Working Paper

Exact Observability of Nonstationary Hyperbolic Systems with Scanning Finite-Dimensional Observations

A.Yu. Khapalov

WP-92-74
October 1992

IIASA International Institute for Applied Systems Analysis ▫ A-2361 Laxenburg ▫ Austria
Telephone: +43 2236 715210 ▫ Telex: 079 137 iiasa a ▫ Telefax: +43 2236 71313

Exact Observability of Nonstationary Hyperbolic Systems with Scanning Finite-Dimensional Observations

A. Yu. Khapalov

WP-92-74
October 1992

IIASA International Institute for Applied Systems Analysis ◻ A-2361 Laxenburg ◻ Austria
Telephone: +43 2236 715210 ◻ Telex: 079 137 iiasa a ◻ Telefax: +43 2236 71313

Foreword

The paper considers the problem of exact observability for nonstationary linear hyperbolic systems with nonstationary interior observations in the case when outputs lie in $L^{\infty}_{n+1}(0,\theta)$, where n stands for the spatial dimensionality of the system in question. Three types of scanning observations are considered and the existence of measurement curves and set-valued maps for sensors that ensure exact observability is established. The proposed method is based on a priori energy estimates of instantaneous type for solutions. It allows to obtain required measurement curves that are continuous in $[0,\theta]$ or measurement maps that are lower semi-continuous in $[0,\theta]$ with respect to Lebesgue measure.

Exact Observability of Nonstationary Hyperbolic Systems with Scanning Finite-Dimensional Observations.

A.Yu. Khapalov
International Institute for Applied Systems Analysis
A-2361 Laxenburg , Austria

Key words. exact observability, nonstationary hyperbolic systems, scanning observations.

AMS(MOS) subject classifications. primary 93B07; secondary 93C20.

1. Introduction, statement of problem.

Let Ω be a bounded domain of an n-dimensional Euclidean space R^n with a boundary $\partial\Omega$. We consider the following homogeneous problem for the nonstationary hyperbolic equation:

(1.1)
$$y_{tt}(x,t) = A(t)y(x,t),$$

$$t \in T = (0,\theta)\ ,\ ; x \in \Omega \subset R^n, \quad x = (x_1,\ldots,x_n)', \quad Q = \Omega \times T, \quad \Sigma = \partial\Omega \times T,$$

$$y \mid_\Sigma = 0,$$

$$y \mid_{t=0} = y_0, \qquad y_t \mid_{t=0} = y_1$$

with unknown initial conditions y_0 and y_1. We assume that $A(t)$ is *time-varying* and uniformly coercive,

$$(1.2) \qquad A(t) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x,t) \frac{\partial}{\partial x_j} - \sum_{i=1}^{n} a_i(x,t) \frac{\partial}{\partial x_i} - a(x,t),$$

$$\nu_1 \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\, \xi_i \xi_j \le \nu_2 \sum_{i=1}^{n} \xi_i^2 \quad \text{for } \forall \xi_i \in R^1, \quad x \in Q,$$

$$a_{ij}(x,t) = a_{ji}(x,t), \ i,j = 1,\dots,n, \qquad \nu_1 = \text{const} > 0, \quad \nu_2 = \text{const} > 0.$$

We further assume that all the coefficients in (1.2) are sufficiently smooth and

$$(1.3) \qquad |\, a_{ijt},\ a_i,\ a,\ a_{ijtt},\ a_{ijx} \,| \le \mu_1, \quad \mu_1 = \text{const} > 0.$$

The main aim of the paper is to study the problem of exact observability for the system (1.1)-(1.3) with respect to the *"energy" space* $H = H_0^1(\Omega) \times L^2(\Omega)$ in the case when observations are *finite dimensional at every instant of time.* The latter is typical in physical situations.

We shall consider three types of interior scanning observations that may be represented in the following general form:

$$(1.4) \qquad z(t) = \mathbf{G}(t) \begin{pmatrix} y_x(\cdot,t) \\ y_t(\cdot,t) \end{pmatrix}, \quad t \in T.$$

Here $z(\cdot)$ is an $(n+1)$-dimensional output, $y_x = (y_{x_1}, \dots, y_{x_n})'$ and $\mathbf{G}(t)$ stands for an observation operator. Each type of observations requires a corresponding regularity of solutions. In order to provide it, in every particular case we introduce the *set of admissible initial conditions* that enables to ensure a needed smoothness. We denote this set by H_0, $(H_0 \subset H)$ and endow H_0 with the "energy" metric, namely,

$$E^{1/2}(y(\cdot,0)) = \Big(\int_{\Omega} (y_x^2(x,0) + y_t^2(x,0))\, dx\Big)^{1/2}, \quad y_x^2 = \| y_x \|_{R^n}^2 .$$

For the sake of simplicity we shall use below the notation $|\,(\cdot)\,|$ for $\|\,(\cdot)\,\|_{R^n}$.

The system described by (1.1)-(1.4) is said to be *exactly observable* (see [2,4]) on $\{H_0,\ B\}$, where B stands for the space for outputs, if

$$(1.5) \qquad \exists\gamma > 0 \text{ such that } \| \mathbf{G}(\cdot)(y_x(\cdot,\cdot), y_t(\cdot,\cdot))' \|_B \geq \gamma\, E^{1/2}(y(\cdot,0))$$

for any y that satisfies (1.1)-(1.3) and such that $\{y(\cdot,0), y_t(\cdot,0)\} \in H_0$.

The problem of exact observability for stationary hyperbolic systems mostly with different types of infinite-dimensional (if the dimensionality of x is higher than 1) observations at every instant of time has been studied by a number of authors by means of the Hilbert Uniqueness Method introduced by Lions [9, 10]. This method supposes that all the spaces in question are Hilbert. The case of finite-dimensional at every instant of time interior observations has been studied in the stationary setting by Lop Fat Ho [11] for the one-dimensional wave equation with outputs in $L^2(T)$ (although [11] mainly deals with outputs from $L^2((x_1, x_2) \times T)$). El Jai and others [3] have discussed how HUM can be linked with the pointwise sensor structure. Due to finite speed of propagation, observations received from the stationary sensors are able to provide the system with exact observability only if the duration of observations is big enough.

In the present paper we establish the existence of *scanning observations* (they are natural when studying nonstationary processes) of three types with $(n+1)$*-dimensional images* at every instant of time that make the *nonstationary* system (1.1)-(1.3) (in general, of arbitrary spatial dimensionality) be exactly observable when the space for outputs is $L^\infty_{n+1}(T) = \underbrace{L^\infty(T) \times \ldots \times L^\infty(T)}_{n+1}$ (e.g., not Hilbert and not separable). This space is rather natural in applications when working with measurement data. The applied techniques is based on the "energy" estimates for solutions (we remind that the nonstationary system (1.1) is *not conservative*) that are of *instantaneous type.* This approach has been used for the wave equation in [6] and for parabolic systems in [7]. Another approach to observability problem with scanning observations has been suggested for parabolic systems by Martin [12], who essentially used the stationary property of the system in question.

The paper is organized as follows. Section 2 deals with a number of preliminary results concerning the correctness of observations. We begin then (Section 3) by studying the one dimensional nonstationary hyperbolic equation with scanning pointwise observations:

$$(1.6) \qquad z(t) = \begin{pmatrix} y_x(\bar{x}^0(t), t) \\ y_t(\bar{x}^1(t), t) \end{pmatrix}, \quad t \in [0, \theta].$$

In this case measurements are taken in the domain Ω along prescribed measurable curves $\bar{x}^i(\cdot)$, $i = 0, 1$. In order to provide a necessary smoothness of solutions, we assume that

$$(1.7) \qquad H_0 = (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega).$$

We prove the existence of measurement curves in (1.6) that make the system (1.1)-(1.3), (1.7) be exactly observable. These curves may be selected to be continuous in $[0,\theta)$.

In Section 4 we consider observations of the following type:

$$(1.8) \qquad z(t) = \begin{pmatrix} \text{meas}^{-1}\{S_{h_0(t)}(\bar{x}^0(t))\cap\Omega\}\int_{S_{h_0(t)}(\bar{x}^0(t))\cap\Omega} y_x(x,t)\,dx \\ \text{meas}^{-1}\{S_{h_1(t)}(\bar{x}^1(t))\cap\Omega\}\int_{S_{h_1(t)}(\bar{x}^1(t))\cap\Omega} y_t(x,t)\,dx \end{pmatrix}, \quad t\in[0,\theta],$$

where $S_{h_i(t)}(\bar{x}^i(t))$, $i=0,1$ is the Euclidean neighborhood in R^n of radius $h_i(t)$ of point $\bar{x}^i(t)$,

$$S_{h_i(t)}(\bar{x}^i(t)) = \{x \mid \| x-\bar{x}^i(t)\|_{R^n} < h_i(t)\},$$

the measurable functions $\bar{x}^i(t)$, $h_i(t)>0$ are given. We extend the results of Section 3 for the case of the observations (1.8) when $x\in R^1$ under the same regularity conditions for solutions. However, we do not manage here to establish the required existence results for the general case when the spatial dimensionality of the system in question is higher than 1 (at least under the assumed regularity). In order to do the latter, in Section 5 we introduce *generalized scanning observations,* as follows:

$$(1.9) \qquad z(t) = \begin{pmatrix} \frac{1}{\text{meas}\{\Omega_1^0(t)\}}\int_{\Omega_1^0(t)} v_1^0(x,t)y_{x_1}(x,t)dx \\ \vdots \\ \vdots \\ \frac{1}{\text{meas}\{\Omega_n^0(t)\}}\int_{\Omega_n^0(t)} v_n^0(x,t)y_{x_n}(x,t)dx \\ \frac{1}{\text{meas}\{\Omega^1(t)\}}\int_{\Omega^1(t)} v^1(x,t)y_t(x,t)dx \end{pmatrix}, \quad t\in[0,\theta],$$

where $\{\Omega_p^0(t)\}_{p=1}^n$, $\Omega^1(t)$ are given set-valued maps from $[0,\theta]$ into the set of all measurable subsets of Ω,

$$t\to\Omega_p^0(t)\subset\Omega,\ \ t\in[0,\theta],$$

$$t\to\Omega^1(t)\subset\Omega,\ \ t\in[0,\theta],$$

and $\{v_p^0(x,t)\}_{p=1}^n$, $v^1(x,t)$ are given measurable functions of sign type, so as

$$| v_p^0(x,t) |, \ | v^1(x,t) | = +1 \quad a.e. \quad \text{in } \Omega, \ t \in [0,\theta], \quad p = 1, \dots .$$

It is assumed here and everywhere below that, if one or more sets in (1.9) are of zero-measure at some instants of time, the corresponding coordinates in (1.9) are omitted (we do not make observations at these instants) and substituted for zero. The last type of observations may be considered as a next step in the *generalization* of scanning observations that corresponds the regularity of the system (1.1)-(1.3) in the general case, when measurement curves can not be well-defined. The maps $\{\Omega_p^0(\cdot)\}_{p=1}^n$, $\Omega^1(\cdot)$ play here the role of measurement curves. However, in order to ensure the enclosure of the range of observation operator in (1.9) into $L_{n+1}^\infty(T)$, we set

$$H_0 = H \bigcap \{\{y(\cdot,0), y_t(\cdot,0)\} \mid y_{x_p}, \ y_t \in L^\infty(Q), \ p = 1, \dots, n\}. \tag{1.10}$$

We study properties of the observations (1.9) and establish the existence of set-valued maps $\{\Omega^0(\cdot)\}_{p=1}^n$, $\Omega^1(\cdot)$ that make the system (1.1)-(1.3), (1.10), (1.9) be exactly observable. These maps may be selected to be lower semi-continuous with respect to Lebesgue measure in $[0,\theta]$.

2. Preliminary results.

It is known that under the conditions (1.2)-(1.3) the system (1.1) admits solutions from the "energy" class [8]:

$$\begin{pmatrix} y \\ y_t \end{pmatrix} \in C([0,\theta]; H)$$

and the "energy" estimate holds,

$$\int_\Omega (y^2(x,t) + y_t^2(x,t) + y_x^2(x,t))dx \ \le \ c(t) \int_\Omega (y_0^2(x) + y_{0x}^2(x) + y_1^2(x))dx.$$

Here $c(t)$ is a monotone positive non-decreasing function that is defined only by the constants from (1.2)-(1.3) [8]. In turn, this implies the existence of such a constant $\hat{c} > 0$ that

$$(2.1) \qquad E^{1/2}(y(\cdot,0)) \le \hat{c}\, E^{1/2}(y(\cdot,t)), \;\; \forall t \in [0,\theta]$$

for any solution of the system (1.1), since, due to Poincaré's inequality,

$$\| y(\cdot,t) \|_{H_0^1(\Omega)} \le \text{const} \, (\int_\Omega y_x^2(x,t)dx)^{1/2},$$

the standard norm in $H_0^1(\Omega) \times L_2(\Omega)$ is equivalent to the "energy" one.

Thus, we may conclude that if all the sets $\{\Omega_p^0(t)\}_{p=1}^n$, $\Omega^1(t)$, $t \in [0,\theta]$ are measurable, the observations of type (1.9) are well-defined at every instant of time (for (1.8) it is always so) under the general assumptions of Section 1. In order to ensure the enclosure of the corresponding ranges of the observation operators in (1.8), (1.9) into $L_{n+1}^\infty(T)$, we have to add additional requirements on the dynamics of measurement maps that will be specified in Sections 4, 5.

Assume now that $x \in \Omega \subset R^1$ and $H_0 = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. In addition to (1.2)-(1.3), we assume that

$$(2.2) \qquad \Omega = (0,1), \quad | \, a_{1t}, \;\; a_t \, | \le \mu_2, \quad \mu_2 = \text{const} > 0.$$

In this case [8],

$$y \in H^2(Q),$$

and, consequently, by virtue of the embedding theorems [13], $H^1(\Omega) \subset C(\bar{\Omega})$. The latter implies that both $y_x(x,t)$, $y_t(x,t)$ are of the Carathéodory type and for any pair of measurable functions $\{\bar{x}^0(t), \bar{x}^1(t)\}$, $t \in T$ the observations (1.6) are well-defined [5, 1]. We recall further that for any solution y of (1.1)-(1.3), (1.7), (2.2) we have the following estimate [8]:

$$\| y_{xx}(\cdot,t) \|_{L^2(\Omega)}^2 + \| y_{tt}(\cdot,t) \|_{L^2(\Omega)}^2 + \| y_{tx}(\cdot,t) \|_{L^2(\Omega)}^2 \le$$

$$(2.3) \qquad \le c_1(\theta) \, (\| y_0 \|_{H^2(\Omega)}^2 + \| y_1 \|_{H_0^1(\Omega)}^2) \quad \text{a.e. in } T,$$

where $c_1(\theta)$ also depends upon $\nu_1,\ \nu_2,\ \mu_1,\ \mu_2$. The last estimate yields

$$\begin{pmatrix} y_x \\ y_t \end{pmatrix} \in L_2^\infty(T; C(\bar{\Omega}))$$

and

$$\| y_x(\bar{x}^0(\cdot),\cdot) \|^2_{L^\infty(T)} + \| y_t(\bar{x}^1(\cdot),\cdot) \|^2_{L^\infty(T)} \le \text{const } (\| y_0 \|^2_{H^2(\Omega)} + \| y_1 \|^2_{H_0^1(\Omega)}) \tag{2.4}$$

for any pair of measurable curves $\{\bar{x}^0(\cdot),\ \bar{x}^1(\cdot)\}$.

Thus, we conclude that the observations of type (1.6) are well-defined under the assumptions (1.2), (1.3), (2.2), (1.7), so as the range of the observation operator (1.6) belongs to $L_2^\infty(T)$.

Remark 2.1. Set

$$\mathbf{K} = \mathbf{G}(\cdot)\mathbf{S}(\cdot), \quad \mathbf{K}: H_0 \to L^\infty_{n+1}(T),$$

where

$$\mathbf{S}(t)\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} y_x(\cdot,t) \\ y_t(\cdot,t) \end{pmatrix}.$$

Then we may say that in the present paper we study the property of *bounded invertibility* of the operator $\mathbf{K}$ with respect to the "energy" norm and the constant γ in (1.5) may be selected as follows

$$\gamma = \frac{1}{\| \mathbf{K}^{-1} \|}.$$

3. The one dimensional hyperbolic equation: pointwise observations.

In this section we consider the system (1.1)-(1.3), (1.6) assuming that the conditions (1.7), (2.2) are fulfilled. The main result here is the following

Theorem 3.1. There exist measurement curves $\{x^0(\cdot),\ x^1(\cdot)\}$ that provide the system (1.1)-(1.3), (1.7), (2.2), (1.6) with exact observability.

We begin the proof of Theorem 3.1 with an auxiliary result. Namely, we show how for any given solution y one can construct measurement curves that ensure the inequality (1.5) for this particular solution. In addition we show that, in order to do this, it is sufficient to make observations in any a priori given subset (of positive measure) of T.

Let y be an arbitrary solution of system (1.1)-(1.3), (1.7), (2.2) and $\beta > 0$ be given. Let r be an arbitrary measurable subset of the time interval $(0, \theta)$ such that $\text{meas}\{r\} > 0$. We set next

$$e_0 = \{(x,t) \in \Omega \times r \mid y_x^2(x,t) \geq \underset{(x,t)\in\Omega\times r}{\text{vrai max}}\, y_x^2(x,t) - \beta\}, \tag{3.1}$$

$$e_1 = \{(x,t) \in \Omega \times r \mid y_t^2(x,t) \geq \underset{(x,t)\in\Omega\times r}{\text{vrai max}}\, y_t^2(x,t) - \beta\}. \tag{3.2}$$

It is not hard to see that both sets e_0 and e_1 are non-empty and have positive measures for any $\beta > 0$.

We introduce now two set-valued maps as follows

$$t \to F_0(t) = \{x \mid (x,t) \in e_0\}, \quad t \in r^0,$$

$$t \to F_1(t) = \{x \mid (x,t) \in e_1\}, \quad t \in r^1,$$

where $r^i = \;\mathit{dom}\, F_i(t) = \{t \mid \{x|\,(x,t) \in e_i\} \neq \emptyset\}, \; i = 0,1.$

By virtue of the embedding theorems [13], the sets $F_0(t)$ and $F_1(t)$ are closed almost everywhere accordingly in r^0 and r^1. Applying the theorem on measurable selection [5, 1], we conclude that there exists such a pair of measurable functions $\{\bar{x}^i(t), t \in r^0;\; \bar{x}^1(t), t \in r^1\}$ that

$$\bar{x}^0(t) \in F_0(t) \quad a.e. \text{ in } r^0, \tag{3.3}$$

$$\bar{x}^1(t) \in F_1(t) \quad a.e. \text{ in } r^1. \tag{3.4}$$

We may easily extend the above functions for the whole interval T (assuming only $\bar{x}^0(t)$, $\bar{x}^1(t) \in \Omega$), so as

$$(3.5) \qquad y_x^2(\bar{x}^0(t),t) \ \geq \operatorname*{vrai\,max}_{(x,t)\in\Omega\times r} y_x^2(x,t) \ - \ \beta, \ \ a.e. \ in \ r^0,$$

$$(3.6) \qquad y_t^2(\bar{x}^1(t),t) \ \geq \operatorname*{vrai\,max}_{(x,t)\in\Omega\times r} y_t^2(x,t) \ - \ \beta, \ \ a.e. \ in \ r^1.$$

Note next that

$$(3.7) \qquad \operatorname*{vrai\,max}_{t\in r} E(y(\cdot,t)) \ = \ \operatorname*{vrai\,max}_{t\in r} \int_\Omega (y_x^2(x,t)+y_t^2(x,t))dx \ \leq$$

$$\leq \ \text{meas}\,\{\Omega\} \operatorname*{vrai\,max}_{(x,t)\in\Omega\times r} (y_x^2(x,t)+y_t^2(x,t)).$$

Finally, combining (2.1), (3.5)-(3.7) yields the needed estimate:

$$(3.8) \qquad E^{1/2}(y(\cdot,0)) \ \leq \ \hat{c}\,\text{meas}^{1/2}\{\Omega\}\,(\| \, y_x(\bar{x}^0(\cdot),\cdot) \, \|^2_{L^\infty(r^0)} + \| \, y_t(\bar{x}^1(\cdot),\cdot) \, \|^2_{L^\infty(r^1)} \ +2\beta)^{1/2}.$$

Thus, we arrive at

Lemma 3.1. Given solution y of the system (1.1)-(1.3), (1.7), (2.2) and given $\beta > 0$ and $r \subset T$, meas $\{r\} > 0$, an arbitrary pair of measurable curves $\{\bar{x}^0(t), \bar{x}^1(t)\}$, $t \in T$ constructed along the lines (3.1)-(3.4) ensures the estimate (3.8).

Remark 3.1. We stress that the estimate (3.8) is uniform over the set of all solutions of the system in question.

Proof of Theorem 3.1. For any solution of the system (1.1)-(1.3), (1.7), (2.2) Lemma 3.1 allows us to construct (on an arbitrary subset of T) measurement curves that ensure (3.8). This, taking into account Remark 3.1, gives us a hint to seek required curves for the set of all solutions among the ones that consist of non-overlapping pieces each of which, in turn, ensures (3.8) for some particular solution.

Let Y stand for the set of all the solutions of the system (1.1)-(1.3), (1.7), (2.2). We recall that

$$Y \subset H^2(Q).$$

Take any positive value δ. Since H_0 in our case is separable, we can select in Y, by virtue of (2.1) and (2.3), a *countable* δ-net,

$$Y^\delta = \{y^k\}_{k=1}^\infty, \quad y^k \in Y,$$

in such a way that for any $y \in Y$ there exists an element y^{k*}, so as

$$\| y - y^{k*} \|_{H^2(Q)} \leq \delta \tag{3.9}$$

and

$$E^{1/2}(y(\cdot,t) - y^{k*}(\cdot,t)) \leq \delta, \quad \forall t \in [0,\theta]. \tag{3.10}$$

It is clear that Y^δ may be selected in infinitely many ways (see Remark 5.6 below). Moreover, due to (2.3), we may suppose that (3.10) is completed by the estimates

$$\| y_x - y_x^{k*} \|_{L^\infty(T;\, C(\bar\Omega))} \leq \delta, \quad \| y_t - y_t^{k*} \|_{L^\infty(T;\, C(\bar\Omega))} \leq \delta. \tag{3.11}$$

We proceed now to the construction of a pair of measurement trajectories that ensure the estimate (1.5).

Let $\{t_k\}_{k=1}^\infty$ be an arbitrary monotone sequence in T, so as

$$0 < t_1 < t_2 < \ldots < t_k < \ldots < \theta.$$

It is clear that there exists $t^* \in [0,\theta]$,

$$t^* = \lim_{k\to\infty} t_k.$$

¿From Lemma 3.1 it follows that for each k there exists a pair of curves $\{\bar x^{0k}(\cdot),\ \bar x^{1k}(\cdot)\}$ that ensure the estimates

$$E^{1/2}(y^k(\cdot,0)) = \left(\int_\Omega (y_x^{k2}(x,0) + y_t^{k2}(x,0))\, dx\right)^{1/2} \leq \tag{3.12}$$

$$\leq \hat{c}\ \text{meas}^{1/2}\{\Omega\}\,(\|\, y_x^k(\bar{x}^{0k}(\cdot),\cdot)\,\|^2_{L^\infty(t_k,t_{k+1})} + \|\, y_t^k(\bar{x}^{1k}(\cdot),\cdot)\,\|^2_{L^\infty(t_k,t_{k+1}))}\ + 2\beta)^{1/2}.$$

Let $\{\bar{x}^0(\cdot),\ \bar{x}^1(\cdot)\}$ be an arbitrary pair of measurable trajectories defined on $[0,\theta]$ such that

$$\text{(3.13)}\qquad \begin{aligned} \bar{x}^0(t) &= \bar{x}^{0k}(t),\ \ t\in(t_k,t_{k+1}), \quad k=1,\ldots, \\ \bar{x}^1(\cdot) &= \bar{x}^{1k}(\cdot),\ \ t\in(t_k,t_{k+1}), \quad k=1,\ldots. \end{aligned}$$

Let us show that $\{\bar{x}^0(\cdot),\ \bar{x}^1(\cdot)\}$ satisfy the necessary requirements. Take any solution y of the system (1.1)-(1.3), (1.7), (2.2) and let y^k be an element in Y^δ for which the estimates (3.10)-(3.11) are fulfilled (for the sake of simplicity we omit the symbol " $*$ " here). Hence, in particular,

$$\text{(3.14)}\qquad \|\, y_x - y_x^k\,\|_{L^\infty((t_k,t_{k+1});\,C(\bar{\Omega}))} \leq \delta, \qquad \|\, y_t - y^k\,\|_{L^\infty((t_k,t_{k+1});\,C(\bar{\Omega}))} \leq \delta.$$

¿From (3.10) (adjusted for our pair of functions), it follows

$$E^{1/2}(y(\cdot,0)) \leq E^{1/2}(y^k(\cdot,0)) + \delta.$$

The last estimate and (3.12)-(3.13) imply

$$E^{1/2}(y(\cdot,0)) \leq$$

$$\leq \hat{c}\ \text{meas}^{1/2}\{\Omega\}\,(\|\, y_x^k(\bar{x}^0(\cdot),\cdot)\,\|^2_{L^\infty(t_k,t_{k+1})} + \|\, y_t^k(\bar{x}^1(\cdot),\cdot)\,\|^2_{L^\infty(t_k,t_{k+1})}\ + 2\,\beta)^{1/2}\ +\ \delta.$$

Now, by virtue of (3.14), we arrive at

$$E^{1/2}(y(\cdot,0)) \leq \hat{c}\ \text{meas}^{1/2}\,\{\Omega\}\,(2\,\|\, y_x(\bar{x}^0(\cdot),\cdot)\,\|^2_{L^\infty(t_k,t_{k+1})} +$$

$$\text{(3.15)}\qquad +\ 2\,\|\, y_t(\bar{x}^1(\cdot),\cdot)\,\|^2_{L^\infty(t_k,t_{k+1})}\ + 4\delta^2\ +\ 2\,\beta)^{1/2}\ +\ \delta.$$

In other words, we obtain the estimate (1.5) with

$$(3.16) \qquad \gamma = \frac{1}{\hat{c}\ \mathrm{meas}^{1/2}\{\Omega\}(2+4\delta^2\ +\ 2\,\beta)^{1/2}\ +\ \delta}\,.$$

This completes the proof of Theorem 3.1.

Corollary 3.1. Measurement curves in Theorem 3.1 may be selected to be continuous in $[0,\theta)$.

Indeed, set $t^* = \theta$. Then, taking into account that for any solution y both $y_x(x,t)$ and $y_t(x,t)$ are continuous in x for almost all $t \in [0,\theta]$ (we recall that $y \in H^2(Q)$), we can modify the above proof in order to obtain the needed assertion.

Remark 3.2. From (3.16) and Remark 2.1 we deduce that

$$\| \mathbf{K}^{-1}(\cdot) \| \leq \hat{c}\ \mathrm{meas}^{1/2}\{\Omega\}\,(2+4\delta^2\ +\ 2\,\beta)^{1/2}\ +\ \delta.$$

Remark 3.3. In the proof of Theorem 3.1 the set of all the solutions of the system (1.1) plays an important role. However, by virtue of the linearity of the equations (1.1) and (1.6) in order to prove Theorem 3.1 (in general, for another constant γ), this set may be replaced by the set of all those solutions that have unit "energy" norm at the initial instant.

4. Spatially-averaged observation operators of type (1.8).

We begin by studying the properties of the observations of type (1.8).

Let us recall first that $y \in C([0,\theta];H)$ and, hence, for any $t \in [0,\theta]$ the sets of Lebesgue points of the functions (in x) $y_x(x,t)$ and $y_t(x,t)$ have a full measure in Ω, namely,

$$(4.1) \qquad \mathrm{meas}\{\Omega \setminus \Omega(y_x(\cdot,t))\} = 0, \quad \mathrm{meas}\{\Omega \setminus \Omega(y_t(\cdot,t))\} = 0, \quad \forall t \in [0,\theta],$$

where (we recall that Ω is open)

$$\Omega(y_x(\cdot,t)) = \{x \mid \lim_{h\to 0} \mid \mathrm{meas}^{-1}\{S_h(x)\} \int_{S_h(x)} y_x(s,t)ds - y_x(x,t) \mid = 0\},$$

$$\Omega(y_t(\cdot,t)) = \{x \mid \lim_{h\to 0} \mid \text{meas}^{-1}\{S_h(x)\} \int\limits_{S_h(x)} y_t(s,t)ds - y_t(x,t) \mid = 0\}.$$

Let $F(t)$ be a set-valued map from T into the set of all the measurable subsets of Ω. We shall say that $F(t)$ is continuous (see [1]) with respect to Lebesgue measure at $t = t^*$ if

$$\text{meas}\{F(t^*)\,\Delta\, F(t^*+\Delta t)\} \to 0, \qquad \Delta t \to 0,$$

where $A\Delta B$ stands for the symmetric difference:

$$A\Delta B = (A\backslash B)\bigcup(B\backslash A).$$

Remark 4.1. The continuity of $F(t)$ at $t = t^*$ implies the continuity of the function $f(t) = \text{meas}\{F(t)\}$ at $t = t^*$.

Lemma 4.1. Let $\Omega^*(t)$, $t \in [0,\theta]$ be a continuous with respect to Lebesgue measure set-valued map with images that are of positive measure for all $t \in [0,\theta]$. Then, for any solution y of the system (1.1)-(1.3) the following functions

$$\frac{1}{\text{meas}\{\Omega^*\}} \int\limits_{\Omega^*(t)} y_x(x,t)dx, \qquad \frac{1}{\text{meas}\{\Omega^*\}} \int\limits_{\Omega^*(t)} y_t(x,t)dx, \quad t \in [0,\theta] \tag{4.2}$$

are continuous in time.

Proof. Indeed, since the solutions of the system (1.1) are functions that are continuous in time in the "energy" norm, the functions in (4.2) are continuous as superpositions of continuous functions. We obtain, for example,

$$\mid \frac{1}{\text{meas}\{\Omega^*(t)\}} \int\limits_{\Omega^*(t)} y_{x_p}(x,t)dx - \frac{1}{\text{meas}\{\Omega^*(t+\Delta t)\}} \int\limits_{\Omega^*(t+\Delta t)} y_{x_p}(x,t+\Delta t)dx \mid \leq$$

$$\leq \frac{1}{\text{meas}\{\Omega^*(t)\}} \int\limits_{\Omega^*(t)} \mid y_{x_p}(x,t) - y_{x_p}(x,t+\Delta t) \mid dx +$$

$$+ \mid \frac{1}{\text{meas}\{\Omega^*(t)\}} - \frac{1}{\text{meas}\{\Omega^*(t+\Delta t)\}} \mid \int\limits_{\Omega^*(t+\Delta t)} \mid y_{x_p}(x,t+\Delta t) \mid dx +$$

$$+ \frac{1}{\text{meas}\{\Omega^*(t)\}} \int\limits_{\Omega^*(t)\Delta\Omega^*(t+\Delta t)} \mid y_{x_p}(x,t+\Delta t) \mid dx.$$

From here it follows

$$| \frac{1}{\text{meas}\{\Omega^*(t)\}} \int\limits_{\Omega^*(t)} y_{x_p}(x,t)dx - \frac{1}{\text{meas}\{\Omega^*(t+\Delta t)\}} \int\limits_{\Omega^*(t+\Delta t)} y_{x_p}(x,t+\Delta t)dx | \leq$$

$$\leq \text{meas}^{-1/2}\{\Omega^*(t)\} \parallel y_{x_p}(\cdot,t) - y_{x_p}(\cdot,t+\Delta t) \parallel_{L^2(\Omega)} +$$

$$+ (| \text{meas}^{-1}\{\Omega^*(t)\} - \text{meas}^{-1}\{\Omega^*(t+\Delta t)\} | \text{meas}^{1/2}\{\Omega^*(t+\Delta t)\} +$$

$$+ \text{meas}^{-1}\{\Omega^*(t)\} \text{meas}^{1/2}\{\Omega^*(t) \Delta \Omega^*(t+\Delta t)\}) \parallel y_{x_p}(\cdot,t+\Delta t) \parallel_{L^2(\Omega)} .$$

All the other terms in (4.2) may be evaluated in the same manner. This implies the conclusion of Lemma 4.1, since we recall that $y \in C([0,\theta]; H)$.

Lemma 4.2. Given solution y of the system (1.1)-(1.3) and given $\beta > 0$ and an interval $r \subset T$, there exist such a set of simple measurable functions $h_i(\cdot)$, $\bar{x}^i(\cdot)$, $i = 0,1$ and an interval $r^* \subset r$ that the following estimates are fulfilled in r^* :

$$(4.3) \quad (\text{meas}^{-1}\{S_{h_0(t)}(\bar{x}^0(t)) \cap \Omega\} \int\limits_{\{S_{h_0(t)}(\bar{x}^0(t))\cap\Omega\}} y_x(x,t)dx)^2 \geq \underset{(x,t)\in\Omega\times r}{\text{vrai max}}\, y_x^2(x,t) - \beta, \; t \in r^*,$$

$$(4.4) \quad (\text{meas}^{-1}\{S_{h_1(t)}(\bar{x}^1(t))\cap\Omega\} \int\limits_{\{S_{h_1(t)}(\bar{x}^1(t))\cap\Omega\}} y_t(x,t)dx)^2 \geq \underset{(x,t)\in\Omega\times r}{\text{vrai max}}\, y_t^2(x,t) - \beta, \quad t \in r^*.$$

Proof. Indeed, due to (4.1), we can specify an instant $t = t^*$ and associated pairs $\{h_i > 0,\ \bar{x}^i \in \Omega\}$, $i = 0,1$ that ensure the estimates

$$(\text{meas}^{-1}\{S_{h_0}(\bar{x}^0) \cap \Omega\} \int\limits_{\{S_{h_0}(\bar{x}^0)\cap\Omega\}} y_x(x,t^*)dx)^2 \geq \underset{(x,t)\in\Omega\times r}{\text{vrai max}}\, y_x^2(x,t) - \beta/2,$$

$$(\text{meas}^{-1}\{S_{h_1}(\bar{x}^1) \cap \Omega\} \int\limits_{\{S_{h_1}(\bar{x}^1)\cap\Omega\}} y_t(x,t^*)dx)^2 \geq \underset{(x,t)\in\Omega\times r}{\text{vrai max}}\, y_t^2(x,t) - \beta/2.$$

Hence, applying Lemma 4.1 yields the required assertion.

We extend now the results of Section 3 for the case of observation operators of type (1.6).

Theorem 4.1. There exists a class of functions $\{x^0(\cdot),\ x^1(\cdot),\ h_0(\cdot),\ h_1(\cdot)\}$ that make the system (1.1)-(1.3), (1.7), (2.2), (1.8) be exactly observable.

Proof. Since the estimates (4.3), (4.4) play the role of the relations (3.5), (3.6) of the previous section, we enable to prove first the assertion of Lemma 3.1 (adjusted for our case) for the observations of type (1.8). Then, noting that

$$\text{meas}^{-1}\{S_{h_i(t)}(\bar{x}^i(t)) \cap \Omega\} \int_{\{S_{h_i(t)}(\bar{x}^i(t))\cap\Omega\}} 1\, dx \;=\; 1, \quad \forall t \in T,\ i = 0, 1,$$

we obtain that the inequalities (3.14) imply

$$\| \mathbf{G}(\cdot) \begin{pmatrix} y_x - y_x^k \\ y_t - y_t^k \end{pmatrix} \|_{L_2^\infty(t_k, t_{k+1})} \;\leq\; \sqrt{2}\, \delta,$$

where $\mathbf{G}(\cdot)$ stands for the observation operator in (1.8). Since the latter plays a crucial role in deriving (3.15), we may establish the required assertion of Theorem 4.1.

Remark 4.2. Measurement maps in Theorem 4.1 may be selected piecewise constant with a countable number of pieces or continuous with respect to Lebesgue measure in $[0, \theta)$.

The existence of a δ-net (with respect to the "energy" norm) in the set of all the solutions of the system (1.1)-(1.3), (1.7), (2.2) plays an important role in the proofs of Theorems 3.1, 4.1. An addition, and that is *critical,* such a net *simultaneously* must generate, due to (1.4), the associated net in the set of outputs (see (3.11)). We remind that the space for outputs $L_2^\infty(T)$ is *not separable* and, therefore, the construction of an appropriate net in the above is essentially based on *a priori estimates of instantaneous type* for solutions of the system in question. In the case of one dimensional spatial variable the construction of a suitable net has been achieved by applying the embedding theorems and a priori estimates (2.3)-(2.4). However, when the dimensionality of the spatial variable exceeds 1, we do not manage to obtain the required result at least under the regularity conditions (1.2), (1.3), (1.7), (2.2). A possible way to overcome this difficulty is to increase the regularity of the system (1.1). For example, we may restrict ourselves by *classical solutions.* In this case we can construct a proper net using the *separability* of the space of continuous functions. Another way is to introduce *more general type of scanning observations* in order to make them be *adequate* to the regularity of system in question. In the next section we shall pursue this way.

Remark 4.3. In the sequel of this section we point out a possible (direct) extension of Theorems 3.1, 4.1 for the case of n-dimensional spatial variable:

Let Y_I be an arbitrary countable set of solutions to the system (1.1)-(1.3),

$$Y_I = \{y_i\}_{i=1}^{\infty}.$$

Given $\delta > 0$, we consider an arbitrary set $Y_{(\delta)}$ for which Y_I is a δ-net in the sense of (3.10), (3.11). Then, the assertion of Theorem 4.1 may be extended for the set $Y_{(\delta)}$ with

$$H_0 = \{\{y(\cdot,0),\ y_t(\cdot,0)\}\ \mid\ y \in Y_{(\delta)}\}.$$

5. Generalized spatially-averaged observations.

The main result of this section is

Theorem 5.1. There exist set-valued maps $\{\Omega_p^0(t)\}_{p=1}^n$, $\Omega^1(t)$, $t \in [0,\theta]$ that make the system (1.1)-(1.3), (1.10), (1.9) be exactly observable.

The assertion of Theorem 5.1 includes the statement that required maps ensure the enclosure of the range of the corresponding observation operator into $L_{n+1}^{\infty}(T)$.

We begin again by studying the properties of observations (1.9). Set

$$\Omega_{p+}^0(t) = \{x \mid v_p^0(x,t) = 1,\ x \in \Omega_p^0(t)\ \},\ \ \Omega_{p-}^0(t) = \{x \mid v_p^0(x,t) = -1,\ x \in \Omega_p^0(t)\ \},\quad t \in [0,\theta],$$

$$\Omega_+^1(t) = \{x \mid v^1(x,t) = 1,\ x \in \Omega^1(t)\ \},\qquad \Omega_-^1(t) = \{x \mid v^1(x,t) = -1,\ x \in \Omega^1(t)\ \},\quad t \in [0,\theta].$$

Then we may rewrite the formula (1.9) in the form

$$z(t) = \begin{pmatrix} \text{meas}^{-1}\{\Omega_1^0(t)\}\ \int_{\Omega_{1+}^0(t)} y_{x_1}(x,t)dx \\ \vdots \\ \vdots \\ \text{meas}^{-1}\{\Omega_n^0(t)\}\ \int_{\Omega_{n+}^0(t)} y_{x_n}(x,t)dx \\ \text{meas}^{-1}\{\Omega^1(t)\}\ \int_{\Omega_+^1(t)} y_t(x,t)dx \end{pmatrix} - \tag{5.1}$$

$$
- \begin{pmatrix} \text{meas}^{-1}\{\Omega_1^0(t)\} \int_{\Omega_{1-}^0(t)} y_{x_1}(x,t)dx \\ \vdots \\ \vdots \\ \text{meas}^{-1}\{\Omega_n^0(t)\} \int_{\Omega_{n-}^0(t)} y_{x_n}(x,t)dx \\ \text{meas}^{-1}\{\Omega^1(t)\} \int_{\Omega_-^1(t)} y_t(x,t)dx \end{pmatrix}, \quad t \in [0,\theta].
$$

Lemma 5.1. Let $\{\Omega_{p\pm}^0(\cdot)\}_{p=1}^n$, $\Omega_\pm^1(\cdot)$ in (5.1) be continuous with respect to Lebesgue measure in $[0,\theta]$ and all the sets $\{\Omega_p^0(t)\}_{p=1}^n$, $\Omega^1(t)$ be of positive measure for all $t \in [0,\theta]$. Then outputs of the system (1.1)-(1.3), (1.9) are continuous in $[0,\theta]$.

The assertion of Lemma 5.1 immediately follows from Lemma 4.1.

The following class of measurement maps plays a crucial role below.

Assumption 5.1. Let $F(t)$ be a set-valued map from T into the set of all the measurable subsets of Ω. We shall say that $F(t)$ satisfies Assumption 5.1 at $t = t^*$ if

1. it is *lower semi-continuous* [1] at $t = t^*$ with respect to Lebesgue measure:
$\forall \ \nu_1 > 0 \quad \exists \, \nu_2 > 0 \quad$ such that

$$
\text{meas}\{F(t^*) \setminus F(t^* + \Delta t)\} \le \nu_1, \quad \forall \, \Delta t, \quad |\, \Delta t \,| \le \nu_2; \tag{5.2}
$$

2.

$$
\text{meas}\{F(t^*)\} > 0. \tag{5.3}
$$

Remark 5.1.

i) If $F(t)$ satisfies Assumption 5.1 everywhere in $[0,\theta]$, then the function $f(t) = \text{meas}\{F(t)\}$ is lower semi-continuous in $[0,\theta]$.

ii) Let T^* be a closed subinterval of T. Then measurement maps satisfying Assumption 5.1 in T^* ensure the enclosure of outputs of the system (1.1)-(1.3), (1.9) into $L_{n+1}^\infty(T^*)$.

We introduce now measurement maps that satisfy Assumption 5.1 in $[0,\theta]$ and allow us then to extend the result of Lemma 3.1 for the case of generalized observations (1.9).

Let y be an arbitrary solution of the system (1.1)-(1.3) and $\beta > 0$ be given. We set for $t \in [0,\theta]$, $p = 1,\ldots,n$:

$$(5.4) \qquad r^0_{p+}(y,t) =$$

$$= \{x \in \Omega \mid y_{x_p}(x,t) > 0,\ y_{x_p}(x,t) > \operatorname{meas}^{-1/2}\{\Omega\} \min_{t\in[0,\theta]} \| y_{x_p}(\cdot,t) \|_{L^2(\Omega)} - \beta\},$$

$$(5.4)' \qquad r^0_{p-}(y,t) =$$

$$= \{x \in \Omega \mid y_{x_p}(x,t) < 0,\ | y_{x_p}(x,t) | > \operatorname{meas}^{-1/2}\{\Omega\} \min_{t\in[0,\theta]} \| y_{x_p}(\cdot,t) \|_{L^2(\Omega)} - \beta\},$$

$$(5.5) \qquad r^1_{+}(y,t) =$$

$$= \{x \in \Omega \mid y_t(x,t) > 0,\ y_t(x,t) > \operatorname{meas}^{-1/2}\{\Omega\} \min_{t\in[0,\theta]} \| y_t(\cdot,t) \|_{L^2(\Omega)} - \beta\},$$

$$(5.5)' \qquad r^1_{-}(y,t) =$$

$$= \{x \in \Omega \mid y_t(x,t) < 0,\ | y_t(x,t) | > \operatorname{meas}^{-1/2}\{\Omega\} \min_{t\in[0,\theta]} \| y_t(\cdot,t) \|_{L^2(\Omega)} - \beta\}.$$

It is not hard to see that for every $t \in [0,\theta]$ all the sets

$$r^0_p(y,t) = r^0_{p+}(y,t) \bigcup r^0_{p-}(y,t) =$$

$$= \{x \mid | y_{x_p}(x,t) | > \operatorname{meas}^{-1/2}\{\Omega\} \min_{t\in[0,\theta]} \| y_{x_p}(\cdot,t) \|_{L^2(\Omega)} - \beta\}, \quad p = 1,\ldots,n,$$

$$r^1(y,t) = r^1_{+}(y,t) \bigcup r^1_{-}(y,t) =$$

$$= \{x \mid | y_t(x,t) | > \operatorname{meas}^{-1/2}\{\Omega\} \min_{t\in[0,\theta]} \| y_t(\cdot,t) \|_{L^2(\Omega)} - \beta\}$$

are of positive measure and the condition (5.3) is fulfilled for them in $[0,\theta]$. Furthermore, from continuity of the functions $y_{x_p}(\cdot,t)$, $y_t(\cdot,t)$ in t in the norm of $L^2(\Omega)$ it follows that for any $\varepsilon > 0$

$$(5.6) \qquad \operatorname{meas}\{x \mid | y_x(x,t+\Delta t) - y_x(x,t) | \geq \varepsilon\} \to 0, \quad \text{when} \quad \Delta t \to 0,$$

$$(5.7) \qquad \operatorname{meas}\{x \mid | y_t(x,t+\Delta t) - y_t(x,t) | \geq \varepsilon\} \to 0, \quad \text{when} \quad \Delta t \to 0.$$

In turn, the relations (5.6), (5.7) and the fact that all the inequalities in (5.4)-(5.5)$'$ are strict imply for $\forall t \in [0,\theta]: \ \forall\nu_1 > 0 \ \ \exists\nu_2 > 0$ such that

$$(5.8) \qquad \text{meas}\{r_p^0(y,t) \setminus r_p^0(y,t+\Delta t)\} \le \nu_1, \quad \forall\Delta t, \quad |\Delta t| \le \nu_2, \quad p=1,\ldots,$$

$$(5.9) \qquad \text{meas}\{r^1(y,t) \setminus r^1(y,t+\Delta t)\} \le \nu_1, \quad \forall\,\Delta t, \quad |\Delta t| \le \nu_2.$$

Hence, the maps $\{r_p^0(y,t)\}_{p=1}^n$, $r^1(y,t)$ satisfy the condition (5.2) in $[0,\theta]$ and we obtain

Lemma 5.2. Let y be an arbitrary solution of the system (1.1)-(1.3) and $\beta > 0$, a closed interval $r \subseteq [0,\theta]$ be given. Let $\{\Omega_{p\pm}^0(y,t)\}_{p=1}^n$, $\Omega^1(y,t)$, $t \in T$ satisfy the following relations:

$$(5.10) \qquad \Omega_{p\pm}^0(y,t) = r_{p\pm}^0(y,t), \quad t \in r, \quad p = 1,\ldots,n,$$

$$(5.11) \qquad \Omega_\pm^1(y,t) = r_\pm^1(y,t), \quad t \in r,$$

where $r_{p\pm}^0(y,t)$, $r_\pm^1(y,t)$ are constructed along (5.4)-(5.5)$'$. Then $\{\Omega_p^0(y,t)\}_{p=1}^n$, $\Omega^1(y,t)$ satisfy Assumption 5.1 in r.

Remark 5.2. Let y be an arbitrary solution of the system (1.1)-(1.3). Then we may deduce from (5.6), (5.7) that for any sequence $\{y_i\}_{i=1}^\infty$ of solutions of the system (1.1)-(1.3) that converges to y in the norm of $C([0,\theta];H)$ the following estimates are fulfilled

$$\varliminf_{i\to\infty} \text{meas}\{r_p^0(y_i,t)\} \ge \min_{t\in[0,\theta]} \text{meas}\{r_p^0(y,t)\}, \quad p = 1,\ldots,$$

$$\varliminf_{i\to\infty} \text{meas}\{r^1(y_i,t)\} \ge \min_{t\in[0,\theta]} \text{meas}\{r^1(y,t)\},$$

uniformly over $t \in [0,\theta]$.

Lemma 5.3. Let y be an arbitrary solution of the system (1.1)-(1.3) and $\beta > 0$, a closed interval $r \subseteq [0,\theta]$ be given. Then the maps $\{\Omega_{p\pm}^0(t)\}_{p=1}^n$, $\Omega_\pm^1(t)\}$, $t \in r$, defined by the relations (5.10) and (5.11), ensure the estimate

$$E(y(\cdot,0)) \le \hat{c}^2 \,\text{meas}\{\Omega\}\, (2\sum_{p=1}^{n}(\text{meas}^{-1}\{\Omega_p^0(y,t)\} \int\limits_{\Omega_p^0(y,t)} \mid y_{x_p}(x,t)\mid dx)^2 +$$

$$+ 2\,(\text{meas}^{-1}\{\Omega^1(y,t)\} \int\limits_{\Omega^1(y,t)} \mid y_t(x,t)\mid dx)^2 + 2(n+1)\,\beta^2), \quad \forall t \in r. \tag{5.12}$$

Proof. Taking into account (2.1), we obtain

$$E(y(\cdot,0)) \le \hat{c}^2\, E(y(\cdot,t)), \;\; \forall t \in r,$$

and then, by (5.4)-(5.5)′,

$$E(y(\cdot,0)) \le \hat{c}^2 \min_{t\in[0,\theta]} \int\limits_{\Omega} (y_x^2(x,t) + y_t^2(x,t))dx \le$$

$$\le \hat{c}^2 \,\text{meas}\{\Omega\}\, (2 \sum_{p=1}^{n} \underset{x\in r_p^0(y,t)}{\text{vrai min}}\, y_{x_p}^2(x,t) + 2\,\underset{x\in r_1(y,t)}{\text{vrai min}}\, y_t^2(x,t) + 2\,(n+1)\,\beta^2), \quad \forall t \in r. \tag{5.13}$$

¿From (5.13) we immediately obtain the needed result.

Remark 5.3. The constants in (5.12) do not depend upon y and r, β.

Proof of Theorem 5.1. Below we follow, in fact, the scheme of the proof of Theorem 3.1, although the argument essentially differs in the technical aspect.

Let Y stand again for the set of all solutions to the system (1.1)-(1.3). Select an arbitrary monotone sequence $\{\delta_j\}_{j=1}^{\infty}$,

$$\delta_1 > \ldots > \delta_j > \ldots > 0.$$

Specify next for each j, a δ_j-net

$$Y^{\delta_j} = \{y_{kj}\}_{k=1}^{\infty}, \quad y_{kj} \in Y,$$

so as for any solution y of the system (1.1)-(1.3) and $j = 1, \ldots$ there exists such an element y_{k*j} that

$$E^{1/2}(y(\cdot,t) - y_{k*j}(\cdot,t)) \le \delta_j, \quad \forall t \in [0,\theta]. \tag{5.14}$$

In turn, the estimate (5.14) implies

$$(5.15)\qquad \text{meas}\{x \mid \mid y_x(x,t) - y_{k*jx}(x,t) \mid \geq \sqrt{\delta_j}\} \leq \delta_j, \quad \forall t \in [0,\theta],$$

$$(5.16)\qquad \text{meas}\{x \mid \mid y_t(x,t) - y_{k*jt}(x,t) \mid \geq \sqrt{\delta_j}\} \leq \delta_j, \quad \forall t \in [0,\theta].$$

We stress that such a net may be selected in infinitely many ways (see Remark 5.6 below).

Select $\beta > 0$. Then, due to Lemma 5.2, for any y we can construct the sets $\{r^0_{p\pm}(y,t)\}_{p=1}^n$, $r^1_\pm(y,t)$, $t \in [0,\theta]$ that are defined by (5.4)-(5.5)$'$ and satisfy Assumption 5.1 everywhere in $[0,\theta]$. This means, in particular, that

$$\min_{t\in[0,\theta]} \text{meas}\{r^0_p(y,t)\} > 0, \quad p = 1,\dots,n,$$

$$\min_{t\in[0,\theta]} \text{meas}\{r^1(y,t)\} > 0.$$

Let $\{\varepsilon_l\}_{l=1}^\infty$ be a monotone decreasing sequence of positive numbers such that

$$\lim_{l\to\infty} \varepsilon_l = 0.$$

Split the set of all the solutions of the system (1.1)-(1.3) as follows :

$$Y = \bigcup_{l=1}^\infty Y_l,$$

$$Y_l = \{y \mid \varepsilon_{l-1} \geq \min_{t\in[0,\theta],\, p=1,\dots,n} \text{meas}\{r^0_p(y,t)\}, \ \min_{t\in[0,\theta]} \text{meas}\{r^1(y,t)\} > \varepsilon_l\}.$$

Take an arbitrary monotone sequence of instants $\{t^j\}_{j=1}^\infty$,

$$0 < t^1 < t^2 < \dots < t^j < \dots < \theta.$$

We select next in each interval (t^j, t^{j+1}) an arbitrary monotone sequence of instants $\{t^j_k\}_{k=1}^\infty$, so as

$$t^j = t_1^j < t_2^j < \ldots < t_k^j < \ldots < t^{j+1}, \quad \lim_{k\to\infty} t_k^j = t^{j+1}.$$

Let $\{\Omega_p^0(t)\}_{p=1}^n$, $\Omega^1(t)$, $t \in [0,\theta]$ be arbitrary maps that in $[0,t^1)\bigcup[\lim_{j\to\infty} t^j,\ \theta]$ satisfy Assumption 5.1 and

$$\Omega_p^0(t) = r_p^0(y_{kj},t), \quad t \in [t_k^j, t_{k+1}^j), \quad k,j = 1,\ldots, \quad p = 1,\ldots,n, \tag{5.17}$$

$$\Omega^1(t) = r^1(y_{kj},t), \quad t \in [t_k^j, t_{k+1}^j), \quad k,j = 1,\ldots . \tag{5.18}$$

Let us show that the observations (1.9) (or (5.1)), (5.17), (5.18) with

$$v_p^0(x,t) = \begin{cases} +1, & x \in r_{p+}^0(y_{kj},t), \quad t \in [t_k^j, t_{k+1}^j), \\ -1, & x \in r_{p-}^0(y_{kj},t), \quad t \in [t_k^j, t_{k+1}^j), \end{cases} \quad k,j = 1,\ldots, \quad p = 1,\ldots,n, \tag{5.19}$$

$$v^1(x,t) = \begin{cases} +1, & x \in r_+^1(y_{kj},t), \quad t \in [t_k^j, t_{k+1}^j), \\ -1, & x \in r_-^1(y_{kj},t), \quad t \in [t_k^j, t_{k+1}^j), \end{cases} \quad k,j = 1,\ldots$$

provide the system (1.1)-(1.3), (1.10), (1.9) with exact observability.

We note first that the range of the corresponding observation operator belongs to $L_{n+1}^\infty(T)$. Take any solution y of the system (1.1)-(1.3), (1.10) and assume that $y \in Y_l$, so as

$$\min_{t\in[0,\theta]} \text{meas}\{r_p^0(y,t)\} > \varepsilon_l, \quad p = 1,\ldots,n, \tag{5.20}$$

$$\min_{t\in[0,\theta]} \text{meas}\{r^1(y,t)\} > \varepsilon_l. \tag{5.21}$$

Remark next that for any k_j, $j = 1,\ldots$ such that (5.14)-(5.16) are fulfilled with k_j substituted for $k*$, the following chain of estimates holds:

$$\left| \frac{1}{\text{meas}\{r_p^0(y_{k_j j},t)\}} \int_{r_p^0(y_{k_j j},t)} v_p^0(x,t)(y_{x_p}(x,t) - y_{k_j j x_p}(x,t))dx \right| \le$$

$$\leq \frac{\sqrt{\delta_j}}{\text{meas}\{r_p^0(y_{k_jj},t)\}} \text{meas}\{x \mid x \in r_p^0(y_{k_jj},t), \ | y_x(x,t) - y_{k_jjx}(x,t) | \leq \sqrt{\delta_j}\} +$$

$$+ \frac{\text{meas}^{1/2}\{x \mid x \in r_p^0(y_{k_jj},t), \ | y_x(x,t) - y_{k_jjx}(x,t) | > \sqrt{\delta_j}\}}{\text{meas}\{r_p^0(y_{k_jj},t)\}} E^{1/2}(y(\cdot,t) - y_{k_j}(\cdot,t)) \leq$$

$$\leq \sqrt{\delta_j}(1 + \frac{\delta_j}{\text{meas}\{r_p^0(y_{k_jj},t)\}}), \quad \forall t \in [t_{k_j}^j, t_{k_j+1}^j), \quad p = 1, \dots .$$

Combining the last estimate and (5.20) yields (via Remark 5.2) that, beginning from some $j = j*$, the following estimates are fulfilled:

$$| \frac{1}{\text{meas}\{r_p^0(y_{k_jj},t)\}} \int\limits_{r_p^0(y_{k_jj},t)} v_p^0(x,t)(y_{x_p}(x,t) - y_{k_jjx_p}(x,t))dx | \leq$$

$$\leq \sqrt{\delta_j}(1 + \frac{\delta_j}{0.5\,\varepsilon_l}), \quad \forall t \in [t_{k_j}^j, t_{k_j+1}^j), \quad p = 1, \dots .$$

In a similar way we may obtain the chain of inequalities for y_t. Thus, for any given $\alpha > 0$ there exists such an element $y_{k_{j_*}j*}$, $j_* = j(\alpha)$ that (in addition to the estimates (5.14)-(5.16)) for the pair y, $y_{k_{j_*}j*}$, we can write:

$$\| \mathbf{G}_*(t)(y(\cdot,t) - y_{k_{j_*}j*}(\cdot,t)) \|_{R^{n+1}} \leq \alpha, \quad \forall t \in [t_{k_{j_*}}^{j*}, t_{k_{j_*}+1}^{j*}),$$

where $\mathbf{G}_*(\cdot)$ stands for observations (1.9), (5.17)-(5.19). In other words

$$\| \mathbf{G}_*(t) y_{k_{j_*}j}(\cdot,t)) \|_{R^{n+1}} \leq \| \mathbf{G}_*(t) y(\cdot,t) \|_{R^{n+1}} + \alpha, \quad \forall t \in [t_{k*}^{j*}, t_{k_{j_*}+1}^{j*}). \tag{5.22}$$

In turn, from (5.14) and Lemma 5.3 we obtain

$$E^{1/2}(y(\cdot,0)) \leq E^{1/2}(y_{k_{j_*}j*}(\cdot,0)) + \delta_{j*} \leq \hat{c} \ \text{meas}^{1/2}\{\Omega\} \ (2(n+1)\,\beta^2 +$$

$$2 \| \mathbf{G}_*(\cdot) y_{k_{j_*}j*}(\cdot,\cdot) \|^2_{L^\infty_{n+1}(t_{k_{j_*}}^{j*}, t_{k_{j_*}+1}^{j*})})^{1/2} + \delta_{j*}, \quad \delta_{j*} = \delta_{j*} \to 0 \text{ when } \alpha \to 0.$$

Finally, (5.22) and the latter yields

$$E^{1/2}(y(\cdot,0)) \le \hat{c}\ \text{meas}^{1/2}\{\Omega\}\,(2\,(n+1)\beta^2 + 4\,\|\,\mathbf{G}_*(\cdot)y(\cdot,\cdot)\,\|^2_{L^\infty_{n+1}(T^\bullet)} + 2\alpha^2)^{1/2} + \delta_{j(\alpha)},$$

which is valid for $\forall\alpha > 0$. The last estimate completes the proof of Theorem 5.1.

Corollary 5.1. Measurement maps that satisfy Theorem 5.1 may be selected lower semicontinuous in $[0,\theta]$.

Proof. Since the number of all the functions $\{y_{kj}\}_{k,j=1}^{\infty}$ forming a sequence of δ_j-nets in the proof of Theorem 5.1 is countable, instead of the above type of choice of instants $\{t_k^j\}_{k,j=1}^{n}$ we can make a selection that has the only limit point $t = \theta$. Then we can construct required maps in (t_k^j, t_{k+1}^j), $k,j = 1,\dots$ in the same manner as in the above. Setting all the sets $\{\Omega^0_{p\pm}(t_k^j)\}_{p=1}^n, \Omega^1_\pm(t_k^j)$, $\{\Omega^0_{p\pm}(\theta)\}_{p=1}^n, \Omega^1_\pm(\theta)$ to be non-empty, but of zero-measure, yields the needed assertion of Corollary 5.1.

Remark 5.4. Given $\beta > 0$, arbitrary maps $\{\Omega^0_{p\pm}(\cdot)\}_{p=1}^n, \Omega^1_\pm(\cdot)$ constructed along the lines (5.17)-(5.19) ensure the estimate

$$\|\,\mathbf{K}^{-1}(\cdot)\,\| \le \hat{c}\ \text{meas}^{1/2}\{\Omega\}\,(2(n+1)\,\beta^2 + 4)^{1/2}.$$

Remark 5.5. From the proof of Theorem 5.1 it follows that, in order to ensure exact observability in a prescribed finite-dimensional subspace spanned by a finite number of solutions of the system (1.1)-(1.3), it is sufficient to specify a finite number of pieces of type (5.17)-(5.19).

Remark 5.6. For constructing suitable δ-net we may use Galerkin's method.

Concluding remarks.

The problem of exact observability with *scanning* sensors has been discussed for the *nonstationary* hyperbolic systems. Three types of scanning observations have been considered and the existence of required measurement curves and maps that ensure exact observability with $L^\infty_{n+1}(T)$ to be taken for the space for outputs has been established. At every instant of time all of the above types of observations provide *finite-dimensional* outputs that is critical in physical situations. The method of proofs, to some extent, is an analogue of Galerkin's method which is widely used when working with the existence results in the theory of generalized solutions of PDE's. Here we have the case when the space for outputs $L^\infty_{n+1}(T)$ is not Hilbert. Therefore, we construct a *countable* δ*-net* (it plays a role of "basis") in the *set* of solutions that, in turn, must generate a similar net in the set of outputs. In order to do this *a priori energy estimates of instantaneous type* for solutions have been used. The proposed method allows to obtain measurement curves that are continuous in $[0, \theta)$ and measurement maps that are lower semi-continuous in $[0, \theta]$ with respect to Lebesgue measure.

References.

[1] J.-P. Aubin, H. Frankowska, *Set-Valued Analysis,* Birkhauser, 1990.

[2] R.F. Curtain, and A.J. Pritchard, *Infinite Dimensional Linear Systems Theory,* Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[3] A. El Jai, A. Belfekih, J. Bouyaghroumni, Observability and Sensors for Hyperbolic Systems, *Intern. J. Systems Sci.,* 1991, 22, No. 7, pp. 1255-1265.

[4] A. El Jai and A.J. Pritchard, Sensors and Actuators in the Analysis of Distributed Systems, J. Wiley, New York, 1988.

[5] A.F. Fillipov, Classical Solutions of Differential Equations with Multivalued Right-Hand Side, *SIAM J. Control,* 5 (1967), pp. 609-621.

[6] A. Yu. Khapalov, Observability of Hyperbolic Systems with Interior Moving Sensors, *Proc. 10th Intern. Conf. on Analysis and Optimization of Systems,* INRIA, Sophia-Antipolis, June 9-12, 1992 (to appear).

[7] A. Yu. Khapalov, Continuous Observability of Parabolic Systems with Interior Scanning Sensors, the paper submitted for publication in the *SIAM Journal on Control and Optimization,* (022737-1).

[8] O.H. Ladyzhenskaya, *Boundary Value Problems of Mathematical Physics,* Nauka, Moscow, 1973.

[9] J.-L. Lions, *Contrôlabilité exacte des systèmes distribués,* C. R. Acad. Sci. Paris Sér. I Math., 302 (1986), pp. 471-475.

[10] J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed systems,* John Von Neumann Lecture, Boston, MA, July, 1986; SIAM Rev., 30 (1988), pp. 1-68.

[11] Lop Fat Ho, Exact Controllability of the One-Dimensional Wave Equation with Locally Distributed Control, *SIAM J. Control and Opt.,* Vol. 28, No. 3, 1990, pp. 733-748.

[12] J-C. E. Martin, Controllability and Observability of Parabolic Systems – An Addendum to Two Recent Papers of Y. Sakawa, *SIAM J. Control and Optimization,* Vol. 15, No. 3, May 1977, pp. 363-366.

[13] S.L. Sobolev, *On Certain Applications of Functional Analysis in Mathematical Physics,* Novosibirsk, 1982.