# Working Paper

## Ellipsoidal Techniques: The Problem of Control Synthesis

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# Foreword

This is the first of a series of papers giving an early account of the application of ellipsoidal techniques to various problems in modeling dynamical systems. The problem of control synthesis for a linear system under bounded controls was selected as the first simple application of these techniques. The second paper extends these results to the case where unknown but bounded disturbances are present. The third deals with guaranteed state estimation – also to be interpreted as a tracking problem – again under unknown but bounded disturbances.

# Contents

# Ellipsoidal Techniques: the Problem of Control Synthesis

*A.B. Kurzhanski, I. Vályi*

## Introduction

This paper introduces a technique for solving the problem of control synthesis with constraints on the controls. Although the problem is treated here for linear systems only, the synthsized system is driven by a nonlinear control strategy and is therefore generically nonlinear. Taking a scheme based on the notion of extremal aiming strategies of N. N. Krasovski, the present paper concentrates on constructive solutions generated through ellipsoidal-valued calculus and related approximation techniques for set-valued maps. Namely, the primary problem which originally requires an application of set-valued analysis is substituted by one which is based on ellipsoidal-valued functions. This yields constructive schemes applicable to algorithmic procedures and simulation with computer graphics.

## 1 The Problem of Control Synthesis

Consider a control system

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t), u(t) \in \mathcal{R}^n, \qquad t_0 \le t \le t_1, \tag{1}$$

with controls $u$ being subjected to a constraint

$$u(t) \in \mathcal{P}(t), \qquad t_0 \le t \le t_1,$$

where $\mathcal{P}(t)$ is a continuous set-valued function with values $\mathcal{P}(t) \in \mathrm{conv}\mathcal{R}^n$ (the set of all convex compact subsets of $\mathcal{R}^n$). The function $f(t, x, u)$ is such that the respective set-valued map

$$\mathcal{F}(t, x) = \{\cup f(t, x, u) | u \in \mathcal{P}(t)\}$$

is continuous in $t$ and upper-semicontinuous in $x$. Let $\mathcal{M} \in \mathrm{conv}\mathcal{R}^n$ be a given set. The problem of control synthesis will consist in specifying a set-valued function $\mathcal{U} = \mathcal{U}(t, x)$, $(\mathcal{U}(t, x) \subset \mathcal{P}(t))$ – "the synthesizing control strategy" – which would ensure that all the solutions $x(t, \tau, x_\tau) = x[t]$ to the equation

$$\dot{x}(t) \in f(t, x(t), \mathcal{U}(t, x(t))), \qquad t_0 \le t \le t_1, \tag{2}$$

that start at some given *position* $\{\tau, x_\tau\}$, ($\tau \in [t_0, t_1]$, $x_\tau = x(\tau)$), would reach the terminal set $\mathcal{M}$ at the given instant of time $t = t_1$ - provided $x_\tau \in \mathcal{W}(\tau, \mathcal{M})$, where the *solvability set* $\mathcal{W}(\tau, \mathcal{M})$ is the set of states from which the solution to the problem does exist at all. Here we kept the notation $f$ for the set-valued function defined as $f(t, x, \mathcal{U}) = \{\cup f(t, x, u) | u \in \mathcal{U}\}$.

We presume

$$\mathcal{W}(\tau, \mathcal{M}) \neq \phi, \qquad t_0 \leq t \leq t_1,$$

The strategy $\mathcal{U}(t, x)$ must belong to a class $\Upsilon$ of *feasible feedback strategies*, which would ensure that the synthesized system (a *differential inclusion*) does have a solution defined throughout the interval $[t_0, t_1]$.

We now recall a technique that allows to determine $\mathcal{U}(t, x)$ once the problem satisfies some preassigned conditions that will be listed below.

For a given instant $\tau \in [t_0, t_1)$ consider the "largest" set $\mathcal{W}(\tau, \mathcal{M})$ of states $x(\tau) = x_\tau$ from which the problem of control synthesis is resolvable in a given class $\Upsilon$. Having defined $\mathcal{W}(\tau, \mathcal{M})$ for any instant $\tau$, we come to a set-valued function

$$\mathcal{W}[\tau] = \mathcal{W}(\tau, \mathcal{M}), \quad t_0 \leq \tau \leq t_1; \quad \mathcal{W}[t_1] = \mathcal{M}.$$

The following simplest conditions, [2], ensure that the function $\mathcal{W}[\tau]$ is convex compact valued and continuous in $t$.

**Lemma 1.1** *Assume that the set-valued mapping $\mathcal{F}(t, x)$ is upper semicontinuous in $x$ for all $t$, continuous in $t$, with $\mathcal{F}(t, x) \in \text{conv}\mathcal{R}^n$ and*

$$\|\mathcal{F}(t, x)\| \leq k \cdot h(t), \qquad t_0 \leq t \leq t_1,$$

*for some $k > 0$ and $h(t)$ integrable on $[t_0, t_1]$. Also assume that the graph*

$$\text{gr}\, \mathcal{F} = \{(t, x) | t \in [t_0, t_1], x \in \mathcal{F}(t, x)\}$$

*of the mapping $\mathcal{F}(t, x)$ is convex.*

*Then the set $\mathcal{W}[t] \in \text{conv}\mathcal{R}^n$ for $t \in [t_0, t_1]$ and the function $\mathcal{W}[t]$ is continuous in $t$.*

We further assume that $\mathcal{W}[\tau] \in \text{conv}\mathcal{R}^n$.

*The Synthesizing Strategy* is defined then as the following set-valued map

$$\mathcal{U}(t, x) = \begin{cases} \mathcal{P}(t) & \text{if} \quad x \in \mathcal{W}[t] \\ \{u \mid f(t, x, u) = \partial_\ell \rho(-\ell^0 \mid \mathcal{F}(t, x))\} & \text{if} \quad x \notin \mathcal{W}[t]. \end{cases} \tag{3}$$

Here $\ell^0 = \ell^0(t, x)$ is a unit vector that resolves the problem

$$(\ell^0, x) - \rho(\ell^0 \mid \mathcal{W}[t]) = \max \{ (\ell, x) - \rho(\ell \mid \mathcal{W}[t]) \mid \|\ell\| \leq 1 \},$$

where symbol $\rho(\ell \mid \mathcal{W}) = \max\{(\ell, x) | x \in \mathcal{W}\}$ stands for the *support function* of set $\mathcal{W}$ and $\partial_\ell g(\ell, t)$ denotes the *subdifferential* of $g(\ell, t)$ in the variable $\ell$.

Strategy $\mathcal{U}(t, x)$ reflects the rule of "extremal aiming" introduced by N.N. Krasovski [1]. Particularly, it indicates that with $x \notin \mathcal{W}[t]$ one has to choose the unit vector $-\ell^0$ that is directed from $x$ to $s^0$, namely $-\ell^0 = (s^0 - x)\|s^0 - x\|^{-1}$, where $s^0$ is the *metric projection* of $x$ onto $\mathcal{W}[t]$. After that, $\mathcal{U}(t, x)$ is defined as the set of points $u^0 \in \mathcal{P}(t)$ each of which satisfies the "maximum" condition:

$$(-\ell^0, f(t, x, u^0)) = \max\{(-\ell^0, f(t, x, u)) | u \in \mathcal{P}(t)\}, \tag{4}$$

so that $\mathcal{U}(t, x) = \{u^0\}$. The latter procedures are summarized in (3).

**Lemma 1.2** *Once the conditions of Lemma 1.1 are satisfied and the system (1) is linear in* $u$*, the following assertions are true:*

*(i) The set-valued map* $\mathcal{U}(t, x)$ *is convex compact-valued, continuous in* $t$ *and upper semicontinuous in* $x$*. This secures the existence of solutions to the differential inclusion*

$$\dot{x}(t) \in f(t, x(t), \mathcal{U}(t, x(t))) \qquad t_0 \leq t \leq t_1.$$

*(ii) If* $x_\tau \in \mathcal{W}[\tau]$*, for a given* $\tau \in [t_0, t_1)$*, then any solution* $x[t]$ *to the system*

$$\dot{x}(t) \in f(t, x(t), \mathcal{U}(t, x(t))), \quad \tau \leq t \leq t_1, \quad x(\tau) = x_\tau,$$

*satisfies the inclusion* $x[t] \in \mathcal{W}[t], \quad \tau \leq t \leq t_1$*, in particular,*

$$x[t_1] \in \mathcal{W}[t_1] = \mathcal{M}.$$

It is obvious that the crucial element for constructing the synthesized control strategy $\mathcal{U}(t, x)$ is the set-valued function $\mathcal{W}[t]$. It is therefore important to define an *evolution equation* for $\mathcal{W}[t]$, [2].

**Lemma 1.3** *Under the conditions of Lemma 1.1 the set-valued function* $\mathcal{W}[t]$ *satisfies the evolution equation*

$$\lim_{\sigma \to +0} h(\mathcal{W}[t - \sigma], \cup\{(x - \sigma\mathcal{F}(t, x)) | x \in \mathcal{W}[t]\}) = 0, \quad t_0 \leq t \leq t_1 \tag{5}$$

*with boundary condition*

$$\mathcal{W}[t_1] = \mathcal{M}.$$

Here $h(\mathcal{W}', \mathcal{W}'')$ is the *Hausdorff distance* between $\mathcal{W}', \mathcal{W}''$. Namely,

$$h(\mathcal{W}', \mathcal{W}'') = \max\{h_+(\mathcal{W}', \mathcal{W}''), h_-(\mathcal{W}', \mathcal{W}'')\}$$

where

$$h_+(\mathcal{W}', \mathcal{W}'') = \min\{r \geq 0 | \mathcal{W}' \subset \mathcal{W}'' + r\mathcal{S}\},$$

$h_-(\mathcal{W}', \mathcal{W}'') = h_+(\mathcal{W}'', \mathcal{W}')$ are the Hausdorff semidistances and $\mathcal{S}$ is the unit ball in $\mathcal{R}^n$.)

The conditions of Lemmas 1.1 and 1.2 are clearly satisfied for a linear system

$$\dot{x}(t) = A(t)x(t) + u(t), \quad u(t) \in \mathcal{P}(t), \qquad t_0 \leq t \leq t_1,. \tag{6}$$

The evolution equation (5) for determining $\mathcal{W}[t]$ then turns to be as follows

$$\lim_{\sigma \to +0} \sigma^{-1} h(\mathcal{W}[t-\sigma], (I - A(t)\sigma)\mathcal{W}[t] - \sigma\mathcal{P}(t)) = 0, \tag{7}$$

$$t_0 \leq t \leq t_1,$$

(here $I$ is the unit matrix), and

$$\mathcal{W}[t_1] = \mathcal{M}. \tag{8}$$

The aim of this paper is to demonstrate that this theory could be converted into constructive relations that allow algorithmization and online computer simulation. This could be achieved by introducing a calculus for ellipsoidal-valued functions that would serve to approximate the set-valued functions of the theory of the above, (also see [3], §§ 10-12).

It is important to observe that the relations given in the sequel do allow an exact approximation of the solution to the primary problem through ellipsoidal approximations.

We will further concentrate on the linear system (6). By substituting $z(t) = S(t, t_1)x(t)$ and returning to the old notation, without any loss of generality it could be transformed into

$$\dot{x}(t) = u(t), \quad u(t) \in \mathcal{P}(t), \quad t_0 \leq t \leq t_1, \quad x(t_1) \in \mathcal{M}, \tag{9}$$

where $x \in \mathcal{R}^n, \mathcal{P}(t), \mathcal{M} \in \text{conv}\mathcal{R}^n$, the function $\mathcal{P}(t)$ is continuous in $t$ and the matrix valued function $S(t, t_1) \in \mathcal{R}^{n \times n}$ is the solution to the equation

$$\dot{S}(t, t_1) = -S(t, t_1)A(t), \quad t_0 \leq t \leq t_1, \qquad S(t_1, t_1) = I.$$

## 2 The Ellipsoidal Techniques

In this paper we do not elaborate on the ellipsoidal calculus in whole but do indicate the necessary amount of techniques for the specific problem of control synthesis.

We will start with the assumption that $\mathcal{P}(t)$ is an ellipsoidal-valued function and that set $\mathcal{M}$ is an ellipsoid. Namely

$$\mathcal{P}(t) = \mathcal{E}(p(t), P(t)), \qquad t_0 \leq t \leq t_1,$$

$$\mathcal{M} = \mathcal{E}(m, M).$$

where the notations are such that the support function is

$$\rho(\ell \mid \mathcal{E}(a, Q)) = (\ell, a) + (\ell, Q\ell)^{1/2}.$$

With $\det Q \neq 0$ this is equivalent to the inequality

$$\mathcal{E}(a, Q) = \{x \in \mathcal{R} | (x-a)'Q^{-1}(x-a) \leq 1\}.$$

Therefore $a$ stands for the center of the ellipsoid and $Q > 0$ for the symmetric matrix that determines its configuration.

With sets $\mathcal{E}(p(t), P(t))$, $\mathcal{E}(m, M)$ being given we are to determine the tube $\mathcal{W}[t]$ for $t \leq t_1$ under the boundary condition $\mathcal{W}[t_1] = \mathcal{M} = \mathcal{E}(m, M)$. According to the above, the set-valued function $\mathcal{W}[t]$ satisfies the *evolution equation*

$$\lim_{\sigma \to +0} \sigma^{-1} h(\mathcal{W}[t-\sigma], \mathcal{W}[t] - \sigma\mathcal{E}(p(t), P(t))) = 0, \; t_0 \leq t \leq t_1, \quad \mathcal{W}[t_1] = \mathcal{E}(m, M). \tag{10}$$

Obviously

$$\mathcal{W}[t] = \mathcal{E}(m, M) - \int_t^{t_1} \mathcal{E}(p(\tau), P(\tau)) d\tau, \qquad t_0 \leq t \leq t_1, \tag{11}$$

so that $\mathcal{W}[t]$ is similar to *the attainability domain* for system (6) but here it is taken *in backward time*; $\mathcal{W}[t]$ is the set of all states $x_t$ from which it is possible to steer system (6) to the set $\mathcal{E}(m, M)$ in time $t_1 - t$ with open loop control

$$u(\tau) \in \mathcal{P}(\tau), \qquad t \leq \tau \leq t_1,$$

It is clear that although $\mathcal{E}(m, M)$, $\mathcal{E}(p(t), P(t))$ are ellipsoids, the set $\mathcal{W}[t]$, in general, is not an ellipsoid.

Therefore the first problem that does arise here is as follows: is it possible to approximate $\mathcal{W}[t]$, both externally and internally, with ellipsoidal-valued functions?

The answer to the question is affirmative as will be shown in the sequel. We will first state the results for $A(t) \not\equiv 0$ in (6).

Consider the inclusion

$$\dot{x} \in A(t)x + \mathcal{E}(p(t), P(t)), \quad \tau \leq t \leq t_1, \qquad x(t_1) \in \mathcal{E}(m, M) \tag{12}$$

with $\mathcal{W}[\tau] = \mathcal{W}(\tau, \mathcal{M})$ being the set of all states $x_\tau$ from which there exists an open-loop control $u(t) \in \mathcal{E}(p(t), P(t))$ that steers the solution from $x_\tau$ into $\mathcal{E}(m, M)$.

Denote $w(t) \in \mathcal{R}^n$, $\tau \leq t \leq t_1$, to be the solution to the equation

$$\dot{w}(t) = A(t)w(t) + p(t), \quad \tau \leq t \leq t_1, \qquad w(t_1) = m, \tag{13}$$

and $W_S(t) \in \mathcal{R}^{n\times n}$ to be the solution to the matrix equation

$$\begin{aligned}\dot{W}_S(t) &= A(t)W_S(t) + W_S(t)A'(t) - \\ &- S^{-1}(t)[S(t)W_S(t)S'(t)]^{1/2}[S(t)P(t)S'(t)]^{1/2}S^{-1}(t) - \\ &- S^{-1}(t)[S(t)P(t)S'(t)]^{1/2}[S(t)W_S(t)S'(t)]^{1/2}S^{-1}(t),\end{aligned} \tag{14}$$

$$\tau \leq t \leq t_1,$$

$$W_S(t_1) = M,$$

where $S(t)$ is a continuous matrix valued function

$$S(\cdot) : [\tau, t_1] \to \mathcal{R}^{n\times n}$$

with invertible values (the set of all such functions will be denoted as $\Sigma$).

**Theorem 2.1 (Internal Approximation)**

*(i) The following inclusion is true*

$$\mathcal{E}(w(\tau), W_S(\tau)) \subset \mathcal{W}[\tau] \tag{15}$$

*whatever is the function $S(\cdot) \in \Sigma$.*

*(ii) The following equality is true*

$$\overline{\bigcup_{S(\cdot)\in\Sigma} \mathcal{E}(w(\tau), W_S(\tau))} = \mathcal{W}[\tau], \tag{16}$$

*where the symbol $\overline{\mathcal{K}}$ stands for the closure of set $\mathcal{K}$.*

Further on, denote $W_\pi(t)$ to be the solution to the equation

$$\dot{W}_\pi(t) = A(t)W_\pi(t) + W_\pi(t)A'(t) - \pi^{-1}(t)W_\pi(t) - \pi(t)P(t), \ \tau \leq t \leq t_1, \quad W_\pi(t_1) = M, \tag{17}$$

where $\pi(t) > 0$ is a continuous scalar function:

$$\pi(\cdot) : [\tau, t_1] \to (0, \infty)$$

(the class of such functions will be denoted as $\Pi$).

**Theorem 2.2 (External Approximation)**

*(i) The following inclusion is true*

$$\mathcal{W}[\tau] \subset \mathcal{E}(w(\tau), W_\pi(\tau)) \tag{18}$$

*whatever is the function* $\pi(\cdot) \in \Pi$.

*(ii) The following equality is true*

$$\mathcal{W}[\tau] = \bigcap_{\pi(\cdot)\in\Pi} \mathcal{E}(w(\tau), W_\pi(\tau)). \tag{19}$$

Equations (16) (19) are obviously simplified under the condition $A(t) \equiv 0$ (we further presume that it holds). It is therefore clear that the set-valued function $\mathcal{W}[t]$ satisfies the inclusions

$$\mathcal{E}^-[t] = \mathcal{E}(w(t), W_S(t)) \subset \mathcal{W}[t] \subset \mathcal{E}(w(t), W_\pi(t)) = \mathcal{E}^+[t], \quad t_0 \le t \le t_1 \tag{20}$$

whatever are the functions $S(\cdot) \in \Sigma$, $\pi(\cdot) \in \Pi$.

Since $\mathcal{W}[t]$ is the solution to the evolution equation (10) the next question arises: do there exist any two types of evolution equations whose solutions would be $\mathcal{E}^-[t]$ and $\mathcal{E}^+[t]$ respectively?

The answer to this question is given in the following assertion:

Consider the evolution equation

$$\lim_{\sigma\to+0} \sigma^{-1} h_+(\mathcal{E}[t-\sigma], \mathcal{E}[t] - \sigma\mathcal{E}(p(t), P(t))) = 0, \quad t_0 \le t \le t_1, \qquad \mathcal{E}[t_1] = \mathcal{E}(m, M). \tag{21}$$

We will say that function $\mathcal{E}_+[t]$ is a solution to equation (23) if it satisfies (23) almost everywhere and if it is *ellipsoidal-valued* (!).

Also consider the evolution equation

$$\lim_{\sigma\to+0} \sigma^{-1} h_-(\mathcal{E}[t-\sigma], \mathcal{E}[t] - \sigma\mathcal{E}(p(t), P(t))) = 0, \ t_0 \le t \le t_1, \quad \mathcal{E}[t_1] = \mathcal{E}(m, M). \tag{22}$$

We will define $\mathcal{E}_-[t]$ to be a solution to equation (24) if it

- satisfies (24) almost everywhere,
- is *ellipsoidal-valued* and
- is also a *maximal* solution to (24).

The latter means that there exists no other ellipsoidal-valued solution $\mathcal{E}'[t]$ to (24) such that $\mathcal{E}_-[t] \subset \mathcal{E}'[t]$ and $\mathcal{E}_-[t] \not\equiv \mathcal{E}'[t]$ $t_0 \le t \le t_1$.

Each of the equations (23), (24) has a nonunique solution.

**Lemma 2.1** *Whatever are the solutions $\mathcal{E}_+[t], \mathcal{E}_-[t]$ to the evolution equations (23), (24), the following inclusions are true*

$$\mathcal{E}_-[t] \subset \mathcal{W}[t] \subset \mathcal{E}_+[t], \qquad t_0 \le t \le t_1.$$

**Lemma 2.2** *Each of the ellipsoidal-valued functions $\mathcal{E}^-[t] = \mathcal{E}(w(t), W_S(t))$, $(S(\cdot) \in \Sigma)$ is a solution $\mathcal{E}_-[t]$ to equation (24).*

**Lemma 2.3** *Each of the ellipsoidal-valued functions $\mathcal{E}^+[t] = \mathcal{E}(w(t), W_\pi(t))$, $(\pi(\cdot) \in \Pi)$ is a solution $\mathcal{E}_+[t]$ to equation (23).*

To conclude this section we underline that the tube $\mathcal{W}[t]$ can be ***exactly approximated*** by ellipsoids – both internally and externally – according to relations (18), (21). To achieve the exact approximation it is necessary in general to use an infinite variety of ellipsoids (actually, a countable set). The given approach, (see also [4]), therefore goes beyond the suggestions of [5] and [6], where the sums of two or more convex sets were approximated by one ellipsoid.

The ellipsoidal approximations will now be used to devise a synthesized control strategy for solving the problem of the above. This strategy will guarantee the attainability of the terminal set $\mathcal{M}$ in prescribed time.

## 3 Synthesized Strategies for Guaranteed Control

The idea of constructing the synthesizing strategy $\mathcal{U}(t,x)$ for the problem of the above was that $\mathcal{U}(t,x)$ should ensure that all the solutions $x[t] = x(t,\tau,x_\tau)$ to the equation

$$\dot{x}(t) \in \mathcal{U}(t, x(t)), \qquad \tau \le t \le t_1,$$

with initial state $x[\tau] = x_\tau \in \mathcal{W}[\tau]$, would satisfy the inclusion

$$x[t] \in \mathcal{W}[t], \quad \tau \le t \le t_1$$

and would therefore ensure $x[t_1] \in \mathcal{M}$.

We will now substitute $\mathcal{W}[t]$ by one of its internal approximations $\mathcal{E}_-[t] = \mathcal{E}(w(t), W(t))$. The conjecture is that once $\mathcal{W}[t]$ is substituted by $\mathcal{E}_-[t]$, we should just copy the scheme of Section 1, constructing a strategy $\mathcal{U}_-(t,x)$ such that for every solution $x[t] = x(t,\tau,x_\tau)$ that satisfies equation

$$\dot{x}[t] = \mathcal{U}_-(t, x[t]), \quad \tau \le t \le t_1, \quad x[\tau] = x_\tau, \qquad x_\tau \in \mathcal{E}_-[\tau], \tag{23}$$

the following inclusion would be true

$$x[t] \in \mathcal{E}_-[t], \quad \tau \leq t \leq t_1, \tag{24}$$

and therefore

$$x[t_1] \in \mathcal{E}(m, M) = \mathcal{M}.$$

It will be proven that once the approximation $\mathcal{E}_-[t]$ is selected "appropriately", the desired strategy $\mathcal{U}_-(t,x)$ may be constructed again according to the scheme of (3), except that $\mathcal{W}[t]$ will now be substituted by $\mathcal{E}_-[t]$, namely

$$\mathcal{U}(t,x) = \begin{cases} \mathcal{E}(p(t), P(t)) & \text{if } x \in \mathcal{E}_-[t] \\ p(t) - P(t)\ell^0(\ell^0, P(t)\ell^0)^{-1/2} & \text{if } x \notin \mathcal{E}_-[t], \end{cases} \tag{25}$$

where $\ell^0 = \partial_x d(x, \mathcal{E}_-[t])$ at point $x = x(t)$, that is the unit vector that solves the problem

$$(\ell^0, x) - \rho(\ell^0 \mid \mathcal{E}_-[t]) = \max\{(\ell, x) - \rho(\ell \mid \mathcal{E}_-[t]) \mid \|\ell\| \leq 1\}. \tag{26}$$

The latter problem may be solved with more detail (since $\mathcal{E}_-[t]$ is an ellipsoid). Indeed, if $s^0$ is the solution to the minimization problem

$$s^0 = \arg\min\{\|(x - s)\| \mid s \in \mathcal{E}_-[t],\ x = x(t)\} \tag{27}$$

then we can take

$$\ell^0 = x(t) - s^0$$

in (26).

**Lemma 3.1** *Consider a nondegenerate ellipsoid $\mathcal{E} = \mathcal{E}(a, Q)$ and a vector $x \notin \mathcal{E}(a, Q)$, then the subgradient $\ell^0 = \partial_x d(x, \mathcal{E}(a, Q))$ can be expressed through $\ell^0 = x - s^0/\|x - s^0\|$,*

$$s^0 = (I + \lambda Q^{-1})^{-1}(x - a) + a,$$

*where $\lambda > 0$ is the unique root of the equation $h(\lambda) = 0$, with*

$$h(\lambda) = \Big((I + \lambda Q^{-1})^{-1}(x - a), Q^{-1}(I + \lambda Q^{-1})^{-1}(x - a)\Big) - 1.$$

Assume $a = 0$. Then the necessary conditions of optimality for the minization problem

$$\|x - s\| = \min, \quad (s, Q^{-1}s) \leq 1$$

are reduced to the equation

$$-x + s + \lambda Q^{-1}s = 0$$

where $\lambda$ is to be calculated as the root of the equation $h(\lambda) = 0, (a = 0)$.

Since it is assumed that $x \notin \mathcal{E}(0, Q)$, we have $h(0) > 0$. With $\lambda \to \infty$ we also have

$$\Big((I + \lambda Q^{-1})^{-1}x, Q^{-1}(I + \lambda Q^{-1})^{-1}x\Big) \to 0.$$

This yields $h(\lambda) < 0$, $\lambda \geq \lambda_*$ for some $\lambda_* > 0$. The equation $h(\lambda) = 0$ therefore has a root $\lambda^0 > 0$. The root $\lambda^0$ is unique since direct calculation gives $h'(\lambda) < 0$ with $\lambda > 0$. The case $a \neq 0$ can now be given through a direct shift $x \to x - a$.

We will now prove that the *ellipsoidal valued strategy* $\mathcal{U}_-(t, x)$ of (26) does solve the problem of control synthesis, provided we start from a point $x_\tau = x(\tau) \in \mathcal{E}_-[\tau]$, $\tau \leq t \leq t_1$. Indeed, assume $x_\tau \in \mathcal{E}_-[\tau]$ and $x[t] = x(t, \tau, x_\tau)$ to be the respective trajectory. We will demonstrate that once $x[t]$ is a solution to equation (23), then we will have (24). (With isolated trajectory $x[t]$ given, it is clearly driven by a unique control $u[t] = \dot{x}(t)$ a.e. such that $u[t] \in \mathcal{P}(t)$).

Suppose, on the contrary, that the distance $d(x[t_*], \mathcal{E}_-[t_*]) > 0$ for some value $t_* > \tau$. Since $x[\tau] \in \mathcal{E}_-[\tau]$ and since $d[t] = d(x[t], \mathcal{E}_-[t])$ is differentiable, there exists a point $t_{**} \in (\tau, t_*]$ such that

$$\frac{d}{dt} d[t]|_{t=t_{**}} > 0, \quad d[t_{**}] > 0. \tag{28}$$

Calculating

$$d[t] = \max\{(\ell, x(t)) - \rho(\ell \mid \mathcal{E}_-[t]) | \|\ell\| \leq 1\}$$

we observe

$$\frac{d}{dt} d[t] = \frac{d}{dt}\left[(\ell^0, x[t]) - \rho(\ell^0 | \mathcal{E}_-[t])\right]$$

and since $\ell^0$ is a unique maximiser,

$$\frac{d}{dt} d[t] = (\ell^0, \dot{x}[t]) - \frac{\partial}{\partial t}\rho(\ell^0 | \mathcal{E}_-[t]) = (\ell^0, u[t]) - \frac{d}{dt}\left[(\ell^0, w(t)) + (\ell^0, W(t)\ell^0)^{1/2}\right]$$

where $\mathcal{E}_-[t] = \mathcal{E}(w(t), W(t))$.

For a fixed function $S(\cdot)$ we have $\mathcal{E}_-[t] = \mathcal{E}(w(t), W_S(t))$ , where $w(t)$, $W_S(t)$ satisfy the system (15), (16), $(A(t) \equiv 0)$. Substituting this into the relation for the derivative of $d[t]$ and remembering the rule for differentiating a maximum of a variety of functions

$$\frac{d}{dt} d[t] = (\ell^0, u[t]) - (\ell^0, p(t)) - \frac{1}{2}(\ell^0, W_S(t)\ell^0)^{-1/2} \cdot$$

$$\cdot \left(\ell^0, S^{-1}(t)([S(t)W_S(t)S(t)]^{1/2}[S(t)P(t)S'(t)]^{1/2} + [S(t)P(t)S'(t)]^{1/2}[S(t)W_S(t)S'(t)]^{1/2})S'^{-1}(t)\ell^0\right)$$

or due to the Bunyakovsky-Schwartz inequality

$$\frac{d}{dt} d[t] \leq -(\ell^0, p(t)) + (\ell^0, P(t)\ell^0)^{1/2} + (\ell^0, u[t]),$$

where

$$u[t] \in \mathcal{E}(p(t), P(t))$$

and

$$u[t] \in \mathcal{U}_-(t, x).$$

For the case $x \notin \mathcal{E}_-(w(t), W_S(t))$ the last relation gives us

$$\frac{d}{dt}d[t]\Big|_{t=t_{**}} = 0$$

which contradicts with (28).

What follows is the assertion

**Theorem 3.1** *Define an internal approximation $\mathcal{E}_-[t] = \mathcal{E}_-(w(t), W_S(t))$ with given parametrization $S(t)$ of (16). Once $x[\tau] \in \mathcal{E}_-[\tau]$ and the synthesizing strategy is $\mathcal{U}_-(t,x)$ of (26), the following inclusion is true:*

$$x[t] \in \mathcal{E}_-[t], \quad \tau \leq t \leq t_1,$$

*and therefore*

$$x[t_1] \in \mathcal{E}(m, M).$$

The ellipsoidal synthesis thus gives a solution strategy $\mathcal{U}_-(t,x)$ for any internal approximation $\mathcal{E}_-[t] = \mathcal{E}_-(w(t), W_S(t))$.

With $x \notin \mathcal{E}_-[t]$, the function $\mathcal{U}_-(t,x)$ is single-valued, whilst with $x \in \mathcal{E}_-[t]$ it is multivalued ($\mathcal{U}_-(t,x) = \mathcal{E}_-[t]$) being therefore upper-semicontinuous in $x$, measureable in $t$ and ensuring the existence of a solution to the differential inclusion (23).

We will now proceed with numerical examples that demonstrate the constructive nature of the solutions obtained above.

## 4 Numerical Examples

We take system (14) to be 4 dimensional, and study it between the initial moment $t_0 = 0$ and final moment $t_1 = 5$.

As the ellipsoids appearing in this problem are *four dimensional*, we present their *two dimensional projections*. The figures are divided into four *windows*, and each shows projections of the original ellipsoids onto the planes spanned by the first and second, third and fourth, first and third, and second and fourth coordinate axes, in a clockwise order starting from bottom left. The drawn segments of coordinate axes corresponding to state variables range from $-10$ to 10 according to the above scheme. In some of the figures, where we show the graph of solutions and of solvability set, the third, skew axis corresponds to time and ranges from 0 to 5.

Let the initial position $\{0, x_0\}$ be given by

$$x_0 = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

the target set $\mathcal{M} = \mathcal{E}(m, M)$ by

$$m = \begin{pmatrix} 0 \\ 5 \\ 5 \\ 0 \end{pmatrix}$$

and

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

at the final moment $t_1 = 5$. We consider a case when the right hand side is constant:

$$A(t) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{pmatrix},$$

describing the position and velocity of two independent oscillators. The restriction $u(t) \in \mathcal{E}(p(t), P(t))$ on the control $u$, is also defined by time independent constraints:

$$p(t) \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$P(t) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so that the controls couple the system. Therefore the class of feasible strategies is such that

$$\Upsilon = \{\mathcal{U}(t, x) | \mathcal{U}(t, x) \subset \mathcal{E}(p(t), P(t))\}.$$

The results to be presented here we obtain by way of discretization. We divide the interval [0, 5] into 100 subintervals of equal lengths, and use the discretized version of (16). Instead of the set valued control strategy (26) we apply a *single valued* selection:

$$u(t,x) = \begin{cases} p(t) & \text{if } x \in \mathcal{E}_-[t] \\ p(t) - P(t)\ell^0(\ell^0, P(t)\ell^0)^{-1/2} & \text{if } x \notin \mathcal{E}_-[t]. \end{cases} \tag{29}$$

again in its discrete version.

We calculate the parameters of the ellipsoid $\mathcal{E}_-[t] = \mathcal{E}_-(w(t), W_S(t))$ by chosing

$$S(t) = P^{-1/2}(t), \quad 0 \le t \le 5$$

in (16).

The calculations give the following internal ellipsoidal estimate $\mathcal{E}_-[0] = \mathcal{E}(w(0), W_S(0))$ of the solvability set $\mathcal{W}(0, \mathcal{M})$:

$$w(0) = \begin{pmatrix} 4.2371 \\ 1.2342 \\ -2.6043 \\ -3.1370 \end{pmatrix},$$

and

$$W_S(0) = \begin{pmatrix} 31.1385 & 0 & 0 & 0 \\ 0 & 31.1385 & 0 & 0 \\ 0 & 0 & 12.1845 & 2.3611 \\ 0 & 0 & 2.3611 & 44.1236 \end{pmatrix}.$$

Now, as is easy to check, $x_0 \in \mathcal{E}_-[0]$ and therefore Theorem 3.1 is applicable, implying that the control strategy of (26) steers the solution of (23) into $\mathcal{M}$, producing

$$x[5] = \begin{pmatrix} 0.0264 \\ 4.9512 \\ 4.0457 \\ -0.0830 \end{pmatrix}$$

as a final state.

Figure 1 shows the graph of the ellipsoidal valued map $\mathcal{E}_-[t]$, $t \in [0, 5]$ and of the solution of

$$\dot{x}[t] = A(t)x[t] + u(t, x[t]), \; 0 \le t \le 5, \quad x[0] = x_0 \tag{30}$$

where we use $u(t, x)$ of (29).

Figure 2 shows the target set $\mathcal{M} = \mathcal{E}(m, M)$, (projections appearing as circles), the solvability set $\mathcal{E}_-[0] = \mathcal{E}(w(0), W_S(0))$ at the initial moment $t = 0$, and the trajectory of the solution of (30).

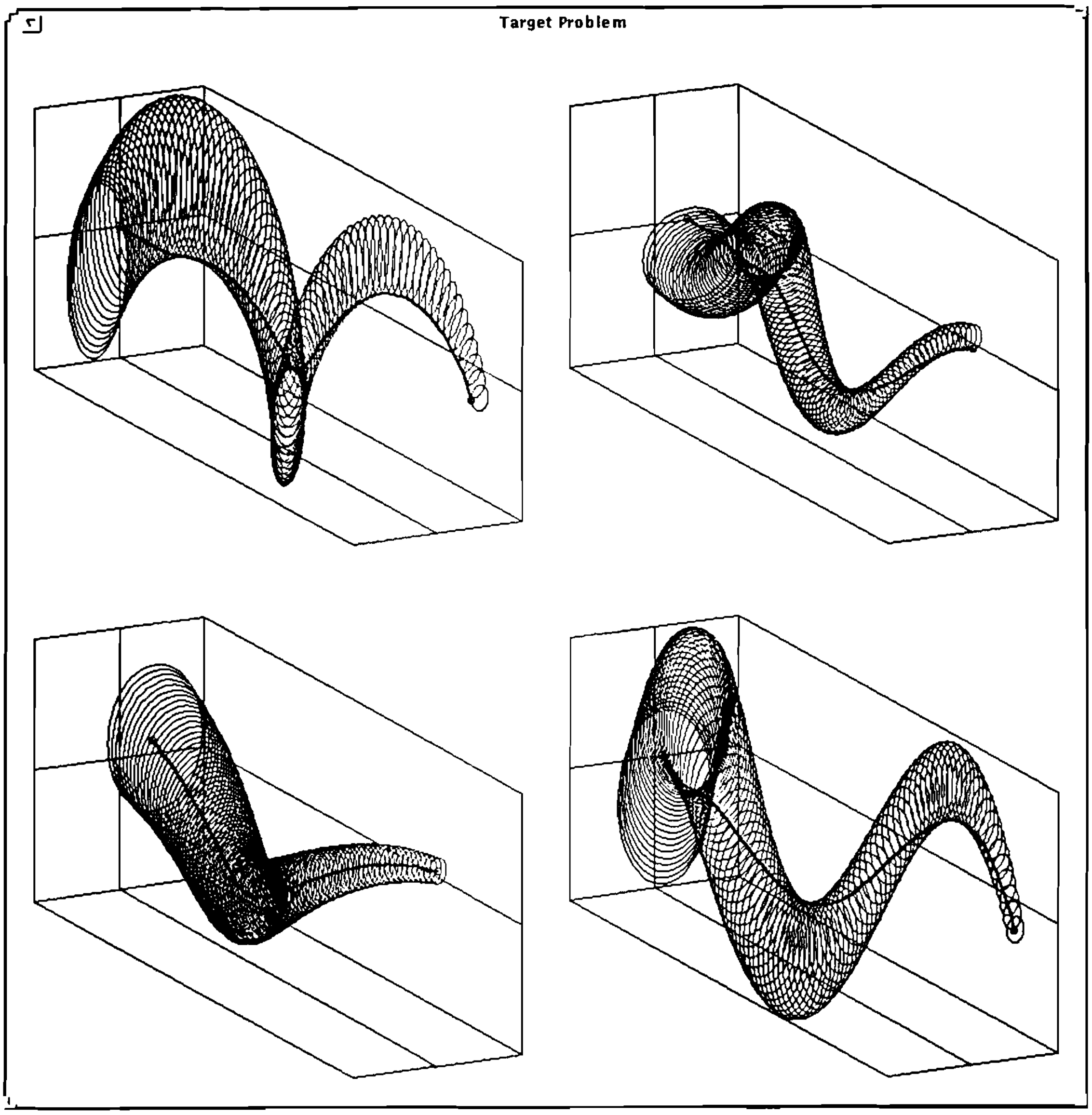


Figure 1: Tube of ellipsoidal solvability sets and graph of solution

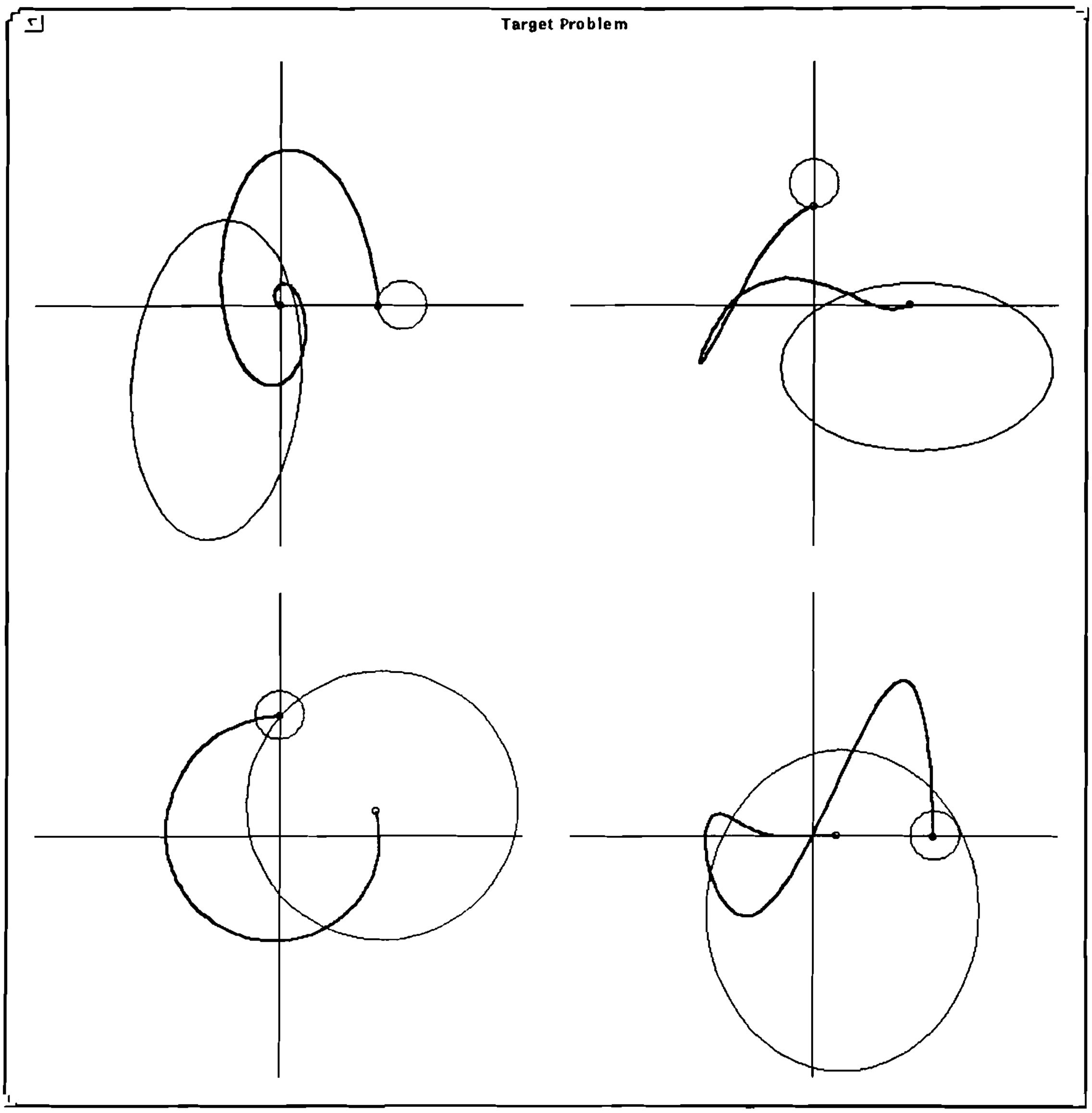


Figure 2: Target set, initial ellipsoidal solvablity set and trajectory in phase space – initial state inside

In the next example we show by way of numerical evidence, what can happens if the initial state $x_0$ does not belong to the ellipsoidal solvability set $\mathcal{E}_-[0]$. Leaving the rest of the data to be the same, we change the initial state $x_0$ in such a way that the inclusion

$$x_0 \in \mathcal{E}_-[0]$$

is hurt, but "not very much", taking

$$x_0 = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

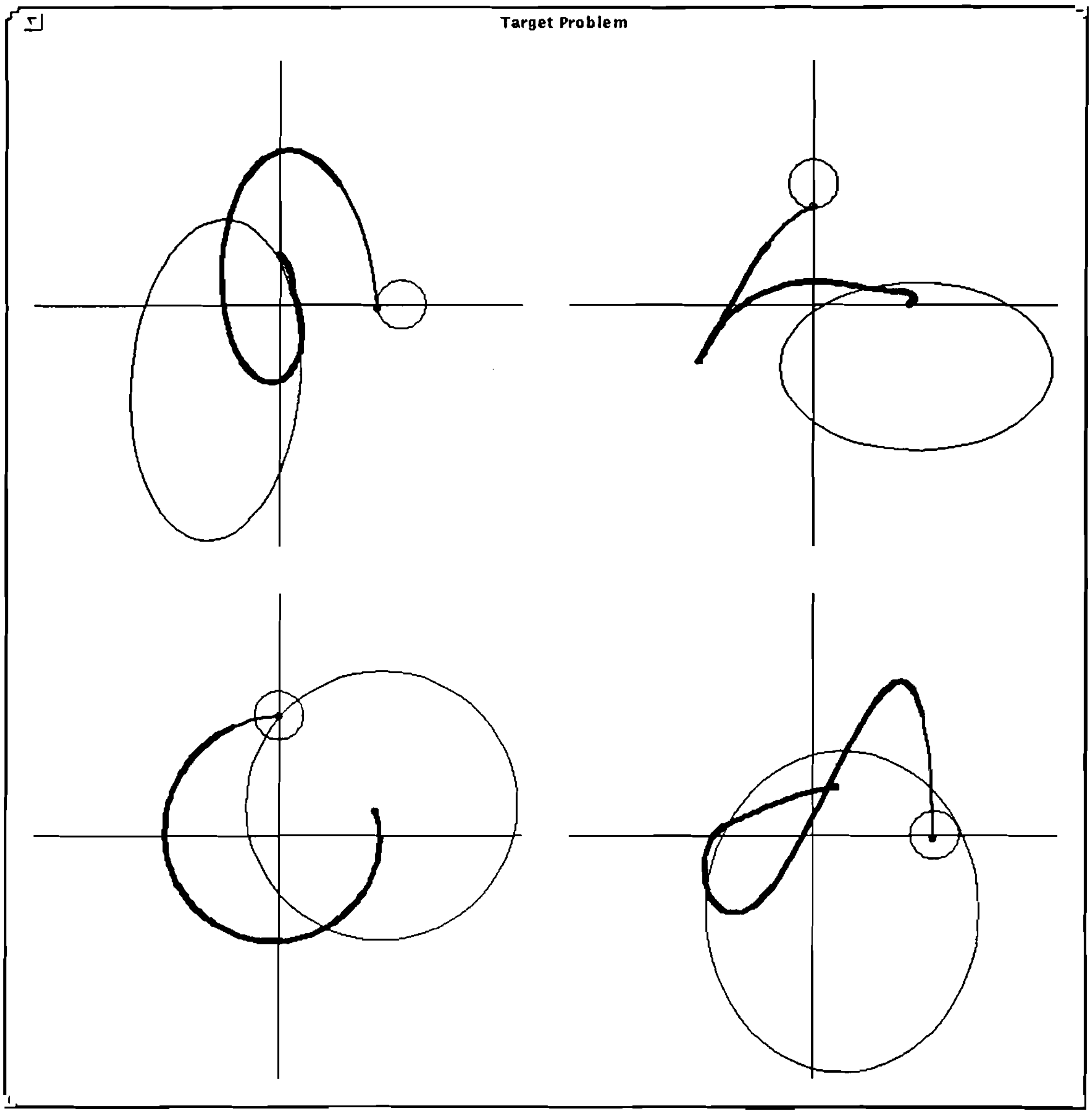


Figure 3: Initial state outside, "but not far away".

Though Theorem 3.1 cannot be used, still we apply formulae (29) and (30). Analogously to Figure 2, Figure 3 shows the phase portrait of the result. The trajectory of the solution to (30) is drawn with a thick line, as long as it is outside of the respective ellipsoidal solvability set, and with a thin line if it is inside. The drawn projections of the initial state are inside, except one, (upper left window). As the illustration shows, at one point in time the trajectory enters the tube $\mathcal{E}_-[t]$, the line changing into thin. After this happens, Theorem 3.1 does take effect, and the trajectory remains inside for the rest of the time interval. In this way we obtain

$$x[5] = \begin{pmatrix} 0.0255 \\ 4.9528 \\ 4.0215 \\ -0.1658 \end{pmatrix}$$

as a final state.

The above phenomenon indicates

- first that for the initial state must be inside the solvability set $\mathcal{W}(0, \mathcal{M})$, that is actually

$$x_0 \in \mathcal{W}(0, \mathcal{M}) \setminus \mathcal{E}_-[0],$$

  as *it was possible* to steer the solution to (29) and (30) into the target set $\mathcal{M}$,

- and second, that *in this particular numerical example* the control rule works beyond the tube $\mathcal{E}_-[t]$.

In the third example, we move the initial state $x_0$ further away, so that the control rule does not work any more, (Figure 4):

$$x_0 = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 3 \end{pmatrix},$$

and obtain as final state

$$x[5] = \begin{pmatrix} 0.0460 \\ 4.9150 \\ 3.3668 \\ -0.5540 \end{pmatrix}.$$

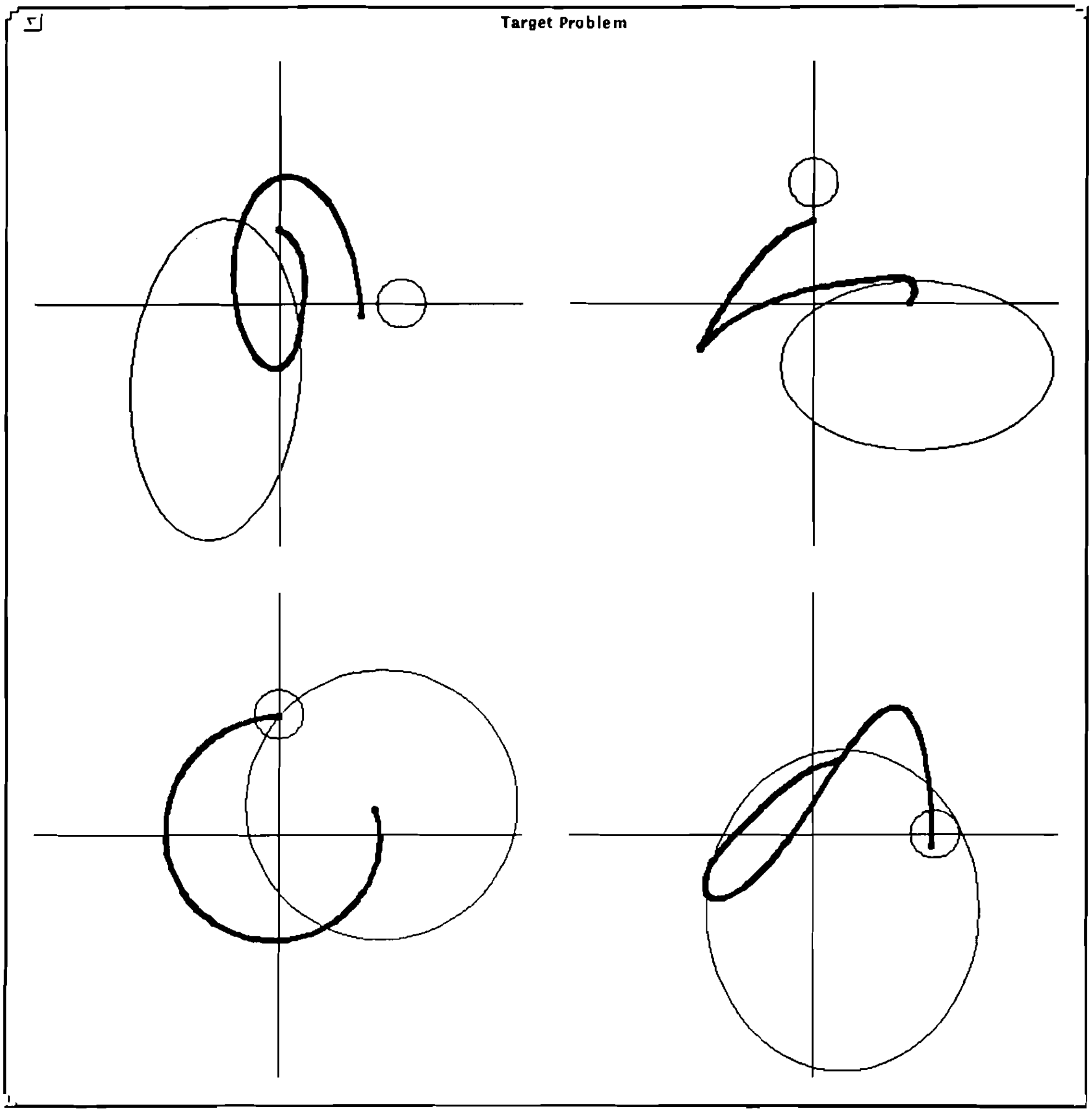


Figure 4: Initial state outside, "far away".

Figures 5 and 6 show the effect of changing the target set. We take the data of the first example except for the matrix $M$ in the target set $\mathcal{M} = \mathcal{E}(m, M)$ by setting the radius to be 2:

$$M = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

resulting in a final state

$$x[5] = \begin{pmatrix} 0.5875 \\ 4.8914 \\ 3.0158 \\ -0.0536 \end{pmatrix}.$$

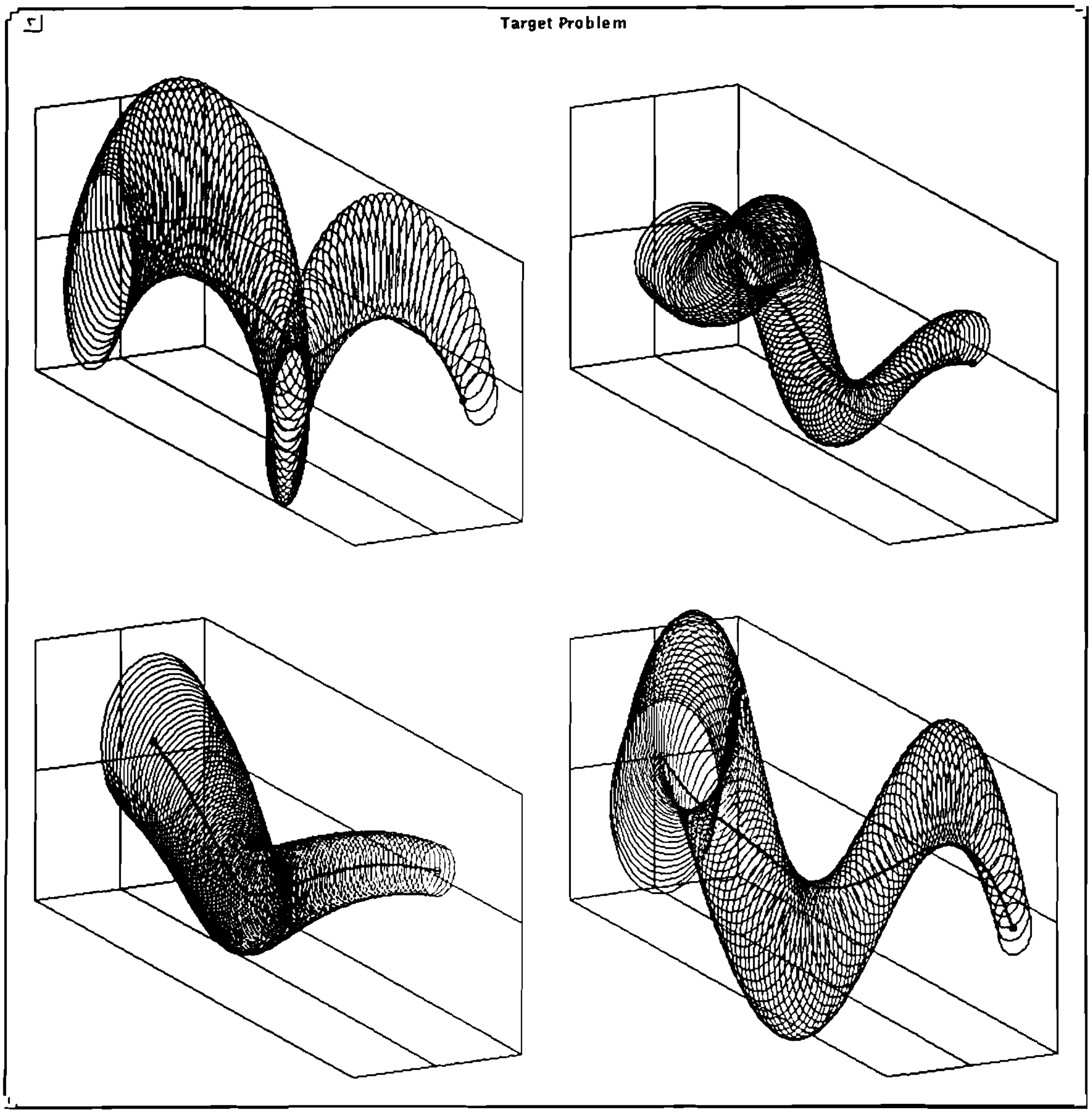


Figure 5: Graph of solution for larger target set

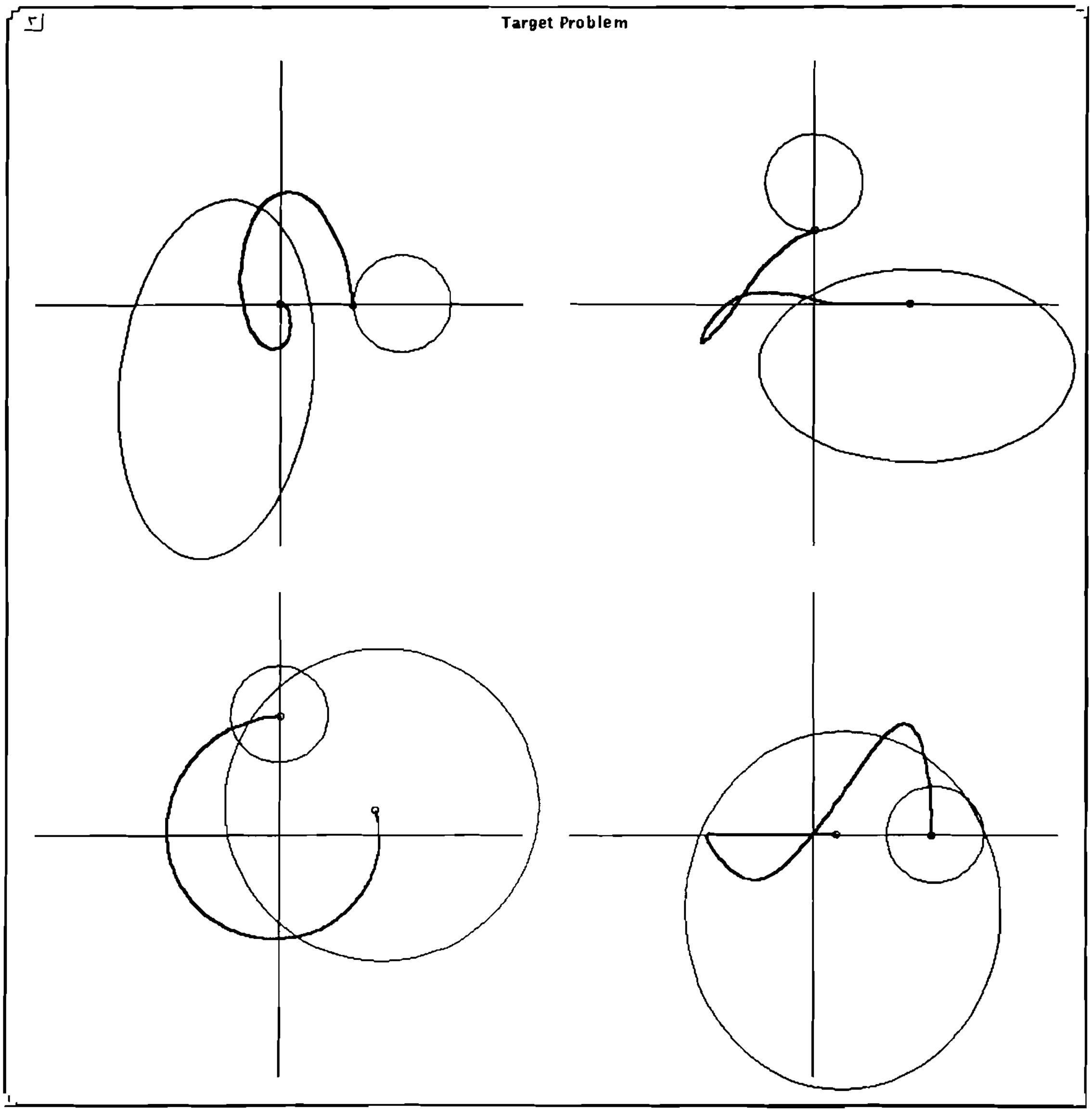


Figure 6: Phase space representation for larger target set

The switching of the control, due to the specific form of (29), is clearly seen in Figure 6. and later in Figure 8.

Taking again the data of the first example, we allow more freedom for the controls, changing the matrix $P(t)$ in the bounding set $\mathcal{P} = \mathcal{E}(p(t), P(t))$ again by setting the radius to be 2:

$$P(t) \equiv \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

with a final state

$$x[5] = \begin{pmatrix} 0.0235 \\ 4.9565 \\ 4.0536 \\ -0.1308 \end{pmatrix}.$$

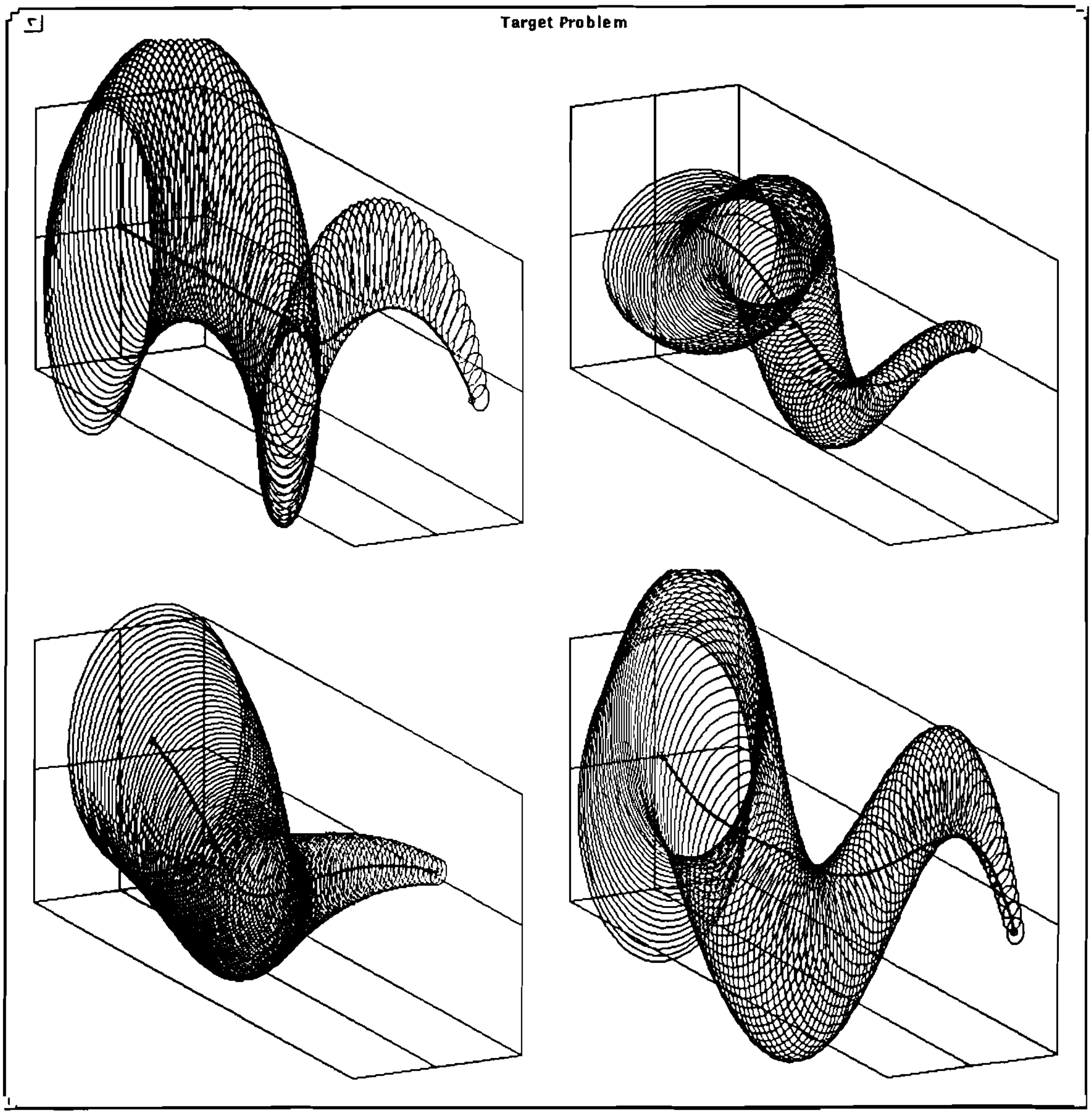


Figure 7: Graph of solution for larger controls

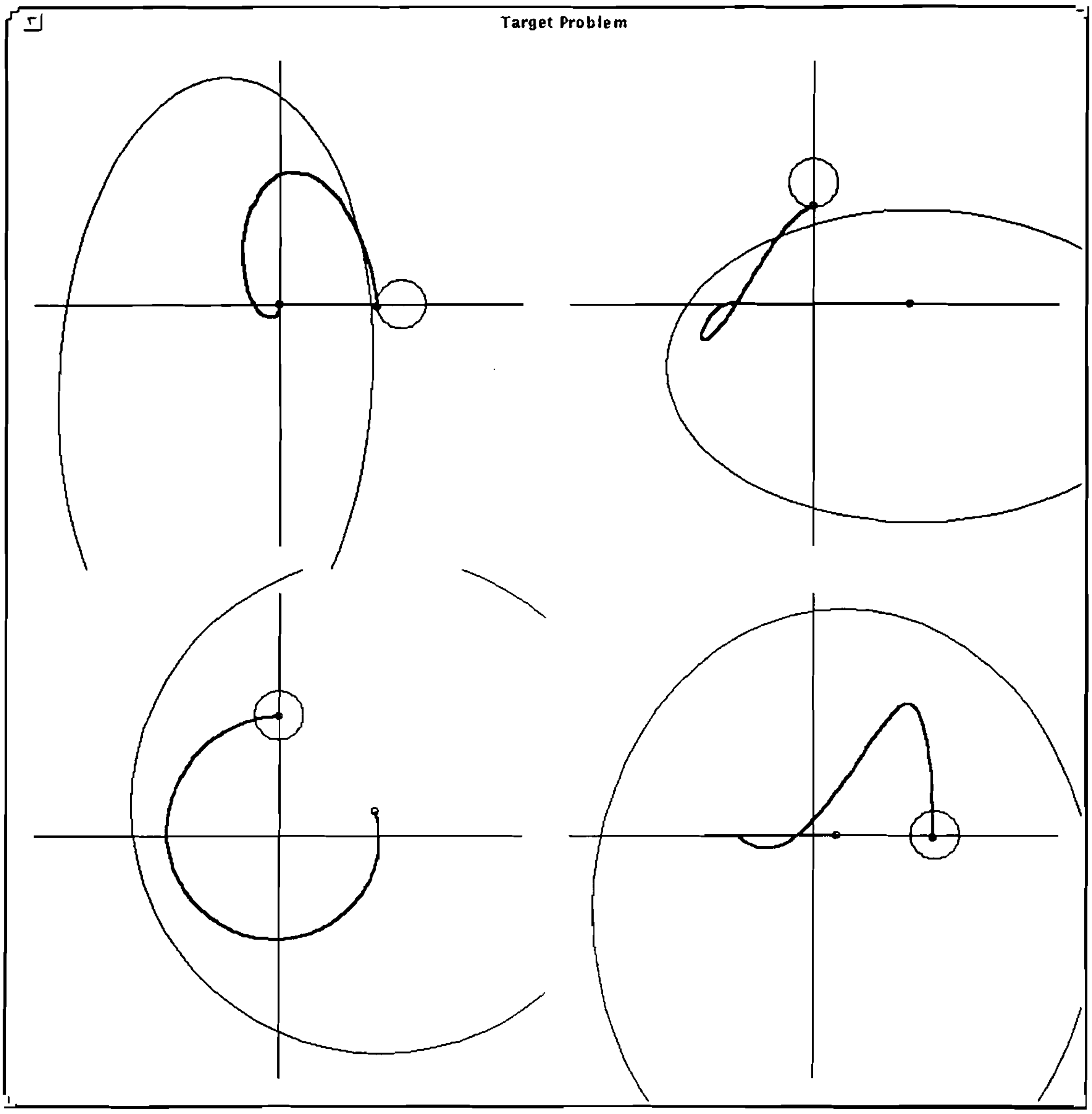


Figure 8: Phase space representation for larger controls

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