nu_1 + \nu_2)(\nu_1^{-1} Q_+^{1,2}[S] + \nu_2^{-1} Q_3)$$

and

$$Q_{\text{ext}}(\pi, S) = S^{-1}[(S Q^{1,2}(\pi) S)^{1/2} - (S Q_3 S)^{1/2}]^2 (S)^{-1}$$

with $S \in \Sigma$, $\pi \in \Pi_+$ and $\nu = (\nu_1, \nu_2) \in \Pi_-(S)$. $\Pi_-(S)$ is constructed according to (13), using the substitutions

$$Q_1 = Q_+^{1,2}[S], \quad Q_2 = Q_3.$$

**Lemma 5.3** *The set $\mathcal{E} \neq \emptyset$ of (14) satisfies the relation (15) whatever are the parameters $\pi \in \Pi_+, S \in \Sigma, \nu \in \Pi_-(S)$.*

Due to Theorems 5.1, 5.2 we now come to

**Theorem 5.3** *The following equalities are true*

$$\overline{\bigcup \{\mathcal{E}(a_1 + a_2 - a_3, Q_{\text{int}}(S, \nu)) | \pi \in \Pi_+, S \in \Sigma, \nu \in \Pi_-(S)\}} =$$

$$= \mathcal{E} =$$

$$= \bigcap \{\mathcal{E}(a_1 + a_2 - a_3, Q_{\text{ext}}(\pi, S)) | \pi \in \Pi_+, S \in \Sigma\}.$$

Through the relations given in this paragraph the discrete time scheme (11) allows a limit transition to the continuous time case.

# 6 Ellipsoidal Techniques: Continuous Time

Returning to the equation

$$\dot{x}(t) \in u(t) - \mathcal{Q}(t), \quad u(t) \in \mathcal{P}(t), \quad t_0 \leq t \leq t_1, \qquad x(t_1) \in \mathcal{M}, \tag{16}$$

we assume that

$$\mathcal{P}(t) = \mathcal{E}(p(t), P(t)), \quad \mathcal{Q}(t) = \mathcal{E}(q(t), Q(t)), \qquad t_0 \leq t \leq t_1$$

and

$$\mathcal{M} = \mathcal{E}(m, M)$$

are ellipsoids.

The set $\mathcal{W}[t] = \mathcal{W}(t, \mathcal{E}(m, M))$ may now be approximated by ellipsoidal solutions $\mathcal{E}_+[t]$ and $\mathcal{E}_-[t]$. Namely if

$$\mathcal{W}[t] = \int_{t,\mathcal{M}}^{t_1} [\mathcal{E}(p(\tau), P(\tau))d\tau \doteq \mathcal{E}(q(\tau), Q(\tau))d\tau] \qquad t_0 \le t \le t_1 \tag{17}$$

or, in other words, if $\mathcal{W}[t]$ is the solution to the "funnel" equation

$$\lim_{\sigma \to +0} \sigma^{-1}h_+(\mathcal{W}[t - \sigma] - \sigma\mathcal{E}(q(t), Q(t)), \mathcal{W}[t] - \sigma\mathcal{E}(p(t), P(t))) = 0, \tag{18}$$

$$t_0 \le t \le t_1,$$

$$\mathcal{W}[t_1] = \mathcal{E}(m, M),$$

then its solution $\mathcal{W}[t]$ does exist but is not bound to be ellipsoidal-valued. Let us introduce two new funnel equations

$$\lim_{\sigma \to +0} \sigma^{-1}h_+(\mathcal{E}[t - \sigma] - \sigma\mathcal{E}(q(t), Q(t)), \mathcal{E}[t] - \sigma\mathcal{E}(p(t), P(t))) = 0, \tag{19}$$

$$t_0 \le t \le t_1,$$

$$\mathcal{E}[t_1] = \mathcal{E}(m, M),$$

and

$$\lim_{\sigma \to +0} \sigma^{-1}h_-(\mathcal{E}[t - \sigma] - \sigma\mathcal{E}(q(t), Q(t)), \mathcal{E}[t] - \sigma\mathcal{E}(p(t), P(t))) = 0, \tag{20}$$

$$t_0 \le t \le t_1,$$

$$\mathcal{E}[t_1] = \mathcal{E}(m, M).$$

A function $\mathcal{E}^+[t]$ will be defined as a solution to (19) if it satisfies (19) almost everywhere and is ellipsoidal-valued.

A function $\mathcal{E}^-[t]$ is defined as a solution to (20) if it

- satisfies (20) almost everywhere,

- is *ellipsoidal-valued*,

- is a *maximal* solution to (20).

The latter means that there exists no other ellipsoidal-valued solution $\mathcal{E}'[t]$ to (20) such that $\mathcal{E}^-[t] \subset \mathcal{E}'[t]$ and $\mathcal{E}^-[t] \ne \mathcal{E}'[t]$.

What follows are the properties of $\mathcal{E}^+[t]$ and $\mathcal{E}^-[t]$.

**Lemma 6.1** *The solution $\mathcal{E}^+[t]$ to (19) and $\mathcal{E}^-[t]$ to (20) do exist and are nonunique.*

**Lemma 6.2** *Whatever are the solutions $\mathcal{E}^+[t]$ to (19) and $\mathcal{E}^-[t]$ to (20), then for the maximal solution to (18), the following inclusions are true*

$$\mathcal{E}^-[t] \subset \mathcal{W}[t] \subset \mathcal{E}^+[t], \qquad t_0 \le t \le t_1. \tag{21}$$

We will now introduce two ellipsoidal-valued functions $\mathcal{E}_+[t] = \mathcal{E}(w(t), W_+(t))$ and $\mathcal{E}_-[t] = \mathcal{E}(w(t), W_-(t))$. Here

$$\dot{w}(t) = p(t) - q(t), \qquad t_0 \le t \le t_1, \tag{22}$$

$$w(t_1) = m$$

further

$$\dot{W}_+(t) = -\nu^{-1}(t)W_+(t) - \nu(t)P(t)+ \tag{23}$$

$$+S^{-1}(t)[S(t)W_+(t)S(t)]^{1/2}[S(t)Q(t)S(t)]^{1/2}S^{-1}(t)+$$

$$+S^{-1}(t)[S(t)Q(t)S(t)]^{1/2}[S(t)W_+(t)S(t)]^{1/2}S^{-1}(t),$$

$$t_0 \le t \le t_1,$$

$$W_+(t_1) = M,$$

and

$$\dot{W}_-(t) = \nu^{-1}(t)W_-(t) + \nu(t)Q(t)- \tag{24}$$

$$-S^{-1}(t)[S(t)W_-(t)S(t)]^{1/2}[S(t)P(t)S(t)]^{1/2}S^{-1}(t)-$$

$$-S^{-1}(t)[S(t)P(t)S(t)]^{1/2}[S(t)W_-(t)S(t)]^{1/2}S^{-1}(t),$$

$$t_0 \le t \le t_1,$$

$$W_-(t_1) = M,$$

For $t \in [t_0, t_1]$, let us denote by $\Pi_+(t)$ and $\Pi_-(t)$ the parameter sets of (12) and (13) constructed from

$$Q_1 = W_-(t), \quad Q_2 = Q(t)$$

and

$$Q_1 = W_+(t), \quad Q_2 = P(t),$$

respectively. The variable $t$ ranging in $[t_0, t_1]$, let then $\Pi_-^*(.)$ stand for the class of all continuous functions $\nu(t) = \nu_1(t)/\nu_2(t)$ with the pair $(\nu_1(t), \nu_2(t)) \in \Pi_-(t)$, $\Pi_+^*(.)$ for the class of all continuous functions $\nu(t) = \nu_1(t)/\nu_2(t)$ with the pair $(\nu_1(t), \nu_2(t)) \in \Pi_+(t)$, and $\Sigma(\cdot)$ for the class of all continuous, symmetrical invertible matrix valued functions $S(\cdot)$.

**Lemma 6.3** *For $S(\cdot) \in \Sigma(\cdot)$ and $\nu(\cdot) \in \Pi_-^*(\cdot)$, each of the ellipsoidal-valued functions $\mathcal{E}(w(t), W_+(t))$ is a solution to (19), further for $S(\cdot) \in \Sigma(\cdot)$ and $\nu(\cdot) \in \Pi_+^*(\cdot)$, $\mathcal{E}(w(t), W_-(t))$ to (19). In addition, the following inclusions hold:*

$$\mathcal{E}(w(t), W_-(t)) \subset \mathcal{W}(t) \subset \mathcal{E}(w(t), W_+(t)), \qquad t_0 \le t \le t_1.$$

And in stronger formulation:

**Theorem 6.1** *For $t_0 \leq t \leq t_1$, the following equalities are true*

$$\overline{\bigcup \{ \mathcal{E}(w(t), W_-(t)) | S(\cdot) \in \Sigma(\cdot), \nu(\cdot) \in \Pi_+^*(\cdot) \}} =$$

$$= \mathcal{W}[t] =$$

$$= \bigcap \{ \mathcal{E}(w(t), W_+(t)) | S(\cdot) \in \Sigma(\cdot), \nu(\cdot) \in \Pi_-^*(\cdot) \},$$

$$t_0 \leq t \leq t_1.$$

Theorem 6.1 indicates that the set-valued "alternated" integral $\mathcal{W}[t]$ allows an exact (internal and external) approximation by ellipsoidal-valued solutions to the evolution equations (19), (20).

# 7    The Approximate "Guaranteed" Strategies

The idea of constructing a synthesizing strategy $\mathcal{U}(t, x)$ for the problem of Section 1 was that $\mathcal{U}(t, x)$ should ensure that all the solutions $x(t, \tau, x_\tau)$ to the differential inclusion

$$\dot{x}(t) \in \mathcal{U}(t, x(t)) - \mathcal{E}(q(t), Q(t)), \quad \tau \leq t \leq t_1, \qquad x_\tau \in \mathcal{W}[\tau],$$

would satisfy the inclusion

$$x(t) \in \mathcal{W}[t], \qquad \tau \leq t \leq t_1,$$

and therefore ensure $x(t_1) \in \mathcal{M}$.

We will now substitute $\mathcal{W}[t]$ by one of its internal approximations $\mathcal{E}_-[t]$. The conjecture is that once $\mathcal{W}[t]$ is substituted by $\mathcal{E}_-[t]$ we should just follow the scheme of Section 3 constructing a strategy $\mathcal{U}_-(t, x)$ such that for every solution $x_-[t] = x_-(t, \tau, x_\tau)$ that satisfies the inclusion

$$\dot{x}(t) \in \mathcal{U}_-(t, x(t)) - \mathcal{E}(q(t), Q(t)) \quad \tau \leq t \leq t_1, \qquad x(\tau) \in \mathcal{E}_-[\tau], \qquad (25)$$

the following inclusion would be true

$$x(t) \in \mathcal{E}_-[t], \qquad \tau < t \leq t_1, \qquad (26)$$

and therefore

$$x(t) \in \mathcal{E}(m, M) = \mathcal{M}$$

would also hold.

The conjecture discussed here is obviously the same as in the absence of an unknown disturbance $v(t)$, or $\mathcal{Q}(t) \equiv \{0\}$, (see [4]), but the solution tube $\mathcal{W}[t]$ and its approximation $\mathcal{E}_-[t]$ are now defined in a far more complicated way, as seen in Sections 2 and 6.

It will be proven that once the approximation $\mathcal{E}_-[t]$ is selected according to Section 6, the strategy $\mathcal{U}(t,x)$ may be again defined due to the scheme of (9) except that $\mathcal{W}[t]$ will now be substituted by $\mathcal{E}_-[t]$ and that the respective relations will be given in a more explicit form, namely

$$\mathcal{U}_-(t,x) = \begin{cases} \mathcal{E}(p(t), P(t)) & \text{if } x \in \mathcal{E}_-[t] \\ p(t) - P(t)\ell^0(\ell^0, P(t)\ell^0)^{-1/2} & \text{if } x \notin \mathcal{E}_-[t], \end{cases} \tag{27}$$

where $\ell^0 = \partial_x d(x, \mathcal{E}_-[t])$ at point $x = x[t]$.

In order to prove that the ellipsoidal-valued strategy $\mathcal{U}_-(t,x)$ of (27) does solve the target problem in the form of a control synthesis, we have to follow the lines of Section 3 in [4].

We will not elaborate the proof in detail but merely underline that the main point is the calculation of the derivative of $d[\cdot]$, where

$$d[t] = d(x_-[t], \mathcal{E}_-[t]) = (\ell^0, x_-[t]) - \rho(\ell^0 \mid \mathcal{E}_-[t])$$

and $\mathcal{E}_-[t] = \mathcal{E}(w(t), W_-(t))$ according to (22), (24) with parametrization $S(\cdot) \in \Sigma(\cdot)$ being given.

By direct calculation, due to (22), (24), we come to

$$\frac{d}{dt} d[t] \leq (\ell^0, u(t)) - (\ell^0, v(t)) - (\ell^0, p(t)) + (\ell^0, P(t))\ell^0)^{1/2} + (\ell^0, q(t)) - (\ell^0, Q(t)\ell^0)^{1/2}$$

where

$$u(t) \in \mathcal{E}(p(t), P(t)), \qquad v(t) \in \mathcal{E}(q(t), Q(t)).$$

For $d[t] > 0, (x_-[t] \notin \mathcal{E}(w(t), W_-(t)))$, the above inequality gives us

$$\frac{d}{dt} d[t] \leq 0.$$

This contradicts with the possibility that a trajectory $x_-[t]$ of (25) would violate the inclusion $x_-[t] \in \mathcal{W}[t]$ — since otherwise there would exist an instant in $(t_0, t_1)$ when the derivative of $d(\cdot)$ is strictly positive.

What follows is the assertion

**Theorem 7.1** *Define an internal approximation $\mathcal{E}_-[t] = \mathcal{E}(w(t), W_-(t))$ of $\mathcal{W}[t]$ with parametrization $S(\cdot) \in \Sigma(\cdot)$. Once $x_\tau = x_-[\tau] \in \mathcal{E}_-[\tau]$ and $x(t, \tau, x_\tau) = x_-[t]$ is a solution to (25), the following relation is true*

$$x_-[t] \in \mathcal{E}_-[t], \qquad \tau < t \leq t_1,$$

*and therefore*

$$x_-[t_1] \in \mathcal{E}(m, M).$$

The "ellipsoidal" synthesis thus produces a solution strategy $\mathcal{U}_-(t,x)$ for any internal approximation $\mathcal{E}_-[t] = \mathcal{E}(w(t), W_-(t))$, driven by any $S(\cdot) \in \Sigma(\cdot)$. The strategy $\mathcal{U}_-(t,x)$ is *ellipsoidal-valued* and satisfies an existence theorem absolutely similar to Lemma 3.1. The differential inclusion (25) is thus correctly defined.

We will now proceed with numerical examples that demonstrate the constructive nature of the solutions obtained above.

# 8 Numerical Examples

Our particular intention first is to illustrate through simulation the effect of introducing an unknown but bounded disturbance $v(t)$ into the system. We do this by considering *a sequence of three problems* where only the size of the bounding sets of the disturbances varies from case to case, starting from no disturbance at all — that is where the sets $\mathcal{Q}(t) = \mathcal{E}(q(t), Q(t)), t \in [t_0, t_1]$ are singletons — to more disturbance allowed so that the problem still remains solvable. The result of this is that in the first case we obtain a "large" internal ellipsoidal estimate of the solvability set $\mathcal{W}[t_0] = \mathcal{W}(t_0, \mathcal{M})$, while in the last it shrinks to be "small". We also indicate the behaviour of isolated trajectories of system (2), in the presence of various given feasible disturbances $v(t) \in \mathcal{E}(q(t), Q(t))$.

For the calculations we use a discrete scheme corresponding to (22), (24), by dividing the time interval — chosen to be $[0, 5]$ — into 100 subintervals of equal lengths. Instead of the set valued control strategy (27) we apply a *single valued* selection:

$$u(t, x) = \begin{cases} p(t) & \text{if } x \in \mathcal{E}_-[t] \\ p(t) - P(t)\ell^0(\ell^0, P(t)\ell^0)^{-1/2} & \text{if } x \notin \mathcal{E}_-[t]. \end{cases} \tag{28}$$

again in its discrete version.

We calculate the parameters of the ellipsoid $\mathcal{E}_-[t] = \mathcal{E}(w(t), W_-(t))$ by chosing the parametrization

$$S(t) = P^{-1/2}(t)$$

and

$$\nu(t) = \frac{\mathrm{Tr}^{1/2}(W_-(t))}{\mathrm{Tr}^{1/2}(Q(t))}$$

in (24). We consider a 4 dimensional system with the initial position $\{0, x_0\}$ given by

$$x_0 = \begin{pmatrix} 2 \\ -10 \\ 1 \\ -6 \end{pmatrix},$$

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at the initial moment $t_0 = 0$ and target set $\mathcal{M} = \mathcal{E}(m, M)$ defined by

$$m = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 10 \end{pmatrix}$$

and

$$M \equiv \begin{pmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 100 \end{pmatrix}$$

at the final moment $t_1 = 5$. We suppose the right hand side to be constant:

$$A(t) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{pmatrix},$$

describing the position and velocity of two independent oscillators. (Through the constraints on the control and disturbance, however, the system becomes coupled.)

The restriction $u(t) \in \mathcal{E}(p(t), P(t))$ on the control and $v(t) \in \mathcal{E}(q(t), Q(t))$ on the disturbance is also defined by time independent constraints:

$$p(t) \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$P(t) \equiv \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The center of the disturbance is the same in all cases:

$$q(t) \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The difference between the three cases $i = 1, 2, 3$ appear in the matrices:

$$Q^{(1)}(t) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q^{(2)}(t) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix},$$

$$Q^{(3)}(t) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 13.1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 13.1 \end{pmatrix}.$$

Clearly, case $i = 1$ is the one treated in [4], but note that in the cases $i = 2, 3$ the data are chosen in such a way that neither the controls, nor the disturbances dominate the other, that is, both $\mathcal{P} \stackrel{\cdot}{-} \mathcal{Q}$ and $\mathcal{Q} \stackrel{\cdot}{-} \mathcal{P}$ are *empty*. Obviously, in these cases the problem can not be reduced to simpler situations without disturbances.

The calculations give the following internal ellipsoidal estimate $\mathcal{E}_{-}^{(i)}[0] = \mathcal{E}(w(0), W_{-}^{(i)}(0))$ of the solvability set $\mathcal{W}^{(i)}(0, \mathcal{M})$, $i = 1, 2, 3$:

$$w(0) = \begin{pmatrix} 2.4685 \\ -8.4742 \\ 1.5685 \\ -5.2087 \end{pmatrix},$$

and

$$W_{-}^{(1)}(0) = \begin{pmatrix} 323.9377 & 30.2735 & 0 & 0 \\ 30.2735 & 341.4382 & 0 & 0 \\ 0 & 0 & 147.0094 & 61.1077 \\ 0 & 0 & 61.1077 & 469.5488 \end{pmatrix},$$

$$W_{-}^{(2)}(0) = \begin{pmatrix} 46.3661 & 25.5502 & 0 & 0 \\ 25.5502 & 66.4791 & 0 & 0 \\ 0 & 0 & 45.3047 & 28.3397 \\ 0 & 0 & 28.3397 & 132.7509 \end{pmatrix},$$

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$$W_-^{(3)}(0) = \begin{pmatrix} 12.2863 & 21.2197 & 0 & 0 \\ 21.2197 & 37.8930 & 0 & 0 \\ 0 & 0 & 33.6241 & 22.3911 \\ 0 & 0 & 22.3911 & 98.7732 \end{pmatrix}.$$

Now, as is easy to check, $x_0 \in \mathcal{E}(w(0), W_-^{(i)}(0))$ for $i = 1, 2, 3$ and therefore Theorem 7.1 is applicable, implying that the control strategy of (27) steers the solution of (25) into $\mathcal{M}$ under any admissible disturbance $v(t) \in \mathcal{E}(q(t), Q^{(i)}(t))$ in all three cases. Also, as it can be proved on the basis of their construction, we have the inclusions

$$\mathcal{E}(w(0), W_-^{(3)}(0)) \subset \mathcal{E}(w(0), W_-^{(2)}(0)) \subset \mathcal{E}(w(0), W_-^{(1)}(0))$$

holding, analogously to the corresponding inclusions between the original (nonellipsoidal) solvability sets $\mathcal{W}^{(i)}(0, \mathcal{M})$.

As the ellipsoids appearing in this problem are *four dimensional*, we present their *two dimensional projections*. The figures are divided into four *windows*, and each shows projections of the original ellipsoids onto the planes spanned by the first and second, third and fourth, first and third, and second and fourth coordinate axes, in a clockwise order starting from bottom left. The drawn segments of coordinate axes corresponding to the state variables range from $-30$ to $30$. The skew axis in Figures 1,2,3 is time, ranging from 0 to 5.

Figures 1,2,3 show the graph of the ellipsoidal valued maps $\mathcal{E}_-^{(i)}[t]$, $t \in [0, 5]$, $i = 1, 2, 3$, respectively, and of the solutions of

$$\dot{x}[t] = A(t)x[t] + u(t, x[t]) - v(t), \quad t_0 \le t \le t_1, \qquad x[0] = x_0 \tag{29}$$

where $u(t, x)$ is defined by (28), with three different choices of the disturbance $v(t)$, one being $v(t) \equiv 0$ and two other — so called extremal bang-bang type — feasible disturbances. The construction of these disturbances is the following. The time interval is divided into subintervals of constant lengths. A value $v$ is chosen randomly at the boundary of $\mathcal{E}(q(t), Q^{(i)}(t))$ and the disturbance is then defined by

$$v(t) = v$$

over all the first interval and

$$v(t) = -v$$

over the second. Then a new value for $v$ is selected and the above procedure is repeated for the next pair of intervals, etc.

The controlled trajectory, that is the solution to (28), (29), is drawn in a thin line if it is inside the current ellipsoidal solvability set, and by a thick line if it it outside. So the statement of Theorem of 7.1 is that the control ensures that a thin line cannot change into thick.
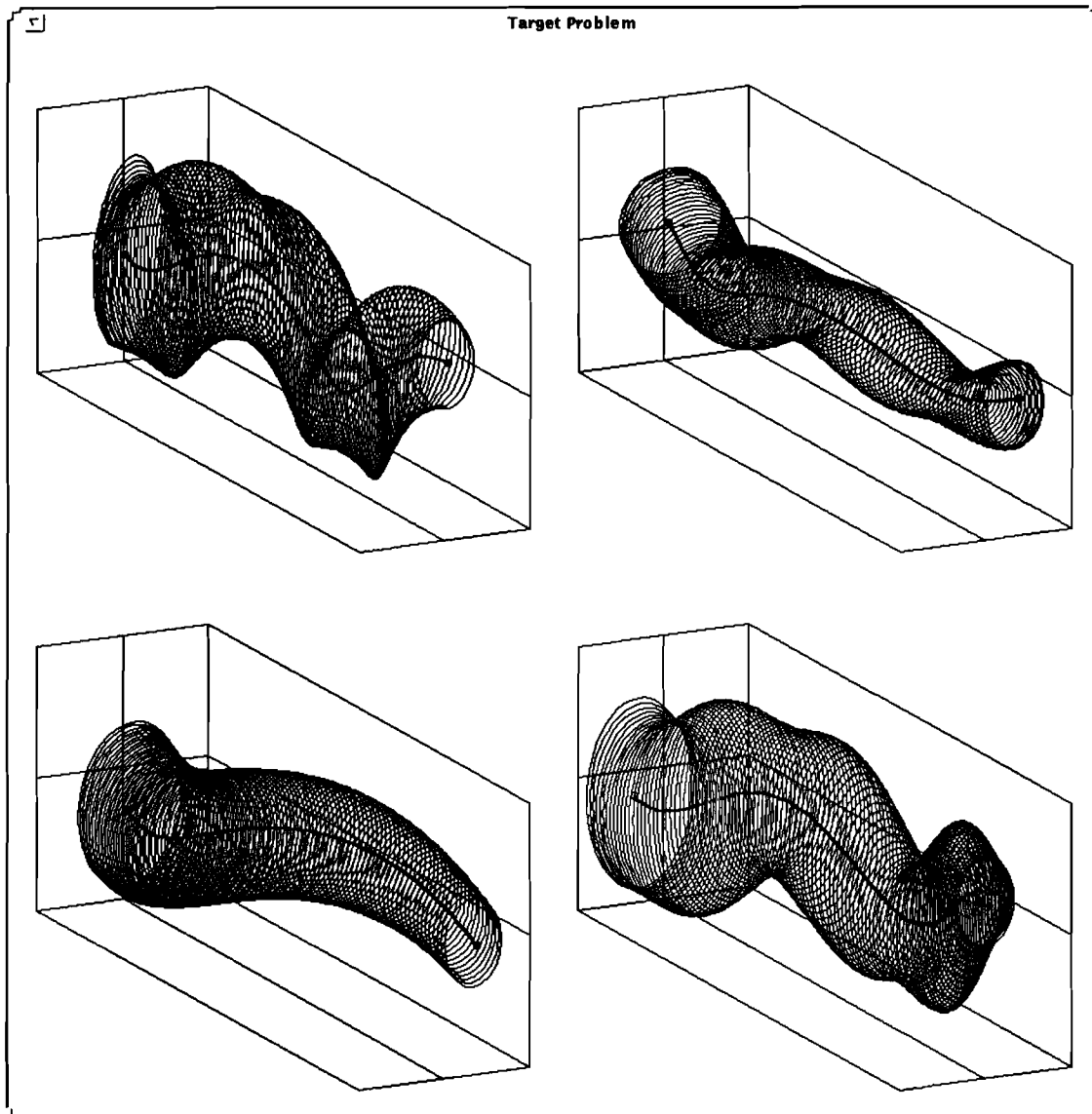
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Figure 1: Tube of ellipsoidal solvability sets and graph of solution, $(i = 1)$.
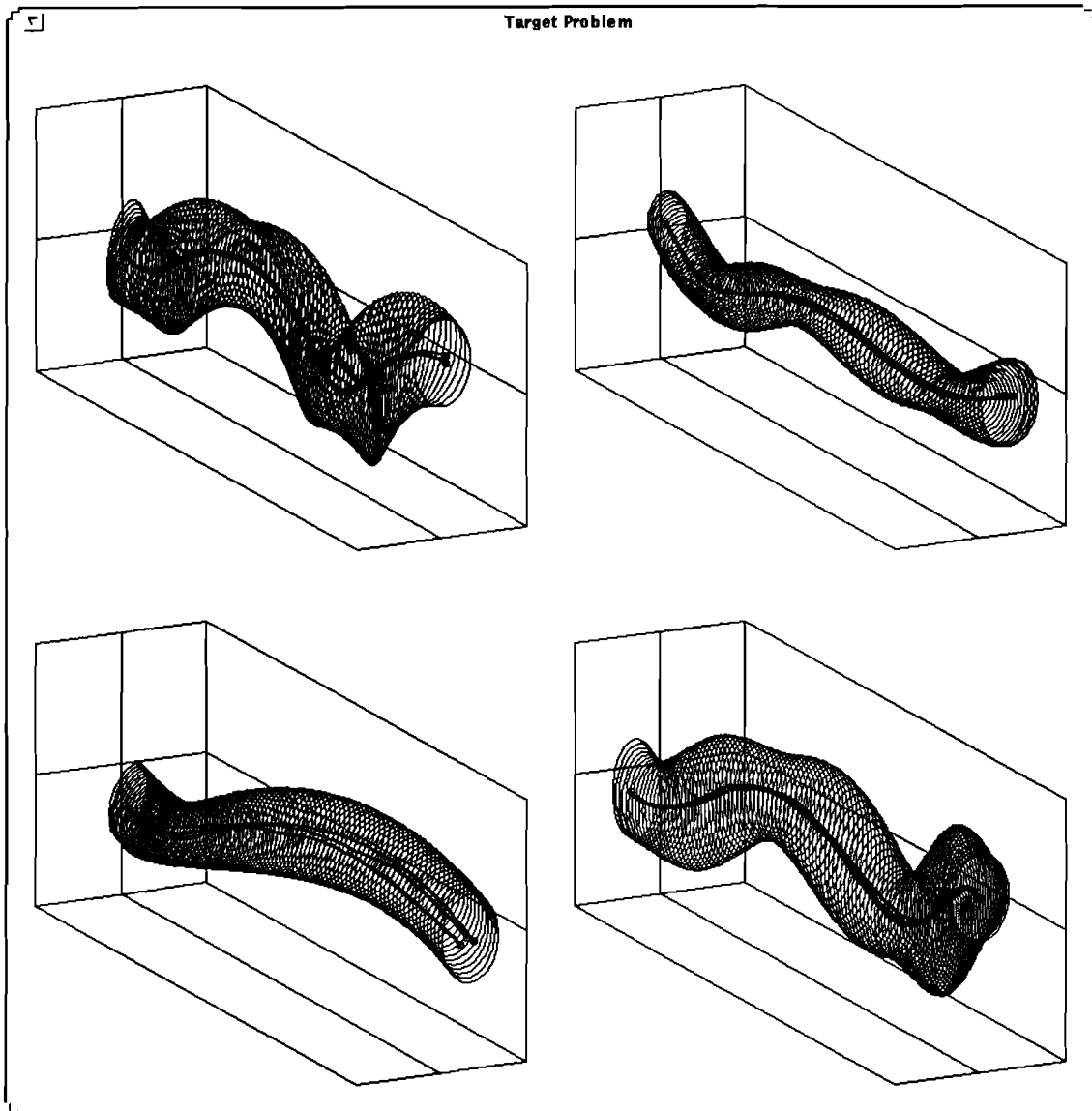
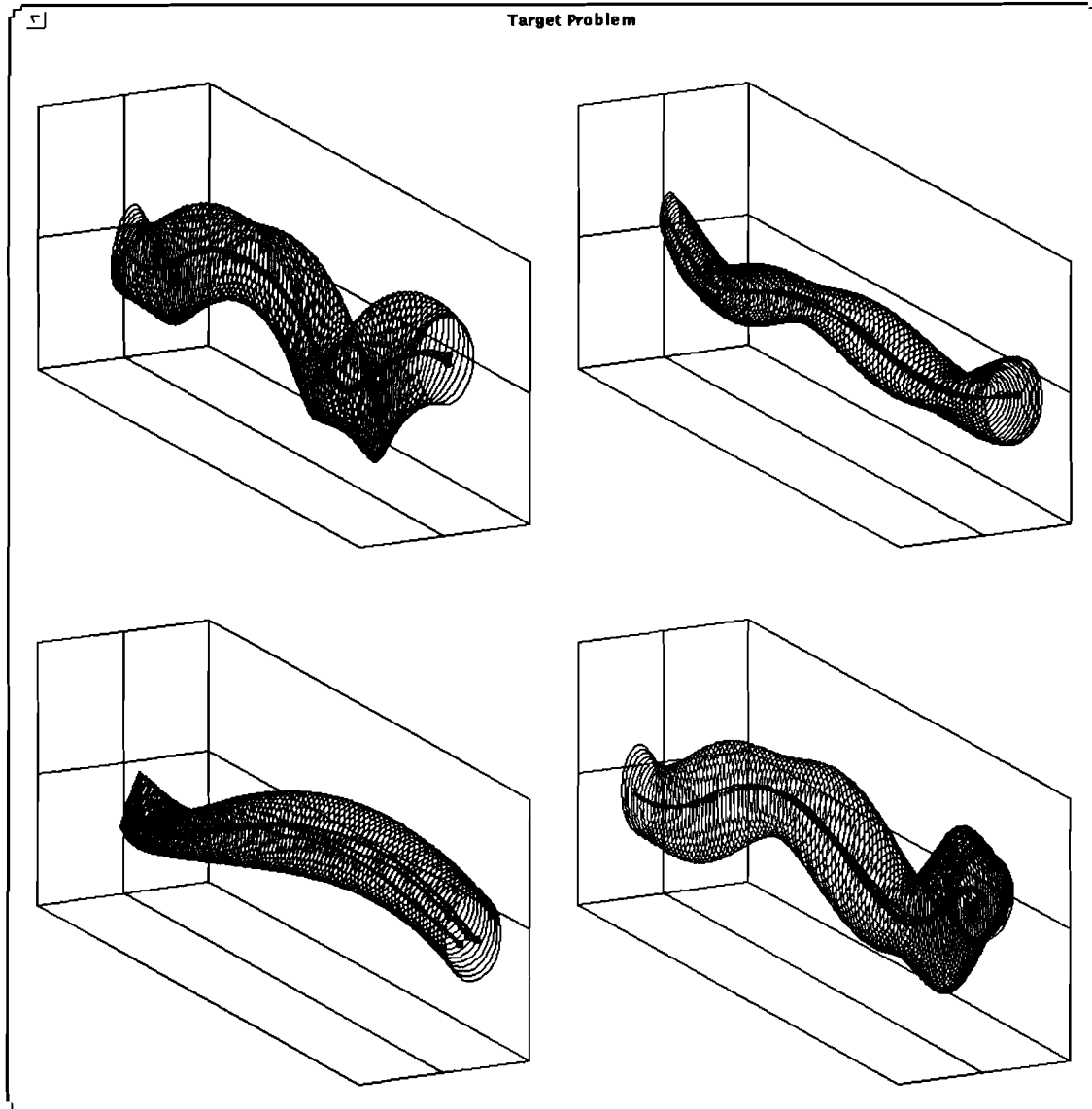Figure 2: Tube of ellipsoidal solvability sets and graphs of solutions, $(i = 2)$.

Figure 3: Tube of ellipsoidal solvability sets and graphs of solutions, $(i = 3)$.

Figures 4,5,6 show the target set $\mathcal{M} = \mathcal{E}(m, M)$, (projections appearing as circles of radius 10), the solvability set $\mathcal{E}_{-}^{(i)}[0] = \mathcal{E}(w(0), W_{-}^{(i)}(0))$ at $t = 0$, and trajectories of the same solutions of (28), (29) in phase space.
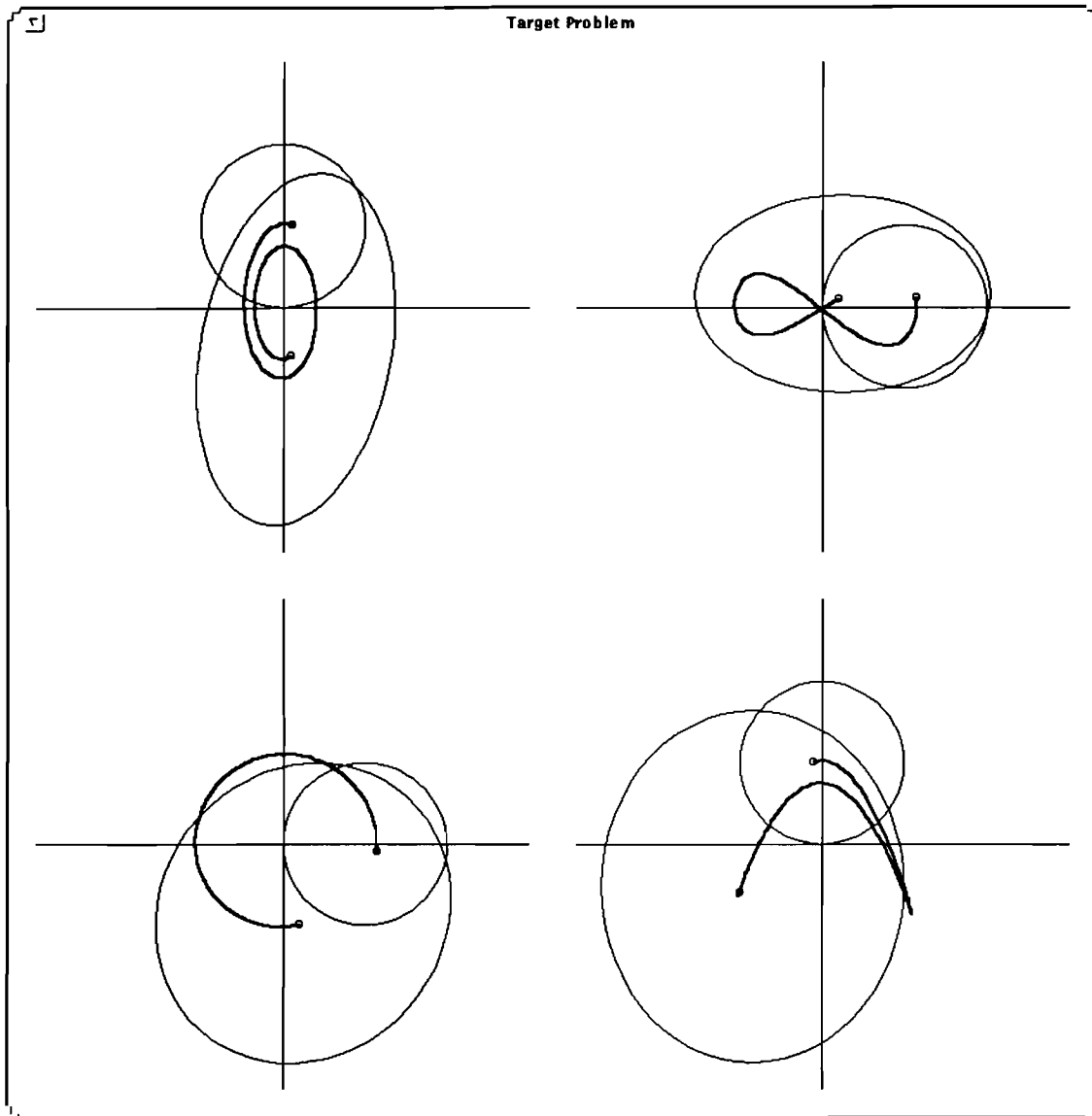
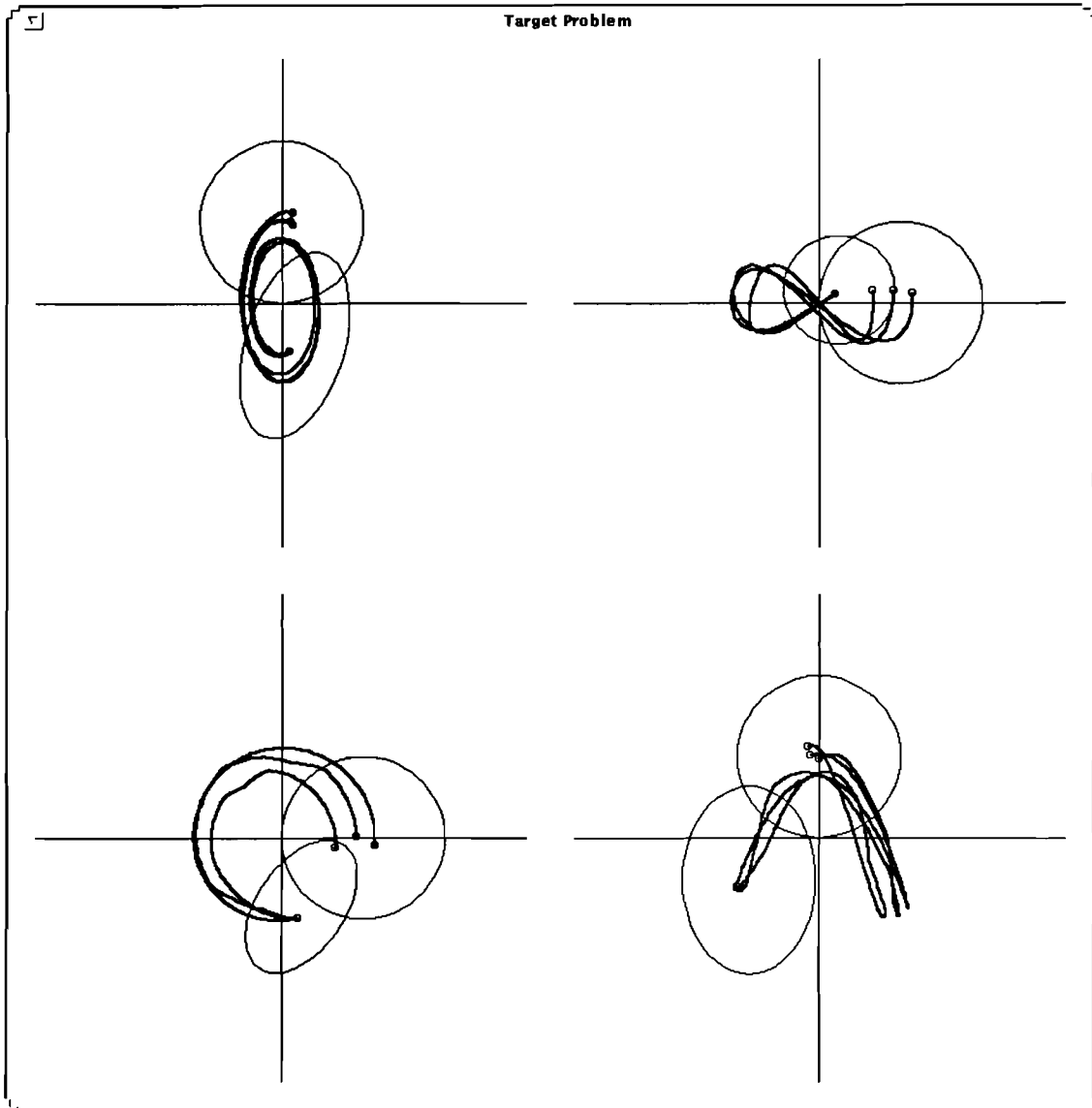Figure 4: Target set, initial ellipsoidal solvability set and trajectory in phase space, ($i = 1$).

Figure 5: Target set, initial ellipsoidal solvability set and trajectories in phase space, ($i = 2$).
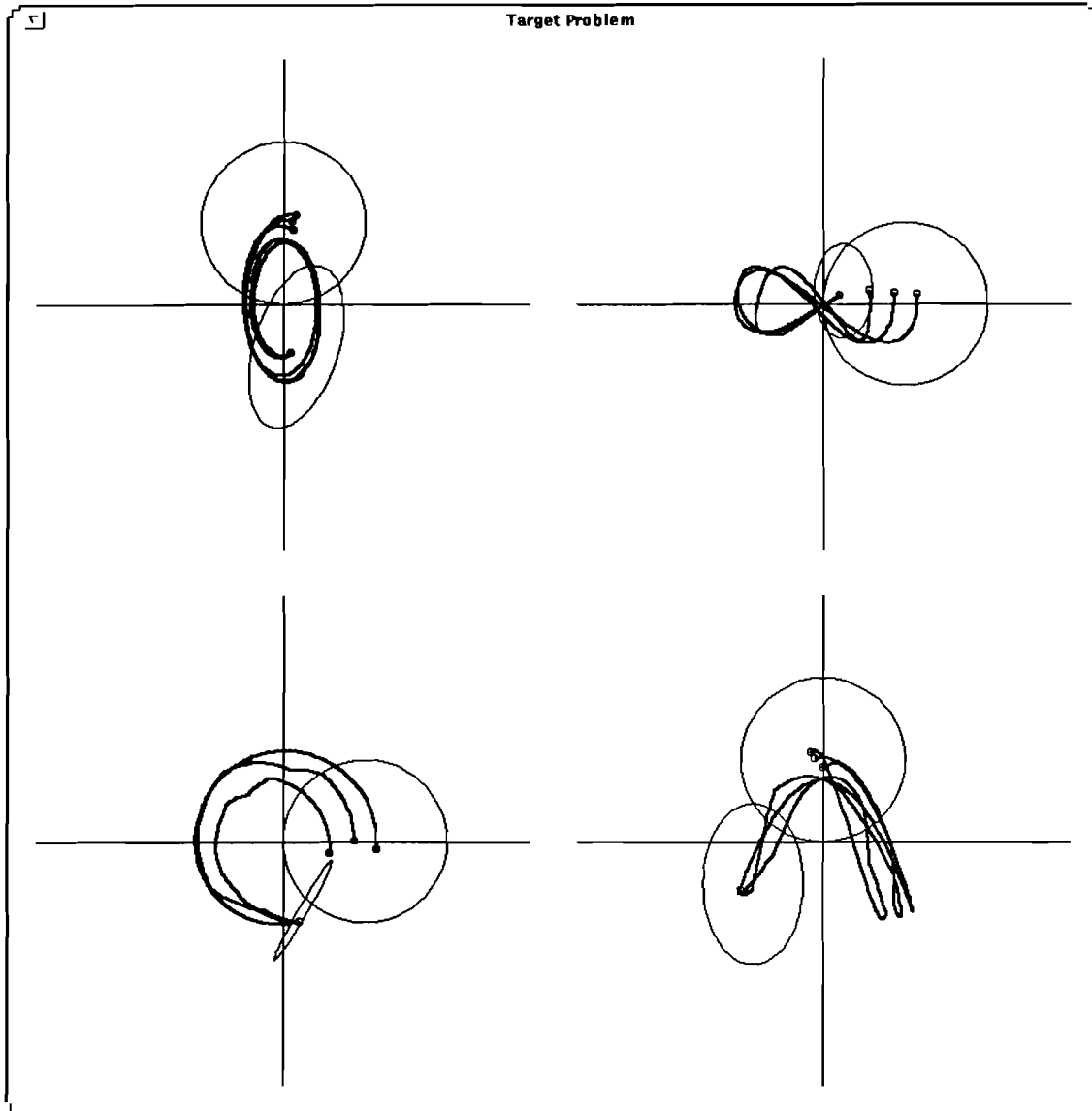
Figure 6: Target set, initial ellipsoidal solvability set and trajectories in phase space, $(i = 3)$.

The ellipsoids $\mathcal{E}_-[0]$ are only subsets of the respective solvability sets $\mathcal{W}(0, \mathcal{M})$, therefore Theorem 7.1 does not and can not make a negative statement, like if the initial state is not contained in $\mathcal{E}_-[t_0]$ then it is not true that the trajectory can be steered into the target set $\mathcal{M}$ under any disturbance $v(t) \in \mathcal{Q}(t)$. However, if the ellipsoidal approximation $\mathcal{E}_-[0] \subset \mathcal{W}(0, \mathcal{M})$ is good enough, then it may occur that such a behaviour can be illustrated on the ellipsoidal approximations.

To show this, we return to the parameter values of the previous examples and change the initial state only, by moving it in such a way that

$$x_0 \in \mathcal{E}_-^{(1)}[0] \setminus \mathcal{E}_-^{(2)}[0] \tag{30}$$
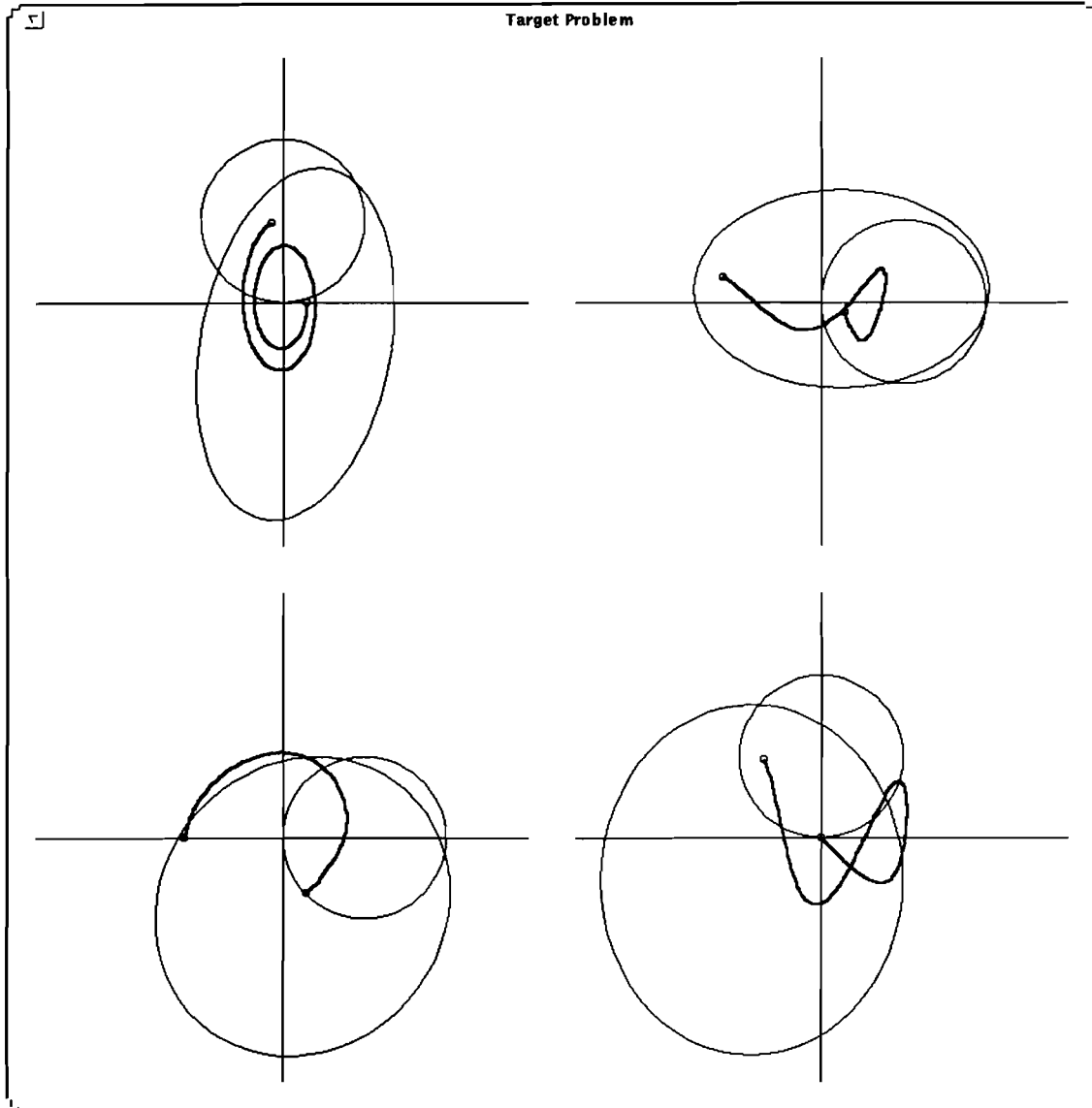
holds, taking

Figure 7: Initial state $x_0$ moved near to the boundary of $\mathcal{E}_-^{(1)}[0]$, outside of $\mathcal{E}_-^{(2)}[0]$, case $i = 1$.

$$x_0 = \begin{pmatrix} -12 \\ 0 \\ 3 \\ 0 \end{pmatrix}.$$

In Figures 7 and 8 it can be seen that relation (30) holds indeed. The trajectory in Figure 7 successfully hits the target set $\mathcal{M}$ at $t = 5$. (This is case $i = 1$, so there is no disturbance.)
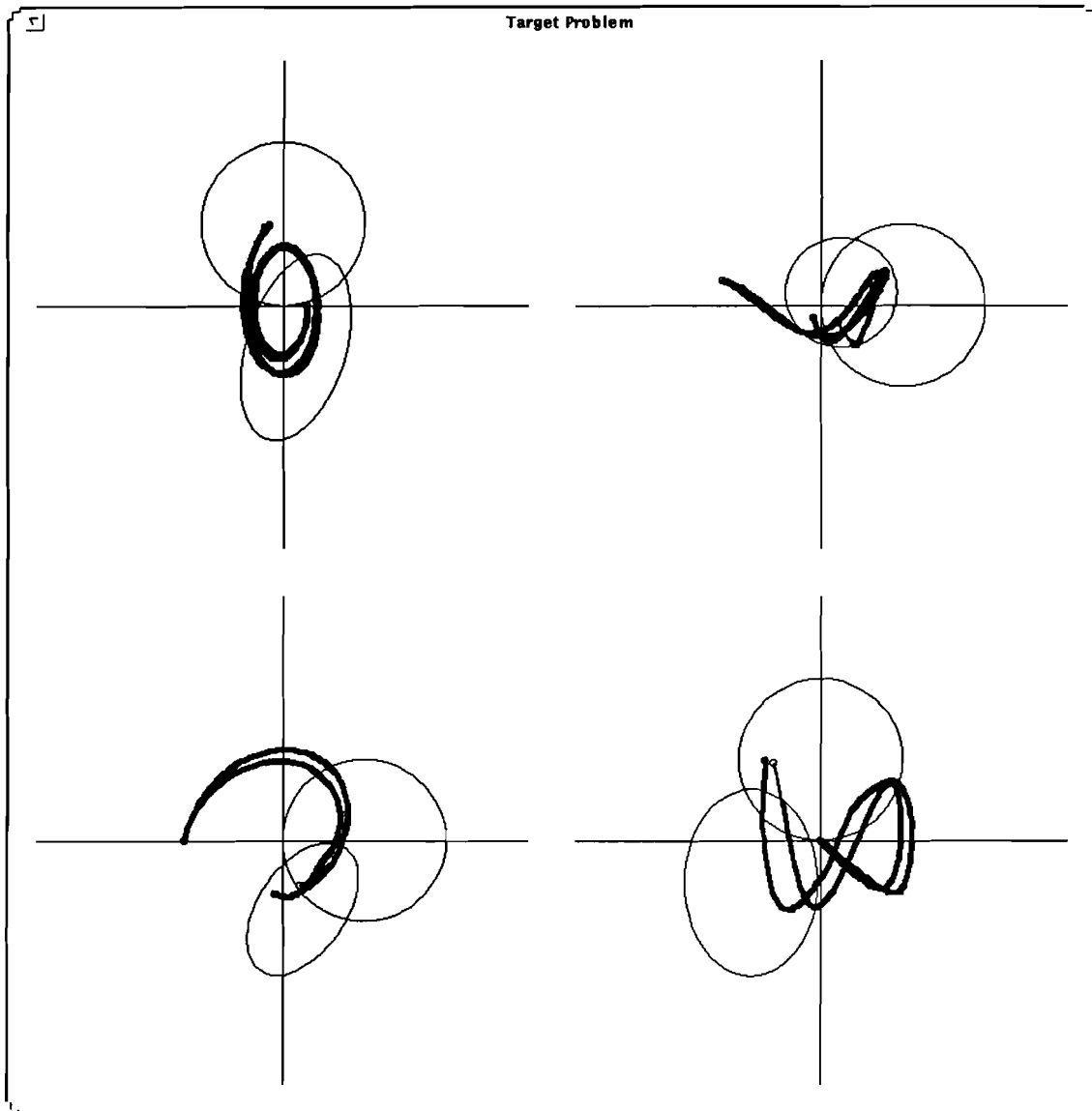
Figure 8: Initial state $x_0$ moved near to the boundary of $\mathcal{E}_-^{(1)}[0]$, outside of $\mathcal{E}_-^{(2)}[0]$, case $i = 2$.

Figure 8 shows two trajectories under two simulated feasible disturbances $v(t) \in \mathcal{E}(q(t), Q(t))$. In one case the control rule defined using the ellipsoidal tube $\mathcal{E}_-^{(2)}[t]$ steers the trajectory into the target $\mathcal{M}$, while under the other disturbance, it does not succeed. (One thick trajectory changing into thin is clearly seen in the right hand side windows, and the projection of the endpoint of the other is outside in the lower left window.) See also the examples in the preceeding paper, [4].

# References

[1] Krasovski, N.N. *The Control of a Dynamic System,* Nauka, Moscow, 1986. (In Russian.)

[2] Pontriagin, L. S. *On Linear Differential Games, I., II.,* Soviet Mathematical Doklady, Vol. 174 No. 6, Vol 175 No. 4, 1967.

[3] Kurzhanski, A.B., Nikonov, O. I. *Funnel Equations and Multivalued Integration Problems for Control Synthesis,* in: B. Jakubczyk, K. Malanowski, W. Respondek Eds. Perspectives in Control Theory, Progress in Systems and Control Theory, Vol. 2, Birkhäuser, Boston, 1990. pp. 143-153.

[4] Kurzhanski, A.B., Vályi, I. *Ellipsoidal Techniques: the Problems of Control Synthesis,* WP-91-003, International Institute for Applied Systems Analysis, Laxenburg, Austria, 1991.

[5] Krasovski, N. N., Subbotin, A. I. *Positional Differential Games,* Nauka, Moscow, 1974. (In Russian.)

[6] Leitman, G. *Feedback and Adaptive Control for Uncertain Dynamical Systems,* in: D. F. Batten, P. F. Lesse Eds. New Mathematical Advances in Economic Dynamics, Croom Helm, London Sydney, 1985. pp. 15-36.

[7] Skowronski J. M. *A Competitive Differential Game of Harvesting Uncertain Resources,* in: D. F. Batten, P. F. Lesse Eds. New Mathematical Advances in Economic Dynamics, Croom Helm, London Sydney, 1985. pp. 105-118.