Working Paper

Multiple Criteria Games Theory and Applications

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MULTIPLE CRITERIA GAMES - THEORY AND APPLICATIONS

Andrzej P. Wierzbicki[1]

Summary.

After a review of basic concepts in multiple criteria optimization, the paper presents a characterization of noncooperative equilibria in multiple criteria games in normal form either by weighted sums or by order-consistent achievement scalarizing functions, for convex and nonconvex cases. Possible applications of multiple criteria games and such characterizations of their equilibria are indicated. The analysis of multiple criteria games night be especially useful when studying reasons of possible conflict escalation processes and ways of preventing them.

1. Introduction.

While the main body of game theory deals with multiple criteria under the assumptions that they either can be scalarized by a given value or utility function or represented by additional players, there are many cases where such assumptions are not justified. This was often pointed out, e.g. by Blackwell [1956], Contini [1966], Yu [1973], Zeleny [1976]. Although the basic concepts of a multi-objective version of game theory were defined and analyzed by Bergstresser and Yu [1977], this theory was not fully developed subsequently.

However, if the aim of investigation is to help in understanding game-like situations - e.g. in multi-objective decision support systems - rather than to predict possible outcomes of a game, then a fully multi-objective extension of game theory is necessary. In realistic game-like situations we often do not precisely know how to aggregate the values or criteria motivating each player into a value or utility function. On the other hand, in order to analyze such multiple criteria games theoretically or to compute their possible solutions, it is useful to introduce an aggregation of criteria by appropriate scalarizing functions.

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Such an aggregation, however, is used only as a tool of analysis and differs from the typical aggregation by a value or utility function. Although scalarizing functions might be interpreted as proxy value functions, they must posses certain additional properties that will be analyzed in detail in this paper. In particular, such scalarizing functions must be parameterized in such a way as to enable an easy scanning and selection of game equilibria, independently of convexity properties.

A specific property of equilibria in multiple criteria games is that there might be not only many of them, but they also might form sets of continuum power. Thus, if one player focuses his/her attention on one such equilibrium but does not communicate with other players, it is almost inevitable that others would select other equilibria. The resulting actions of all players correspond then to a disequilibrium outcome, usually worse than predicted; in repetitive games, this might easily lead to conflict escalation processes. One of main applications of the analysis of noncooperative solutions in multiple criteria games might be related to studying the reasons and ways of preventing conflict escalation.

2. Needed Concepts from Multiple Criteria Optimization.

The basic concepts of multiple criteria optimization are sufficiently well described, see e.g, Sawaragi et al. [1985], Yu [1985], Steuer [1986], Seo and Sakawa [1988]. We recall some of them and add some extensions.

A set X_o of admissible decisions x and a model $f: X_o \rightarrow Y$ of their impacts define the set $Y_o = f(X_o)$ of attainable decision outcomes y contained in an outcome space $\mathbb{Y}$; compact $Y_o \subset \mathbb{Y} = \mathbb{R}^p$ is assumed here. Moreover, it is assumed that domination structures in $\mathbb{Y}$ - see e.g. Yu [1985] - are implied by a positive (nonempty, pointed, closed and convex) cone in this space. The simplest positive cone corresponds to the maximization of all objectives or outcomes:

$$(1) \quad C = \{y \in \mathbb{R}^p: y_i \geq 0, \; i = 1,\ldots p\} = \mathbb{R}^p_+$$

while minimized objectives can be taken into account by changing their signs. Another type of positive cone expresses the assumption that some objectives are stabilized, i.e. kept close to a given reference level:

$$(2) \quad C = \{y \in \mathbb{R}^p: y_i \geq 0, \; i = 1,\ldots p_1; \; y_i = 0, \; i = p_1+ 1,\ldots p\}$$

This cone has an empty interior; for this and other reasons, it is useful to consider ε-conical neighborhoods of a given positive cone C. Such neighborhoods are the interiors of the following extended cones:

$$(3) \quad C(\varepsilon) = \{y \in \mathbb{R}^p: \ \mathrm{dist}(y,C) \le \varepsilon\|y\|\}$$

For this definition, we can use any norm and define dist(y,C) as a Haussdorf distance between y and the set C; this distance can correspond to any other norm (since all norms in $\mathbb{R}^p$ are topologically equivalent). The cone $C(\varepsilon)$ has a nonempty interior for any $\varepsilon > 0$; but it might be not convex even if C is convex. If $C = \mathbb{R}^p_+$ and the norm l_1 is consistently used in (3), then the cone $C(\varepsilon)$ is convex; another, very useful convex cone $C(\varepsilon)$ is obtained if the norm l_1 is used in the right-hand side of the inequality in (3) and an extended Chebyshev norm is used to define the distance. Such a cone can be represented in a number of equivalent forms (see Wierzbicki [1990]). Let $y^{(-)}$ denote a vector with components $y_i^{(-)} = \min(0, y_i)$ which determines the distance of y to $\mathbb{R}^p_+$; then:

$$(4) \quad C(\varepsilon, l_1, l_\infty) = \{y \in \mathbb{R}^p: \ \|y^{(-)}\|_{l_1} + 2\varepsilon\|y^{(-)}\|_{l_\infty} \le \varepsilon\|y\|_{l_1}\} =$$

$$= \{y \in \mathbb{R}^p: \ y = \sum_{j=1}^{p} \lambda_j y^{(j)}, \ \lambda_j \ge 0\} \text{ with } y^{(j)} = (-\varepsilon, -\varepsilon, \ldots 1+(p-1)\varepsilon_{(j)}, \ldots -\varepsilon, -\varepsilon)^T,$$

$$C(\varepsilon, l_1, l_\infty) = \{y \in \mathbb{R}^p: \ -y_j \le \varepsilon \sum_{i=1}^{p} y_i, \ j = 1, \ldots p\} = \{y \in \mathbb{R}^p: \ \min_{1\le i\le p} y_i + \varepsilon \sum_{i=1}^{p} y_i \ge 0\}$$

The last form of this cone is especially important: it indicates that the cone is the zero-level set of the function $\min_{1\le i\le p} y_i + \varepsilon \sum_{i=1}^{p} y_i$.

Given a positive cone, we define usually the weak, strict and strong inequalities in the outcome space as (weak) $y'' \ge_C y' \Leftrightarrow y'' - y' \in C$, (strict) $y'' >_C y' \Leftrightarrow y'' - y' \in \tilde{C} = C\setminus\{0\}$, (strong) $y'' >_{\mathrm{int}C} y' \Leftrightarrow y'' - y' \in \mathrm{int}\, C$. Accordingly, the set of weakly efficient outcomes is defined as:

$$(5) \quad \hat{Y}_o^w = \{\hat{y} \in Y_o: \ Y_o \cap (\hat{y} + \mathrm{int}\, C) = \emptyset\}$$

where $\emptyset$ denotes empty set. Weakly efficient outcomes are the easiest to analyze mathematically, but impractical in applications: for example, if C has the form (2), all attainable outcomes are weakly efficient. The set of efficient outcomes:

$$(6) \quad \hat{Y}_o = \{\hat{y} \in Y_o: \ Y_o \cap (\hat{y} + C\setminus\{0\}) = \emptyset\}$$

is more difficult to analyze; but, as it is known since Geoffrion [1968], this set contains improperly efficient outcomes with unbounded trade-off coefficients, which is also impractical. There are various definitions of trade-off coefficients; a general one (for $C = \mathbb{R}^p_+$, but with Y_o not necessarily smooth nor convex) might be as follows:

(7a) $t_{ij}(\hat{y}) = \sup_{y \in Y^{(j)}(\hat{y})} (y_i - \hat{y}_i)/(\hat{y}_j - y_j), \ \hat{y} \in \hat{Y}_o$, where:

(7b) $Y^{(j)}(\hat{y}) = \{y \in Y_o: y_j < \hat{y}_j, \ y_i \geq \hat{y}_i \text{ for } i \neq j\}$

These are global trade-off coefficients that can be only greater as the local ones for which the supremum is restricted to sequences convergent to $\hat{y}$; in the convex case there is no difference between the global and the local trade-off coefficients.

There are also various definitions of properly efficient outcomes, see e.g. Benson [1977] or Henig [1982]; a most practical one corresponds to such that have a specified prior and finite bound on trade-off coefficients. As suggested by Wierzbicki [1977], [1990], such definition can be obtained with the help of the extended cone $C(\varepsilon)$, e.g. as follows:

(8) $\hat{Y}^{p\varepsilon}_o = \{\hat{y} \in Y_o: Y_o \cap (\hat{y} + \text{int } C(\varepsilon)) = \emptyset\}$

The union of $\hat{Y}^{p\varepsilon}_o$ over all $\varepsilon > 0$ gives the set of properly efficient outcomes $\hat{Y}^p_o$ as defined e.g. by Henig [1982].

Efficient decisions $\hat{x}$ and outcomes $\hat{y}$ can be obtained by maximizing a scalarizing function $s(y,\alpha)$ with $y = f(x)$ over $x \in X_o$, where $s: \mathbb{Y} \times A \to \mathbb{R}^1$, A is a set of parameters, $\alpha \in A$ is a parameter controlling the selection of efficient outcomes. This is **sufficient** for obtaining (weakly, properly) efficient outcomes provided that the scalarizing function s is **strictly monotone** with respect to y, in the sense of the inequality implied by the cone $\tilde{C}$ (or int C, or int $C(\varepsilon)$ - in the last case, this means that $s(y',\alpha) > s(y'',\alpha)$ for all y', y'' such that $y' - y'' \in \text{int } C(\varepsilon)$ and all $\alpha \in A$). However, in practical applications we need also a controllability property of scalarizing functions related to a **necessary** condition of efficiency:

Definition 1. A scalarizing function $s(y,\alpha)$ with $\alpha \in A^n \subseteq A$ gives a (almost) **completely controllable parameterization** or a **complete characterization** of the set of efficient outcomes $\hat{Y}_o$ (or $\hat{Y}^w_o$, or $\hat{Y}^{p\varepsilon}_o$), if for (almost) all $\hat{y} \in \hat{Y}_o$ there exists an $\hat{\alpha} \in A^n$ such that $\hat{y}$ can be obtained by maximizing (over $x \in X_o$ with $y = f(x)$ or, equivalently, over $y \in Y_o$) the function $s(y,\hat{\alpha})$. This parameterization or characterization is **continuously controllable**, if the resulting dependence of $\hat{y}$ on $\hat{\alpha}$ is Lipschitz continuous.

Most powerful tools in examining necessary conditions of optimality are separation theorems. Usually, we apply linear separating functions; we could apply nonlinear ones, but the variety of nonlinear functions is too large to choose a useful class of them, at least in scalar optimization. In vector or multiple criteria optimization, however, there is a useful class of nonlinear separating functions that results in controllable parameterizations of efficient outcomes even in non-convex cases. This class is related to the following basic lemma on **conical separation of sets**; to formulate this lemma, we denote $s(y,\alpha) = r(y)$ and recall that a function $r: \mathbb{Y} \to \mathbb{R}^1$ strictly separates two disjoint sets $Y', Y'' \subset \mathbb{Y}$ at a point $\bar{y} \in Y'$, if $r(y) \le r(\bar{y})$ for all $y \in Y'$ and $r(y) > r(\bar{y})$ for all $y \in Y''$.

Lemma 2. The statement that a function $r: \mathbb{Y} \to \mathbb{R}^1$ strictly separates, at any point $\bar{y} \in Y_o$, the set Y_o and the shifted cone $\bar{y} + \tilde{C}$ (or $\bar{y}$ + int C, or $\bar{y}$ + int C(ε)), is equivalent to three simultaneous statements:

(i) $\bar{y}$ maximizes $r(y)$ over $y \in Y_o$;

(ii) $\bar{y} \in \hat{Y}_o$, it is efficient (or $\bar{y} \in \hat{Y}^w_o$, or $\bar{y} \in \hat{Y}^{p\varepsilon}_o$);

(iii) $r(y)$ is strictly monotone (at least at the point $\bar{y}$, in the sense of the cone $\tilde{C}$, int C, or int C(ε) - i.e. $r(y) > r(\bar{y})$, $\forall\, y - \bar{y} \in \text{int } C(\varepsilon)$).

This lemma is elementary, but might be fundamental to all multiple criteria optimization. Therefore, we give here the proof. The separation property can be rewritten as $r(y) \le r(\bar{y})$ for all $y \in Y_o$, which is equivalent to (i), and also as $r(\bar{y}) < r(y)$ for all $y \in \bar{y} + \tilde{C}$, which is equivalent to (iii). Suppose the separation property holds and $\bar{y}$ is not efficient; then there would exist a point $y' \in Y_o$ such that also $y' \in \bar{y} + \tilde{C}$, which is impossible, since we cannot have $r(y') \le 0$ and $r(y') > 0$ at the same time. Thus, separation implies (ii). Conversely, if (i), (ii), (iii) hold, then $r(y) \le r(\bar{y})$ for all $y \in Y_o$ from (i), and $r(\bar{y}) < r(y)$ for all $y \in \bar{y} + \tilde{C}$ from (iii), which is equivalent to separation.

If the set Y_o is convex, we could use linear separating functions:

$$\text{(9a)} \qquad s(y,\lambda) = \sum_{i=1}^{p} \lambda_i y_i$$

where $\alpha = \lambda \in A$ are interpreted as weighting coefficients. The assumptions on A that result in the appropriate monotonicity of this function depend on the cone we are separating. As it is known, in the case of the cone $\tilde{C}$ only an almost complete characterization is possible (see e.g. Wierzbicki [1986]). In the case of the cones int C or int C(ε), it is sufficient to take their polar cones in order to obtain a complete characterization:

(9b) $A = A^n = C^{\#}(\varepsilon)\setminus\{0\};\ C^{\#}(\varepsilon) = \{\lambda \in \mathbb{R}^p:\ \lambda^T y \geq 0 \text{ for all } y \in C(\varepsilon)\}$

Lemma 3 (Wierzbicki [1990]). Suppose Y_o is convex, $C = \mathbb{R}^p_+$, $C(\varepsilon) = C(\varepsilon, l_1, l_\infty)$, and the scalarizing function (9a) is used with normalized $\bar{\lambda}_i = \lambda_i / \sum_{j=1}^{p} \lambda_j$. Then $\lambda \in A^n$ if and only if $\bar{\lambda} \geq \delta = \varepsilon/(1+p\varepsilon)$; for such λ, if $\hat{y}$ maximizes the function (9a) with respect to $y \in Y_o$, then $\hat{y} \in \hat{Y}^{p\varepsilon}$. Moreover, $t_{ij}(\hat{y}) \leq 1+1/\varepsilon = 1-p+1/\delta$ for all $\hat{y} \in \hat{Y}^{p\varepsilon}$. Conversely, for every properly efficient $\hat{y}$ with such bound on trade-off coefficients, there exists $\hat{\lambda} \in A^n$ such that the maximum of function (9a) is attained at $\hat{y}$.

The specific bound on trade-off coefficients obtained in this lemma does not actually depend on the convexity of Y_o. The (bi)linear scalarizing function (9a) has, however, several drawbacks even in the convex case; for example, the parameterization obtained by this function is not continuously controllable even in the simplest case when Y_o is a polytope (e.g. in the case of multiple criteria linear programming). Thus it is better to use nonlinear functions that separate positive cones; such functions result in characterizations of efficient outcomes also for non-convex or discrete cases. An example of nonlinear functions that have such a conical separation property (for $C = \mathbb{R}^p_+$) is as follows:

$$(10) \qquad s(y,\bar{y}) = \min_{1\leq i\leq p} (y_i - \bar{y}_i) + \varepsilon \sum_{i=1}^{p} (y_i - \bar{y}_i)$$

where the controlling parameters $\alpha = \bar{y}$ are interpreted as **reference or aspiration points** in the objective space; the use of such controlling parameters has many advantages over the more typical use of weighting coefficients and the methods related to them are called **reference point methods** or **aspiration-led methods**. Function (10) can be interpreted in various ways, e.g. as a strict penalty function for constraints $y_i \geq \bar{y}_i$. However, its most important properties are that it is strictly monotone in the sense of the cone int $C(\varepsilon)$ for all $\bar{y} \in \mathbb{R}^p$ and strictly separates int $C(\varepsilon)$ from Y_o if $\bar{y} \in \hat{Y}_o^{p\varepsilon}$.

Theorem 4 (Wierzbicki, [1986]). Suppose $\bar{y} \in A$ is a **reference or aspiration point** and $s:\ Y \times A \to \mathbb{R}^1$ is a continuous **order-consistent scalarizing function**, i.e. it has two following properties:

(a) **monotonicity**: $\forall\ \bar{y} \in A,\ s(y',\bar{y}) > s(y'',\bar{y})\ \forall\ y',y'':\ y'-y'' \in \text{int } C(\varepsilon)$;

(b) **order representation**: $\forall\ \bar{y} \in A,\ \{y \in Y:\ s(y,\bar{y}) \geq 0\} = \bar{y} + C(\varepsilon)$, where $A \geq \hat{Y}_o^{p\varepsilon}$, $\varepsilon \geq 0$ (if $\varepsilon = 0$, $\hat{Y}_o^{p0} = \hat{Y}_o^w$ and the case of weak efficiency is thus included). Then:

(1) For any $\bar{y} \in A$, each maximal point of $s(y,\bar{y})$ over $y \in Y_o$ is in $\hat{Y}_o^{p\varepsilon}$;

(2) For each $\hat{y} \in \hat{Y}_o^{p\varepsilon}$, there exists $\bar{y} \in A$ (e.g. $\bar{y} = \hat{y}$) such that $s(y,\bar{y})$ attains its maximum over $y \in Y_o$ at $\hat{y}$.

Function (10) has both properties (a,b) and is a simplest example of an order-consistent function; under additional assumptions, this function results also in a continuously controllable parameterization (Wierzbicki [1986]). However, Theorem 4 is much more general (it is valid also in Banach spaces provided Y_o is compact) and can be used with many other forms of order-consistent scalarizing functions.

In particular, consider a case when $\mathbb{Y} = \mathbb{R}^p$, C is defined as in (2) and a decision-maker (an user of a multiple criteria optimization package or a decision support system, a player in a multiple criteria game) is imprecise about his/her preferences on decision outcome values but can specify and modify fuzzy membership functions for his/her satisfaction with these outcome values. For fuzzy maximized outcomes, it is necessary to assume that the membership functions are strictly monotone, e.g. (see Granat at al., [1992]):

$$\text{(11a)} \quad \mu_i(y_i,\bar{y}_i,\bar{\bar{y}}_i) = 0.5(\tanh(\alpha_i(y_i - \beta_i)) + 1), \quad i = 1,\ldots p_1;$$

with:

$$\alpha_i = (\bar{\bar{a}}_i - \bar{a}_i)/(\bar{\bar{y}}_i - \bar{y}_i); \quad \beta_i = (\bar{\bar{a}}_i\bar{y}_i - \bar{a}_i\bar{\bar{y}}_i)/(\bar{\bar{a}}_i - \bar{a}_i);$$
$$\bar{a}_i = \text{artanh}(2\bar{\mu}_i - 1); \quad \bar{\bar{a}}_i = \text{artanh}(2\bar{\bar{\mu}}_i - 1)$$

where $\bar{y}_i$ is a controlling parameter interpreted here as a **reservation level** and associated with a given, low value of $\mu_i(\bar{y}_i) = \bar{\mu}_i$ (e.g. $\bar{\mu}_i = 0.1$) while $\bar{\bar{y}}_i$ is an additional controlling parameter interpreted as an **aspiration level**, associated with a given, high value of $\mu_i(\bar{\bar{y}}_i) = \bar{\bar{\mu}}_i$ (e.g. $\bar{\bar{\mu}}_i = 0.9$). Fuzzy minimized outcomes can be treated similarly. Outcomes numbered with $i = p_1 + 1,\ldots p$ can be treated as fuzzy equal[2] goals, with the following membership functions:

$$\text{(11b)} \quad \mu_i(y_i,\bar{y}_i,\bar{\bar{y}}_i) = 1 - \tanh^2(\alpha_i(y_i - \beta_i)), \quad i = p_1 + 1,\ldots p;$$

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[2] For fuzzy equal outcomes, special concepts of M-efficiency have been introduced, see Seo and Sakawa [1988]; such concepts are in fact similar to the proper efficiency with a prior bound for the cone C defined as in (2).

with:

$$\alpha_i = 2\bar{\bar{a}}_i/(\bar{\bar{y}}_i - \bar{y}_i);\ \beta_i = (\bar{\bar{y}}_i + \bar{y}_i)/2$$
$$\bar{\bar{a}}_i = \mathrm{artanh}((1 - \bar{\mu}_i)^{0.5});$$

where $\bar{y}_i$, $\bar{\bar{y}}_i$ are interpreted as **lower and upper reservation levels**, $\mu_i(\bar{\bar{y}}_i) = \mu_i(\bar{y}_i) = \bar{\mu}_i$ (e.g. $\bar{\mu}_i = 0.1$), while the aspiration level with $\mu_i(y_i) = 1$ is located at $y_i = \beta_i$. Because the function (11b) is monotone with respect to $|y_i - \beta_i|$, we can treat the vector membership function $\mu(y,\bar{y},\bar{\bar{y}})$ composed of $\mu_i(y_i,\bar{y}_i,\bar{\bar{y}}_i)$ for $i = 1,\dots p$ as a new outcome vector with the ordering implied by the cone $C = \mathbb{R}^p_+$; hence, the following scalarizing function can be applied:

$$(12)\quad s(\mu(y,\bar{y},\bar{\bar{y}})) = \min_{1\le i\le p} \mu_i(y_i,\bar{y}_i,\bar{\bar{y}}_i) + \varepsilon \sum_{i=1}^{p} \mu_i(y_i,\bar{y}_i,\bar{\bar{y}}_i)$$

This function satisfies the assumptions of Theorem 5 with $\mu(y,\bar{y},\bar{\bar{y}})$ treated as an outcome vector and $\bar{\mu} = \mu(\bar{y},\bar{y},\bar{\bar{y}})$ as the controlling parameter vector; thus, the conclusions of this theorem apply to this function as well. Moreover, the trade-off coefficients between the membership values are bounded by $1 + 1/\varepsilon$ as in Lemma 1. Because of the strict monotonicity properties of the function μ, it is also possible to show that the original outcomes $\hat{y}$ remain efficient at the maxima of (12), although the bounds on their trade-off coefficients are naturally different than $1 + 1/\varepsilon$.

The same conclusions could apply also for piece-wise linear fuzzy membership functions - provided such a function would be strictly monotone. For a classical definition of a membership function, interpreted as a multi-valued logical variable constrained between 0 and 1, there are no piece-wise linear functions that are strictly monotone for all $y_i \in \mathbb{R}^1$. Therefore, in order to preserve the conclusion that all maxima of (12) are also efficient in the original outcome space $\mathbb{Y}$, it is necessary to use the following **extended** (piece-wise linear) **membership functions**:

$$(13a)\quad \tilde{\mu}_i(y_i,\bar{y}_i,\bar{\bar{y}}_i) = \begin{cases} \gamma_i(y_i - \bar{y}_i), & y_i < \bar{y}_i \\ (y_i - \bar{y}_i)/(\bar{\bar{y}}_i - \bar{y}_i), & \bar{y}_i \le y_i \le \bar{\bar{y}}_i \\ 1 + \alpha_i(y_i - \bar{\bar{y}}_i), & \bar{\bar{y}}_i < y_i \end{cases},\quad i = 1,\dots p_1$$

with $\tilde{\mu}_i(\bar{y}_i,\bar{y}_i,\bar{\bar{y}}_i) = 0$, $\tilde{\mu}_i(\bar{\bar{y}},\bar{y}_i,\bar{\bar{y}}_i) = 1$, where α_i, $\gamma_i > 0$ can be defined in relation e.g. to given bounds $y_{i,lo}$, $y_{i,up}$ on y_i, with possibly additional requirements that $\tilde{\mu}_i$ is concave (which is essential in the case of linear models if the maximization of (12) is to be converted into a linear programming problem). In the case of fuzzy equal goals, the corresponding extended membership functions have the form:

$$(13b)\quad \tilde{\mu}_i(y_i,\bar{y}_i,\bar{\bar{y}}_i) = \begin{cases} \gamma_i(y_i - \bar{y}_i), & y_i < \bar{y}_i \\ (y_i - \bar{y}_i)/(\beta_i - \bar{y}_i), & \bar{y}_i \le y_i \le \beta_i \\ (\bar{\bar{y}}_i - y_i)/(\bar{\bar{y}}_i - \beta_i), & \beta_i < y_i \le \bar{\bar{y}}_i \\ \gamma_i(\bar{\bar{y}}_i - y_i), & \bar{\bar{y}}_i < y_i \end{cases}, \quad i = p_1+1,..p$$

with $\gamma_i > 0$, $\beta_i = 0.5(\bar{\bar{y}}_i + \bar{y}_i)$ and $\tilde{\mu}_i(\beta_i,\bar{y}_i,\bar{\bar{y}}_i) = 1$ corresponding to the aspiration level, while $\bar{y}_i$ and $\bar{\bar{y}}_i$ are interpreted as the lower and upper reservation levels with $\tilde{\mu}_i(\bar{y}_i,\bar{y}_i,\bar{\bar{y}}_i) = \tilde{\mu}_i(\bar{\bar{y}}_i,\bar{y}_i,\bar{\bar{y}}_i) = 0$. For interpretations in terms of multi-valued logic, the values of the functions $\tilde{\mu}_i$ can be projected again on the interval [0;1]. However, if used instead of μ_i in the maximization of the order-consistent scalarizing function (12), the extension of their values beyond this interval is essential for preserving the efficiency of the corresponding outcomes y_i in the frequent cases when the decision maker specifies either attainable aspiration levels $\bar{\bar{y}}_i$ or unattainable reservation levels $\bar{y}_i$[3].

3. Noncooperative Solutions in Multiple Criteria Games.

We shall show here how the concepts and properties of conical separation of sets apply to noncooperative solutions of multiple criteria games in normal form.

Denote **players** by capital letters, $K \in \mathbb{K} = \{A,B,..N\}$. The decisions x_K of each player K belong to a specified set of admissible decisions $X_K \subset \mathbb{X}_K$, where $\mathbb{X}_K$ is his/her decision space. The decision spaces can be, generally, any topological spaces; we assume only that all sets of admissible decisions are compact. Thus, the decisions of players can represent as well their mixed strategies, continuous probability distributions in games on a square, etc. We assume, however, that the sets X_K are **independent** of the decisions of other players. Thus, the admissible **multi-decisions** of all players, $x = (x_A,...x_K,...x_N)$, belong to the product of individual

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[3] In practical applications, such an extension turns out to be necessary even if the (theoretically) strictly monotone membership functions (11) are used, in order to preserve their practical monotonicity i.e. to guarantee that their derivatives remain above a given lower bound in an interval $<y_{i,lo};y_{i,up}>$.

admissible decision sets $X = \prod_{K=A}^{N} X_K$.

For an admissible multi-decision, a **multi-outcome** $y = (y_A, \ldots y_K, \ldots y_N)$ is defined. Individual outcomes $y_K \in \mathbb{Y}_K$ are vectors representing individual objectives - interests, values and criteria for each player. We assume that multi-outcomes are defined by a continuous **multi-outcome function** $f : X \rightarrow \mathbb{Y} = \prod_{K=A}^{N} \mathbb{Y}_K$. Thus, the individual outcomes are $y_K = f_K(x_A, \ldots x_K, \ldots x_N)$. In order to stress that the individual outcomes for a player depend both on his/her own decisions x_K and on the decisions of other players x_I, $I \neq K$, it is useful to denote $x_K = u_K$ and to introduce **individualized outcome functions** $\tilde{f}_K : X_K \times X \rightarrow \mathbb{Y}_K$ defined by $\tilde{f}_K(u_K, x) = f_K(x_A, \ldots x_J, u_K, x_L, \ldots x_N)$.

Given an admissible multi-decision $x \in X$, the **set of attainable outcomes** $Y_K(x)$ for player K is defined by all (not only by $u_K = x_K$) his/her own admissible responses or actions $u_K \in X_K$:

$$(14) \qquad Y_K(x) = \tilde{f}_K(X_K, x) = \{y_K \in \mathbb{Y}_K: \; y_K = \tilde{f}_K(u_K, x), \; u_K \in X_K\}$$

while the set of attainable multi-outcomes is $Y = f(X) = \tilde{f}(X, X)$.

Assume that individual outcome spaces $\mathbb{Y}_K = \mathbb{R}^{p_K}$ are finite-dimensional and domination structures in these spaces are implied by positive cones $C_K \subset \mathbb{Y}_K$ of the form (1) or (2). Correspondingly, various types of inequalities and efficiency can be defined in the individual outcome spaces as well as in the multi-outcome space $\mathbb{Y}$ where the positive cone is $C = \prod_{K=A}^{N} C_K$. Note that appropriate sets of efficient outcomes for an individual player K are dependent on the multi-decision x (of other players), e.g. in the case of properly efficient outcomes with a prior bound on trade-off coefficients:

$$(15) \qquad \hat{Y}_K^{p\varepsilon}(x) = \{\hat{y}_K \in Y_K(x): \; Y_K(x) \cap (\hat{y}_K + \text{int } C_K(\varepsilon)) = \emptyset\}$$

with $C_K(\varepsilon)$ defined as in (3) or (4); similarly we define $\hat{Y}_K^{w}(x)$ or $\hat{Y}_K(x)$.

A concept of a noncooperative solution of a multiple criteria game was introduced e.g. by Bergstresser and Yu [1977]: given a multi-decision $x \in X$, any player K would choose only such his/her decisions $u_K \in X_K$ that are in some sense efficient (or Pareto-optimal) with respect to his/her positive cone C_K. According to the three types of efficiency we can define three types of noncooperative solutions to a multiple criteria game.

The basic type shall be called here **noncooperative Pareto-Nash solutions**. The set of such solutions is defined by:

(16) $X^* = \{x^* \in X:\ y^*_K = \tilde{f}_K(x^*_K, x^*) \in \hat{Y}_K(x^*),\ \forall\ K = A,\ldots N\};\ Y^* = f(X^*)$

The most important for applications is another type called here **noncooperative properly Pareto-Nash solutions with a prior bound** (1 + 1/ε on trade-off coefficients). The set of such solutions is defined by:

(17) $X^{*p\varepsilon} = \{x^* \in X:\ y^*_K = \tilde{f}_K(x^*_K, x^*) \in \hat{Y}^{p\varepsilon}_K(x^*), \forall\ K=A,..N\};\ Y^{*p\varepsilon} = f(X^{*p\varepsilon})$

for $\varepsilon > 0$. For $\varepsilon = 0$, (14) defines also **noncooperative weakly Pareto-Nash solutions**. Because $\hat{Y}^{p\varepsilon}_K(x) \subseteq \hat{Y}_K(x) \subseteq \hat{Y}^w_K(x)$ for all $x \in X$, we have also $X^{*p\varepsilon} \subseteq X^* \subseteq X^{*w}$, $Y^{*p\varepsilon} \subseteq Y^* \subseteq Y^{*w}$. If the number of objectives would be $p_K = 1$ for each player K, all three types of noncooperative Pareto-Nash solutions would coincide with the noncooperative Nash equilibria.

An essential distinction between the sets (13), (14) for $p_K > 1$ and the sets of Nash equilibria in single criteria games is that the former often contain sets of continuum power while the latter are usually composed of discrete points (for nondegenerate games, see e.g. Shapley [1974]). Therefore, we must be often satisfied with some reasonable selections of points representing the sets (13) or, especially, (14). In order to facilitate such selections and to obtain characterizations of noncooperative Pareto-Nash solutions, the concept of a scalarizing function can be usefully adopted. For convex $Y_K(x)$ we can use linear separating functions to obtain the following theorem - see Wierzbicki [1990] for the proof of this particular formulation, while earlier results of this type were given by Yu [1973] or Zeleny [1976]:

Theorem 5. Suppose the sets $Y_K(x)$ are convex for all $x \in X$. Then, with $\lambda_K \in \Lambda^s_K = \text{int}\ \mathbb{R}^{p_K}_+$, the (bi-)linear scalarizing functions $s_K(y_K, \lambda_K) = \lambda^T_K y_K$ almost completely characterize the set Y^* of noncooperative Pareto-Nash outcomes: each such outcome corresponds to a noncooperative Nash equilibrium of a proxy single criteria game with payoffs equal to $\lambda^T_K y_K$ for some $\lambda_K \in \Lambda^{nw}_K = \mathbb{R}^{p_K}_+ \setminus \{0\}$ and any noncooperative Nash equilibrium of the proxy game with $\lambda_K \in \Lambda^s_K$ gives outcomes in the set Y^*. With $\lambda_K \in \Lambda^{nw}_K$, these functions completely characterize the set Y^{*w} of noncooperative weakly Pareto-Nash outcomes. With $\lambda_K \in \Lambda^{np\varepsilon}_K = \{\lambda_K \in \text{int}\ \mathbb{R}^{p_K}_+:\ \bar{\lambda}_{Ki} = \lambda_{Ki} / \sum_{j=1}^{p} \lambda_{Kj} \geq \delta_K = \varepsilon/(1+p_K\varepsilon)\}$, they completely characterize the set $Y^{*p\varepsilon}$ of noncooperative properly Pareto-Nash outcomes with a prior bound 1 + 1/ε on trade-off coefficients.

This theorem, however, is unsatisfactory because not only the assumption of convexity of $Y_K(x)$ is rather strong, but also the resulting characterizations are not continuously controllable. Therefore, an application of the concept of conical separation gives results that are not only stronger theoretically, but give also the possibility of influencing the selection of noncooperative Pareto-Nash outcomes continuously by small parameter changes; such results follow from the following application of Theorem 4 to the case of multiple criteria games:

Theorem 6. Suppose $C_K = \mathbb{R}_+^{p_K}$ - each player K = A,...N in a multiple criteria game maximizes his/her all objectives - and order-consistent achievement scalarizing functions $s_K(y_K,\bar{y}_K)$ of a form as in (10) with $C_K(\varepsilon)$ as in (4) are used for a proxy aggregation of these objectives.

(a) If x^*, $y^* = f(x^*)$ are a noncooperative Nash equilibrium of a proxy single criteria game with payoffs $s_K(f_K(x),\bar{y}_K)$ for any $\bar{y}_K \in \mathbb{R}^{p_K}$, then they are also a noncooperative properly Pareto-Nash solution (with the prior bound $1 + 1/\varepsilon$ on trade-off coefficients) of the multiple criteria game.

(b) If x^*, $y^* = f(x^*)$ are a noncooperative properly Pareto-Nash solution (with a prior bound) of the multiple criteria game, then there exists such $\bar{y} = (\bar{y}_A,\ldots\bar{y}_K,\ldots\bar{y}_N)$ - while it is sufficient to take $\bar{y}_K = y^*_K$ - that x^* and $y^* = f(x^*)$ correspond to a noncooperative Nash equilibrium of the proxy single criteria game.

Note that above Theorem applies also - with $\varepsilon = 0$ - to noncooperative weakly Pareto-Nash solutions. An outline of the proof of Theorem 6 (Wierzbicki, [1990]) is as follows.

Denote $r_K(y_k) = s_K(y_K,\bar{y}_K)$ with $y_K \in Y_K(x)$ defined as in (14). If x^*, $y^* = f(x^*)$ are a noncooperative Nash equilibrium of a proxy single criteria game with payoffs $s_K(f_K(x),\bar{y}_K)$, then r_k attains its maximum over $y_K \in Y_K(x^*)$ at y^*_K. Since r_K is strictly monotone in the sense of the inequality implied by the cone int $C(\varepsilon)$, its maximal point is efficient with respect to this cone, that is, properly efficient with a prior bound on trade-off coefficients, $y^*_K \in \hat{Y}^{p\varepsilon}_K(x^*)$. This applies to any K; hence, $x^* \in X^{*p\varepsilon}$ defined as in (17).

Conversely, let $x^* \in X^{*p\varepsilon}$, $y^*_K \in \hat{Y}^{p\varepsilon}_K(x^*)$ and take $\bar{y}_K = y^*_K$, for all K. Then the sets $\bar{y}_K$ + int $C(\varepsilon)$ and $Y^{p\varepsilon}_K(x^*)$ do not intersect and can be strictly separated at $\bar{y}_K = y^*_K$ by the function $r_K(y_k) = s_K(y_K,\bar{y}_K)$. According to Lemma 2, this function attains its maximum over $y_K \in Y_K(x^*)$ at y_K^*. This applies to any K; thus, x^*, $y^* = f(x^*)$ are a noncooperative Nash equilibrium of a proxy single criteria game with payoffs $s_K(f_K(x),\bar{y}_K)$.

When using Theorem 6, we should be also able to compute noncooperative Pareto-Nash solutions parameterized by aspiration points $\bar{y}_K$. However, computing Nash equilibria - especially in non-convex cases - is a difficult task in itself. On the other hand, there exists a technique - as suggested e.g. by Aubin [1979] - of converting the computation of Nash equilibria to a mathematical programming problem, provided we are ready to use nondifferentiable optimization even for the case of differentiable payoffs. But the proxy payoff functions $s(f_K(x),\bar{y}_K)$ in Theorem 6 are anyway nondifferentiable and the nondifferentiable optimization algorithms were considerably advanced in the last decade - see e.g. Kiwiel [1985].

A multi-decision x^* is a Nash equilibrium of the proxy scalarized game, given a controlling parameter vector $\bar{y} = (\bar{y}_A,\ldots\bar{y}_K,\ldots\bar{y}_N)$, if and only if:

$$(17a) \qquad \omega(x^*,\bar{y}) = \min_{x\in X} \omega(x,\bar{y}) = 0$$

where:

$$(17b) \qquad \omega(x,\bar{y}) = \max_{u\in X} \sum_{K=A}^{N} (s_K(\tilde{f}_K(u_K,x),\bar{y}_K) - s_K(f_K(x),\bar{y}_K))$$

4. Applications of Multiple Criteria Games.

Most of existing applications of multiple criteria games are related to concepts of cooperative solutions in such games - see Bergstresser and Yu [1977], Seo and Sakawa [1988], Bronisz et al. [1989]. However, it can be argued that in order to reasonably model and analyze real-life game-like multiple criteria problems, the analysis of possible noncooperative solutions should precede the proposals for cooperative ones. Such an analysis should also include the possibility of special disequilibrium **processes of conflict escalation** type.

The computation of an equilibrium say, of Nash noncooperative type in a single criteria game, is based on the assumption that preferences (payoff functions) of all players are known to each other. In a multiple criteria game we must still assume that at least the type of objectives and the corresponding reference points for each player are known. However, in real-life situations even such information would be strategically guarded; any player would have to guess and assume the objectives and the reference points of other players. If some of the players guessed incorrectly, their decisions would not correspond to any equilibrium and the outcomes of such decisions might be worse than expected. In a repetitive game, this could lead to revised assessments of the preferences of other players, with a psychological tendency to blame them and to retaliate - while the original lack of information should be rather blamed and the retaliation leads to a process of conflict escalation.

An example of such an analysis for a multiple criteria model of a conflict over fishery rights was given in Wierzbicki [1990]. The problem of possible conflicts arising when fishing fleets of various countries compete should be actually modeled by a dynamic, nonlinear and multi-objective, stochastic game. As a dynamic single-objective game it was investigated e.g. by Hamalainen, [1976]; various experimental games with computer simulation have been also used to study the problem. However, in order to see the main reasons of a possible conflict escalation, a rather simplified model of such a game is sufficient, provided it stresses the multi-objective character of the problem.

Consider only two countries A and B with fishing fleets of sizes x^A and x^B. Suppose only such varieties of sea fish are considered that spawn in the rivers either of country A or of country B; therefore, it makes sense to speak about stocks z_A and z_B of fish originating in either of these countries and returning for breeding to the coastal waters and rivers of this country. In the open sea both countries can catch both types of fish; fishing in coastal waters by another country is assumed (for simplicity) to be prohibited.

A very simplified but nonlinear model of fish catches assumes that a part of the joint stock $z_A + z_B$ can be caught in the open sea proportionally to the relative number of fishing boats of respective country, but with catching intensity decreasing exponentially when the joint number of boats increases. A part of the remaining fish of the type z_A can be then caught in the coastal waters of country A, again with exponentially decreasing catching intensity when the number of boats increases; similarly for country B. We assume that country A (similarly B) can influence - e.g. through various fiscal incentives - the number of boats $x_{A,1}$ sent to open sea and $x_{A,2}$ to its coastal waters, where:

$$\text{(18)} \quad 0 \le x_{A,1}; \quad 0 \le x_{A,2}; \quad x_{A,1} + x_{A,2} \le x^A;$$
$$0 \le x_{B,1}; \quad 0 \le x_{B,2}; \quad x_{B,1} + x_{B,2} \le x^B$$

The catch of country A (similarly for B) is approximated by:

$$(19)\quad c_A = c_{A,o} + c_{A,c}$$

where $c_{A,o}$ is the catch in the open sea:

$$(20)\quad c_{A,o} = \frac{x_{A,1}}{x_{A,1} + x_{B,1}} (1 - \exp(-a(x_{A,1} + x_{B,1}))) (z_A + z_B)$$

Thus, only $z_A \exp(-a(x_{A,1} + x_{B,1}))$ of the initial stock z_A remains (where a is an appropriate coefficient) and returns to the coastal waters. Therefore, the catch $c_{A,c}$ in the coastal waters is:

$$(21)\quad c_{A,c} = (1 - \exp(-ax_{A,2})) \exp(-a(x_{A,1} + x_{B,1})) z_A$$

Again, only $z_A \exp(-a(x_{A,1} + x_{A,2} + x_{B,1}))$ of the initial stock z_A remains for breeding. The breeding coefficient $r(z_A \exp(-a(x_{A,1} + x_{A,2} + x_{B,1})))$ is actually a function of this number and of various other random factors; for simplicity, we assume it to be a constant r. Thus, the fish stock next fishing season is:

$$(22)\quad z_A^+ = r \exp(-a(x_{A,1} + x_{A,2} + x_{B,1})) z_A$$

Actually, Eq. (22) and a similar one for the stock z_B should be considered as basic dynamic equations for a long-term dynamic model of the game. On the other hand, we can also consider the stock in next fishing season as one of the objectives of countries A or B in a static, repetitive game. We could formulate various objectives for such a game; but in order to illustrate the issues of conflict escalation, it is sufficient to analyze two objectives for each country.

The first objective for country A (similarly for country B) represents the interests of fishers, who must maintain their fleets; thus, they are primarily interested in the level of fish catch:

$$(23)\quad y_{A,1} = c_A$$

The second objective represents more aggregated interests of country A, including the catch, but also subtracting the relative cost of fishing and taking into account the future value of fish stock expected for the next fishing season:

$$(24)\quad y_{A,2} = c_A - p(x_{A,1} + x_{A,2}) + bz_A^+$$

where p, b are appropriate coefficients (similarly for country B).

When analyzing such a game, it can be shown that fishing in own coastal waters might be profitable for individual fishermen but not for the entire fisher community (fish comes to the coastal waters depleted by the catch in open sea where also the other country is fishing; thus, the total catch decreases if a boat is fishing in coastal waters instead of in open sea) and certainly not good for the aggregated interests of the country. This might lead to internal conflicts; in order to concentrate on external conflicts, let us assume $x_{A,2} = x_{B,2} = 0$.

For the remaining part of the game, the sets of noncooperative Pareto-Nash solutions can be computed for assumed parameter values. These solutions were computed in Wierzbicki [1990] not according to the characterization (17a,b) but with the help of penalty functions for proxy constraints (which is actually an equivalent computational approach under additional assumptions). The problem of representing the results (sets in the four-dimensional space of objectives of countries A and B) can be resolved by drawing on the plane $(y_{A,2}, y_{B,2})$ the level sets of corresponding values $(y_{A,1}, y_{B,1}) = (c_A, c_B)$. An example of such results is given in Fig. 1.

The sets of Pareto-Nash noncooperative equilibria are denoted in Fig. 1 by continuous lines of level sets for c_A, c_B. There are two such sets: a bigger one at reasonably high levels of aggregated objectives $y_{A,2}$, $y_{B,2}$ and a smaller one at much lower levels of these objectives (and higher catches; this smaller set corresponds to overfishing the sea). However, even when starting at a point in the bigger set, a conflict escalation process P1 - P2 - P3 - P4 - P5 can develop. Suppose the fishers of the country B wanted a bigger catch than at the point P1 and the country could not control them; the point thus shifted to P2. All depends now on the reaction of country A. If it tries to communicate with country B, perhaps with the help of an international agency, conflict escalation might be prevented. If, instead, its fishers insist on catching at least as much fish as country B and send more boats, the solution shifts again to disequilibrium P3. After further such "adjustments", the process ends at an equilibrium point P5 - however, at much lower level of aggregated objectives $y_{A,2}$, $y_{B,2}$ and with few fish remaining for next season.

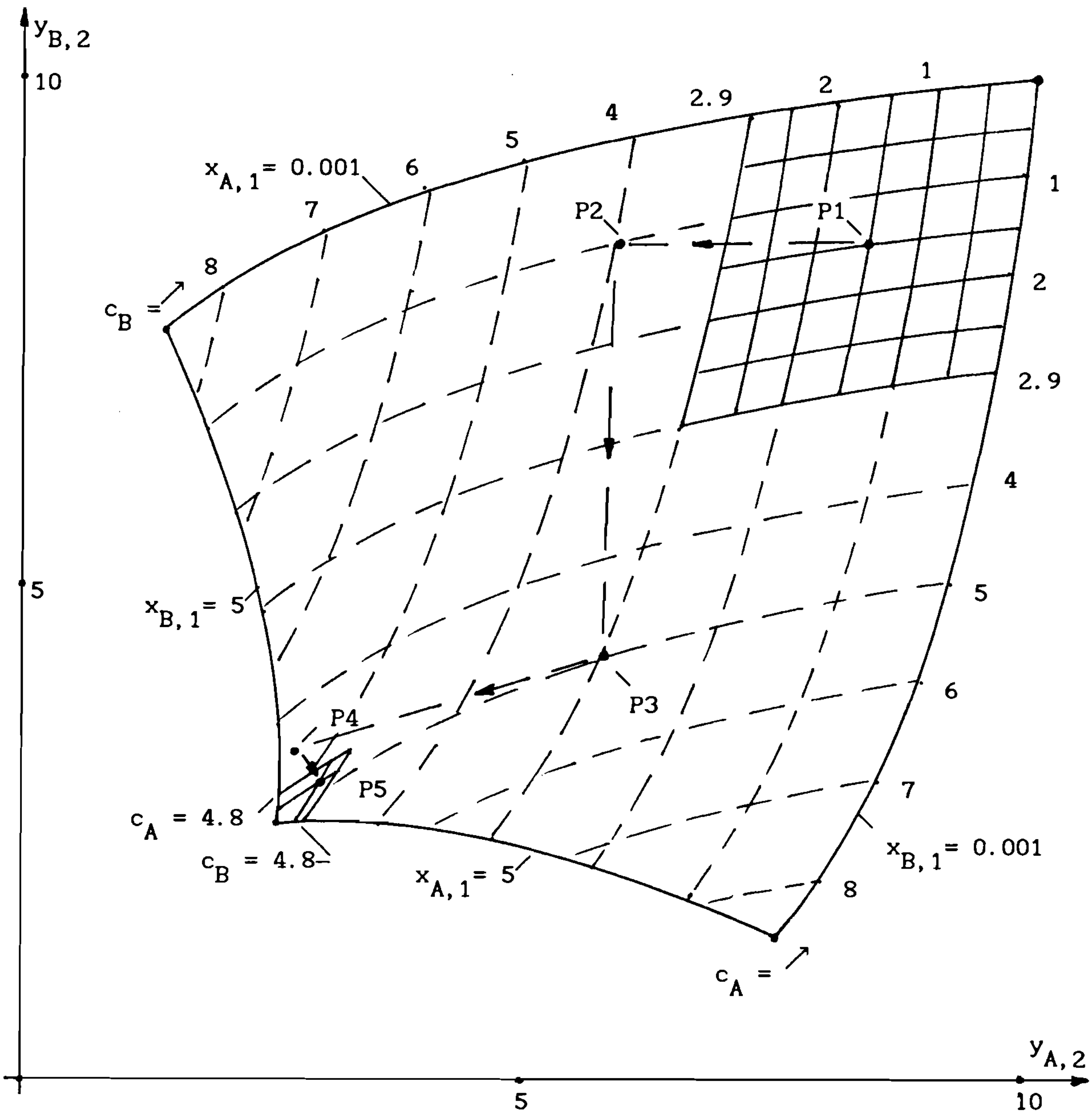

Fig. 1. An example of a conflict escalation process for the fishery game (with a = 0.4, r = 5, $z_A = z_B = 5$, $x^A = x^B = 5$, p = 0.5 and b = 0.4).

Another example of an educational study of conflict escalation processes is an experimental, multiple criteria game "Humble Shall be Rewarded" (Wierzbicki [1992]) with a simple mechanics, where two players show at one time a selected number from 0 to 5 fingers or plastic chips, for 50 rounds - or less, depending on specific stopping rules. The game has, however, a rather complicated payoff structure with two apparent objectives: winning points and money (small financial rewards are given to the players by the organizer of the game) which are not directly correlated. The game has also many rationality traps and novice players might easily start a conflict escalation process; but the stopping rules penalize heavily such players. To play the game well, players must change their strategic perspective, discover and concentrate on other, substitute objectives hidden in the stopping rules of the game; they must learn to avoid conflict escalation, to develop implicit cooperation, to communicate with the opponent (by the single allowed medium - the sequence of numbers of chips played). The game is thus rather challenging - since changing a strategic perspective is a difficult task for most players.

Experiments with this game have shown that there are several distinct strategic perspectives with corresponding distinct substitute objectives and types of strategies that might be discovered by players when they improve their skills. The more creative and adaptable players learn sooner, thus the game might be a good test of such abilities. The game has analogies in real life (honor versus money) and provides a good educational medium for teaching that a change of perspective - which consist in changing the hierarchy of objectives or even adopting new objectives - is often necessary when solving difficult problems, particularly when they include the danger of conflict escalation. Thus, the game is a practical tool for enlarging habitual domains in the sense of Yu [1990].

5. Conclusions.

The paper shows that it is possible to characterize Pareto-Nash noncooperative solutions of a multiple criteria game as Nash equilibria of a proxy single criteria game with payoffs equal to parameterized scalarizing functions, even in nonconvex or discrete cases. The best type of such functions are order-consistent achievement scalarizing functions: beside an appropriate type of monotonicity, they have also a special conical separation property. This characterization makes it possible to compute Pareto-Nash noncooperative solutions by nondifferentiable optimization or other equivalent algorithms and to analyze examples of multiple criteria conflict situations.

Because of the essential multiplicity of noncooperative Pareto-Nash solutions, conflict escalation processes can easily occur in multiple criteria games, particularly if the players do not communicate. When avoiding the danger of conflict escalation, a change of strategic perspective - involving different hierarchies of objectives or even adopting new objectives - is often necessary.

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