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Set-Valued Approach to Hamilton-Jacobi-Bellman Equations

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SET-VALUED APPROACH TO HAMILTON-JACOBI-BELLMAN EQUATIONS

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FOREWORD

Viability and invariance theorems for systems with dynamics depending on time in a measurable way and having time dependent state constraints:

$$x'(t) \in F(t, x(t)), \quad x(t) \in P(t)$$

are proved. In the above $t \rightsquigarrow P(t)$ is an absolutely continuous set-valued map and $(t,x) \rightsquigarrow F(t,x)$ is a set-valued map which is measurable with respect to t and upper semicontinuous (or continuous, or locally Lipschitz) with respect to x. For this aim the infinitesimal generators of reachable maps and the Lebesgue points of set-valued maps are investigated.

The results are applied to define and to study lower semicontinuous solutions of the Hamilton-Jacobi-Bellman equation

$$u_t + H\,(t,\ x,\ u_x) \ = \ 0$$

with the Hamiltonian H measurable with respect to time, locally Lipschitz with respect to x and convex in the last variable.

SET-VALUED APPROACH TO HAMILTON-JACOBI-BELLMAN EQUATIONS

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1 Introduction

This paper is devoted to several viability and invariance theorems for differential inclusions with dynamics measurable with respect to time. Namely we a consider set-valued map $F : [0,T] \times \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ such that for every $x \in \mathbf{R}^d$, $F(\cdot, x)$ is measurable. Regularity assumptions on $F(t,\cdot)$ are upper semicontinuity (in Theorem 3.3), continuity (in Theorem 3.2) and Lipschitz continuity (in section 4.2). The constraints are given by a set-valued map $P : [0,T] \rightsquigarrow \mathbf{R}^d$, called tube, which is supposed to be absolutely continuous. We investigate the existence of solutions to the Cauchy problem

$$\begin{cases} x' & \in & F(t,x) \\ x(t_0) & = & x_0 \\ x(t) & \in & P(t) \text{ for all } t \geq t_0\,, \end{cases}$$

for every $x_0 \in P(t_0)$ and all $t_0 \in [0,T[$.

The first viability result is due to Nagumo [19]. He considered the case of single-valued continuous time independent f and closed time independent constraints $P(t) \equiv K$. In this case the necessary and sufficient condition for the existence of solution to the above problem is

$$\forall\, x \in K, \;\; f(x) \in T_K(x)$$

where $T_K(x)$ denotes the contingent cone to K at x (see for instance [3] for the definition of the contingent cone and its properties). In the case of time independent set-valued map, i.e. $F(t,x) = F(x)$, the necessary and sufficient condition becomes

$$\forall\, x \in K, \;\; F(x) \cap T_K(x) \neq \emptyset$$

(see [6] and also [15] for functional differential inclusions). We also refer to [2] for further improvements, discussions and historical comments on the viability problems.

Studying control systems with state constraints we often deal with dynamics which are merely measurable in time and with constraints which depend upon the time. For this reason it is natural to look for extensions of viability theory to such a case. A partial answer to the viability problem was given in [17], where the viability conditions are expressed in terms of reachable sets, which are not simple to verify.

In this paper we show that the necessary and sufficient condition for viability in the case where $F(t, \cdot)$ is continuous is:

$$\begin{array}{l} \exists\, A \subset [0, T] \text{ of full measure such that} \\ \forall\, t \in A,\ \forall\, x \in P(t),\ \ (\{1\} \times F(t,x)) \cap T_{\mathrm{Graph}(P)}(t,x) \neq \emptyset \end{array}$$

In the case when $F(t, \cdot)$ is Lipschitz continuous the above condition can be given in a weaker form

$$\begin{array}{l} \exists\, A \subset [0, T] \text{ of full measure such that} \\ \forall\, t \in A,\ \forall\, x \in P(t)\ \ (\{1\} \times F(t,x)) \cap \overline{co}\left(T_{\mathrm{Graph}(P)}(t,x)\right) \neq \emptyset \end{array}$$

where $\overline{co}$ stands for the closed convex hull. We emphasize that for upper semicontinuous dynamics such a condition was given in [14] (see also [2, Theorem 3.2.4]). In the Lipschitz case we also provide a necessary and sufficient condition for invariance (improving the results of [2, Theorem 5.3.4]). In all theorems concerning viability and invariance problem we assume that the tube P has closed values and is absolutely continuous, the notion which is introduced and discussed in section 3.

In section 2 we investigate the infinitesimal generators of reachable maps of the differential inclusions

$$\begin{cases} x'(t) \in F(t, x(t), v) \\ x(\tau) = x_\tau\ ,\ x'(\tau) = u \end{cases}$$

where u is an arbitrary but fixed element in $F(\tau, x_\tau, v)$ and v is a parameter. The results we provide improve those from [12,20].
Section 2 is devoted to the Lebesgue points of nonlinear set-valued maps and extends [1,7,16] to the state dependent case. In this way we generalize

results from [20] by relaxing regularity assumptions on x.
In section 3 we investigate the viability and invariance problems.
Finally, in section 4 we provide an application to the Hamilton-Jacobi-Bellman equations arising in optimal control. For the Mayer problem with data measurable in time we show that the associated value function is the unique solution to the equation

$$\begin{cases} -\frac{\partial V}{\partial t} + H(t, x, -\frac{\partial V}{\partial x}) = 0 \\ V(t, \cdot) = g(\cdot), \end{cases}$$

where the lower semicontinuous solution is understood in a generalized sense related to viscosity solutions (comp. [5,8,13]) and H is the Hamiltonian associated to a given control system. If the dynamics are measurable in time, then so is H. Viscosity solutions in this case were also studied in [4,18] and by Ishii. We define viscosity solutions in terms of super and subdifferentials without involving L^1 test functions as it was done in the above papers.

The full proofs of the results given below are to appear elsewhere.

2 Lebesgue Points of Set-Valued Maps

If $t \in [0, T]$ is a Lebesgue point of an integrable function $g : [0, T] \longrightarrow \mathbf{R}^d$, then

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} g(s) ds = g(t) \tag{2.1}$$

Thus (2.1) is satisfied for almost all $t \in [0, T]$. This result has been extended in [1], [7], [16], [20] to situations where set-valued maps replace the function g.

The two lemmas below provide a generalization of Proposition 4.1 from [20] – we do not impose the Lipschitz continuiuty on F with respect to x.

Lemma 2.1 *Assume that* $F : [0, T] \times \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ *has closed convex values and*

$x \rightsquigarrow F(t, x)$ *is upper semicontinuous for almost all* $t \in [0, T]$;

$\|F(t, x)\| \leq \mu(t)$ *for almost all* $t \in [0, T]$ *and all* x, *where* $\mu(\cdot)$ *is integrable.*

Then there exists a set $A \subset [0,T]$ of full measure such that for every $(\tau, x) \in A \times \mathbf{R}^d$

$$\bigcap_{\alpha>0} \operatorname{Limsup}_{h\to 0} \frac{1}{h}\int_{\tau}^{\tau+h} F(s, x+\alpha B)ds \subset F(\tau, x)\,. \tag{2.2}$$

Lemma 2.2 *Assume that $F : [0,T] \times \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ has closed convex values and*

$x \rightsquigarrow F(t,x)$ is continuous for almost all $t \in [0,T]$;

$t \rightsquigarrow F(t,x)$ is measurable for every $x \in \mathbf{R}^d$;

$\|F(t,x)\| \le \mu(t)$ for almost all $t \in [0,T]$ and for all $x \in \mathbf{R}^d$, where $\mu(\cdot)$ is integrable.

Then there exists a set $A \subset [0,T]$ of full measure such that for every $(\tau, x) \in A \times \mathbf{R}^d$ and any measurable function $y : [\tau, \tau+\varepsilon] \longrightarrow \mathbf{R}^d$ such that $\lim_{h\to 0^+} y(\tau+h) = x$ we have:

$$F(\tau, x) \subset \operatorname{Liminf}_{h\to 0^+} \frac{1}{h}\int_{\tau}^{\tau+h} F(s, y(s))ds\,. \tag{2.3}$$

Both the above lemmas together imply the following:

Corollary 2.3 *Under the assumptions of Lemma 2.2 there exists a set $A \subset [0,T]$ of full measure such that for every $(\tau, x) \in A \times \mathbf{R}^d$*

$$F(\tau, x) = \operatorname{Lim}_{h\to 0} \frac{1}{h}\int_{\tau}^{\tau+h} F(s, x)ds \tag{2.4}$$

3 Viability and Invariance Theorems

Consider $T > 0$, a set-valued map $F : [0,T] \times \mathbf{R}^d \rightsquigarrow \mathbf{R}^d$ and the differential inclusion

$$x'(t) \in F(t, x(t)) \text{ almost everywhere} \tag{3.1}$$

Denote by $S_{[t_0,T]}(x_0)$ the set of absolutely continuous solutions of (3.1) defined on $[t_0, T]$ and satisfying the initial condition $x(t_0) = x_0$.

We are interested in the existence of solutions to the differential inclusion (3.1) satisfying constrains of the type $x(t) \in P(t)$, where $P : [0,T] \rightsquigarrow \mathbf{R}^d$ is a set-valued map (we shall call it a tube). The tube $P(\cdot)$ is said to have a viability property if for every $t_0 \in [0,T]$, $x_0 \in P(t_0)$ there is a solution $x \in S_{[t_0,T]}(x_0)$ satisfying $x(t) \in P(t)$ for every $t \in [t_0,T]$. The tube $P(\cdot)$ is called invariant by F if for every $t_0 \in [0,T]$ every solution $x \in S_{[t_0,T]}(x_0)$ starting in the tube (i.e. $x_0 \in P(t_0)$) satisfies $x(t) \in P(t)$ for every $t \in [t_0,T]$.

Definition 3.1 *Let $P : [0,T] \rightsquigarrow \mathbf{R}^d$ be closed-valued. We say that P is left absolutely continuous on $[0,T]$ if the following property holds:*

$$\left\{ \begin{array}{l} \forall \varepsilon > 0\ ,\ \forall \text{compact } K \subset \mathbf{R}^d\ ,\ \exists \delta > 0\ ,\ \forall \Lambda \subset N\ ,\ \forall \{t_i, \tau_i |\ t_i < \tau_i,\ i \in \Lambda\} \\ \sum(\tau_i - t_i) \leq \delta \Longrightarrow \sum e(P(t_i) \cap K,\ P(\tau_i)) \leq \varepsilon \end{array} \right. \tag{3.2}$$

where $e(U,V) = inf\{\varepsilon > 0 |\ U \subset V + \varepsilon B\}$ and N is the set of natural numbers.

We get the definitions of right absolute continuity and absolute continuity by replacing $e(P(t_i) \cap K,\ P(\tau_i))$ in (3.2) respectively by $e(P(\tau_i) \cap K,\ P(t_i))$ and

$$e_K(P(t_i), P(\tau_i)) \ := \ \max\{e(P(t_i) \cap K,\ P(\tau_i))\,,\, e(P(\tau_i) \cap K,\ P(t_i))\}$$

Let $K \subset \mathbf{R}^d$ be a nonempty subset and $x_0 \in K$. Then contingent cone to K at x_0 is defined by

$$v \in T_K(x_0) \Longleftrightarrow \liminf_{h \to 0^+} d\left(v, \frac{K - x_0}{h}\right) = 0$$

See [3, Chapter 4] for many properties of tangent cones.

The contingent derivative $DP(\tau, y)$ of P at $(\tau, y) \in Graph(P)$ is defined as the set-valued map from $\mathbf{R}$ to $\mathbf{R}^d$ whose graph is described by

$$\text{Graph}(DP(\tau, y)) = T_{\text{Graph}(P)}(\tau, y)$$

It is not difficult to prove, using Proposition 5.1.4. from [3], that

$$v \in DP(\tau, y)(1) \Longleftrightarrow \liminf_{h \to 0+} dist\left(v, \frac{P(\tau + h) - y}{h}\right) = 0\ . \tag{3.3}$$

We first address the case when F is measurable in t, continuous in x and has convex compact values.

Theorem 3.2 *Assume that a closed valued map $P : [0,T] \leadsto \mathbf{R}^d$ is left absolutely continuous and $F : [0,T] \times \mathbf{R}^d \leadsto \mathbf{R}^d$ has closed convex values and satisfies*

$$x \leadsto F(t,x) \text{ is continuous for almost all } t \in [0,T] \ ; \tag{3.4}$$

$$t \leadsto F(t,x) \text{ is measurable for every } x \in \mathbf{R}^d; \tag{3.5}$$

$$\|F(t,x)\| \leq \mu(t) \text{ for almost all } t \in [0,T] \text{ and all } x \in P(t) \tag{3.6}$$

where μ is integrable.
Then the following conditions are equivalent:

i) there exists $A \subset [0,T]$ of full measure such that

$$\forall\, t \in A,\ \forall\, x \in P(t),\quad F(t,x) \cap DP(t,x)(1) \neq \emptyset \tag{3.7}$$

ii) for every $t_0 \in [0,T)$ and every $x_0 \in P(t_0)$ there exists a solution $x(\cdot)$ of the problem

$$x' \in F(t,x), \quad x(t_0) = x_0 \tag{3.8}$$

defined on $[t_0,T]$ and satisfying $x(t) \in P(t)$ for all $t \in [t_0,T]$.

The implication (i)$\Rightarrow$(ii) in the above Theorem can be deduce from Lemma 2.2 and the following:

Theorem 3.3 *Assume that P is left absolutely continuous on $[0,T]$, $F : Graph(P) \leadsto \mathbf{R}^d$ has closed convex values, satisfies (3.6) and :*

$$x \leadsto F(t,x) \text{ is upper semicontinuous on } P(t) \text{ for almost } t \in [0,T]\,; \tag{3.9}$$

and for almost all $t \in [0,T]$, for every $x \in P(t)$

$$\forall\, \beta > 0,\ DP(t,x)(1) \cap \operatorname{Liminf}_{h \to 0+} \frac{1}{h} \int_t^{t+h} F(s, x+\beta B)\, ds \neq \emptyset\ . \tag{3.10}$$

Then for every $t_0 \in [0,T]$ and $x_0 \in P(t_0)$ there exists a solution $x(\cdot)$ of the problem (3.8) defined on $[t_0,T]$ and satisfying $x(t) \in P(t)$ for all $t \in [t_0,T]$.

Remark – If for every $(t_0,x_0) \in Graph(P)$, with $t_0 < T$, the inclusion (3.8) has a viable solution in P defined on $[t_0,T]$ and if F satisfies (3.6) then P is left absolutely continuous.

To see the above fix $\varepsilon > 0$, a compact $K \subset \mathbf{R}^d$ and choose $\delta > 0$ such that if $A \subset [0,T]$ and $m(A) < \delta$ then $\int_A \mu(s)ds < \varepsilon$. Let $t_i < \tau_i$,

$\sum(\tau_i - t_i) < \delta$ and — for those i for which $P(t_i) \cap K \neq \emptyset$ — $x_i(\cdot)$ denote a solution of $x' \in F(t,x)$ satisfying $x(t_i) \in P(t_i) \cap K$ and $x_i(t) \in P(t)$ for $t \in [t_i, T]$. Thus we get

$$\sum e(x(t_i), P(\tau_i)) \leq \int_A \mu(s)ds < \varepsilon$$

Since $x(t_i) \in P(t_i) \cap K$ is arbitrary the proof follows (we have $e(P(t_i) \cap K, P(\tau_i)) = 0$ if $P(t_i) \cap K = \emptyset$).

The assumption (3.10) in Theorem 3.3 is not the weakest tangential condition ensuring the viability. We have given it because the roles played by P and F in it are clearly visible and when F is continuous with respect to x it reduces to the simple condition i) in Theorem 3.2.

However, without any changes in the proof of Theorem 3.3, we can replace (3.10) by another condition which is not only sufficient but also necessary. We shall denote, for any $(\tau, y) \in Graph(P)$, $\alpha > 0$, $h > 0$

$$W_{\tau,y,\alpha}(h) = P(\tau + h) - (y + \int_\tau^{\tau+h} F(s, y + \alpha B)ds) .$$

Theorem 3.4 *Let P and F be as in Theorem 3.3 with conditions (3.9), and (3.6) satisfied. Then the assertion of Theorem 3.3 holds if and only if for almost all $t \in [0,T)$, for every $x \in P(t)$, for every $\alpha > 0$*

$$0 \in DW_{t,x,\alpha}(0,0)(1) . \tag{3.11}$$

Remark - If in Theorem 3.3, (or in Theorem 3.4) the tangential condition (3.10) (or respectively (3.11), is satisfied for all $t \in [0,T)$ and $x \in P(t)$ then, in order to have the existence of viable solutions, it is enough to assume, instead of left absolute continuity of P, that $Graph(P)$ is closed from the left in the following sense:

for any $\tau \in (0,T]$, $t_n \to \tau-$, $x_n \in P(t_n)$ such that $x_n \to \xi$, we have $\xi \in P(\tau)$.

The proof of Theorem 3.3 follows the general idea given, it seems, for the first time by Nagumo in [19] in the context of differential equations and time independent constraints and explored later by many authors. Viability on tubes was also studied in [17]. The main difference of Theorem 3.4 with respect to [17] is that we assume our tangency conditions satisfied almost everywhere with respect to t and for all $\alpha > 0$, while in [17] the assumption is made everywhere and $\alpha = 0$. This allows us to deduce the viability condition (3.7) which is a natural extension of the stationary case to the tubes (see [2]). This change requires introduction of left absolute continuity of the map P and modifications of the proof – more accurate definition of the partially ordered family of approximate solutions.

In [17] also a necessary and sufficient condition for viability was given for F bounded and upper semicontinuous in x. But it was expressed in terms of reachable sets of the inclusion $x' \in F(t,x)$, $x(t_0) = x_0$, for all $(t_0, x_0) \in \mathit{Graph}(P)$ — usually very difficult to find. Our condition (3.11) uses the Aumann integrals of the maps $s \rightsquigarrow F(s, x_0 + \alpha B)$ and does not involve reachable sets.

Now we assume in addition that F is locally Lipschitz with respect to x, i.e.

$$\left\{ \begin{array}{l} \forall\, k > 0,\ \exists\, c_k \in L^1(0,T) \text{ such that for almost all } t \in [0,T] \\ F(t,\cdot) \text{ is } c_k(t)\text{-Lipschitz on } kB \end{array} \right. \tag{3.12}$$

Theorem 3.5 *Assume that a tube $P : [0,T] \rightsquigarrow \mathbf{R}^d$ is absolutely continuous, that F has convex compact images and satisfies (3.12), (3.5), (3.6). Then the following three statements are equivalent:*

i) There exists a set $A \subset [0,T]$ of full measure such that

$$\forall\, t \in A,\ \forall\, x \in P(t),\ \ F(t,x) \cap DP(t,x)(1) \neq \emptyset$$

ii) There exists a set $C \subset [0,T]$ of full measure such that

$$\forall\, t \in C,\ \forall\, x \in P(t),\ \ (\{1\} \times F(t,x)) \cap \overline{co}\left(T_{\mathrm{Graph}(P)}(t,x)\right) \neq \emptyset$$

iii) For every $t_0 \in [0,T]$ and $x_0 \in P(t_0)$ there exists $x \in S_{[t_0,T]}(x_0)$ such that $x(t) \in P(t)$ for every $t \in [t_0, T]$.

In this way our theorem generalizes [14] (see also [2, Theorem 3.2.4]).

In the Lipschitz case we also provide a necessary and sufficient condition for invariance (improving the results of [2, Theorem 5.3.4]).

Theorem 3.6 (Invariance) *Assume that a tube $P : [0,T] \rightsquigarrow \mathbf{R}^d$ is absolutely continuous, that F has convex compact images and satisfies (3.12), (3.5), (3.6). Then the following three statements are equivalent:*

i) There exists a set $A \subset [0,T]$ of full measure such that for every $t \in A$ and all $x \in P(t)$ we have

$$F(t,x) \subset DP(t,x)(1)$$

ii) There exists a set $C \subset [0,T]$ of full measure such that for every $t \in C$ and all $x \in P(t)$ we have

$$\{1\} \times F(t,x) \subset \overline{co}\left(T_{\mathrm{Graph}(P)}(t,x)\right)$$

iii) For all $t_0 \in [0,T]$ and $x_0 \in P(t_0)$ every $x \in S_{[t_0,T]}(x_0)$ verifies $x(t) \in P(t)$ for all $t \in [t_0,T]$.

4 Hamilton-Jacobi-Bellman Theory

We first recall generalizations of notions of directional derivatives and gradients for nonsmooth functions which allow to define nonsmooth solutions of the Hamilton-Jacobi-Bellman equation.

Definition 4.1 *Consider an extended function $\varphi : \mathbf{R}^n \mapsto \mathbf{R} \cup \{\pm\infty\}$.*

i) The domain of φ, $\mathrm{Dom}(\varphi)$, is the set of all x_0 such that $\varphi(x_0) \neq \pm\infty$.

ii) The subdifferential and the superdifferential of φ at $x_0 \in \mathrm{Dom}(\varphi)$ are respectively given by

$$\partial_-\varphi(x_0) \;=\; \left\{p \in \mathbf{R}^n \mid \liminf_{x\to x_0} \frac{\varphi(x)-\varphi(x_0)- < p, x-x_0 >}{\|x-x_0\|} \;\geq\; 0\right\}$$

and

$$\partial_+\varphi(x_0) \;=\; \left\{p \in \mathbf{R}^n \mid \limsup_{x\to x_0} \frac{\varphi(x)-\varphi(x_0) - < p, x-x_0 >}{\|x-x_0\|} \;\leq\; 0\right\}$$

iii) The contingent epiderivative and the contingent hypoderivative of φ at $x_0 \in \mathrm{Dom}(\varphi)$ in the direction $u \in \mathbf{R}^d$ are respectively defined by

$$D_{\uparrow}\varphi(x_0)(u) \;=\; \liminf_{h\to 0+,\, u'\to u} \frac{\varphi(x_0+hu')-\varphi(x_0)}{h}$$

and

$$D_{\downarrow}\varphi(x_0)(u) \;=\; \limsup_{h\to 0+,\, u'\to u} \frac{\varphi(x_0+hu')-\varphi(x_0)}{h}$$

It was shown in [3, p. 226] that for all $x_0 \in \mathrm{Dom}(\varphi)$

$$\mathcal{E}p(D_\uparrow \varphi(x_0)) = T_{\mathcal{E}p(\varphi)}(x_0, \varphi(x_0)) \tag{4.1}$$

where $\mathcal{E}p$ denotes the epigraph. Similarly

$$\mathcal{H}yp(D_\downarrow \varphi(x_0)) = T_{\mathcal{H}yp(\varphi)}(x_0, \varphi(x_0)) \tag{4.2}$$

where $\mathcal{H}yp$ denotes the hypograph.

From [12] (see also [3, pp. 249, 253]) we know that

Proposition 4.2 *Let* $\varphi : \mathbf{R}^d \mapsto \mathbf{R} \cup \{\pm\infty\}$ *and* $x_0 \in \mathrm{Dom}(\varphi)$. *Then*

$$p \in \partial_- \varphi(x_0) \iff \forall\, u \in \mathbf{R}^d,\ < p, u > \leq D_\uparrow \varphi(x_0)(u)$$

$$\iff (p, -1) \in \left[T_{\mathcal{E}p(\varphi)}(x_0, \varphi(x_0))\right]^- \quad \text{(the negative polar cone)}$$

and

$$p \in \partial_+ \varphi(x_0) \iff \forall\, u \in \mathbf{R}^d,\ < p, u > \geq D_\downarrow \varphi(x_0)(u)$$

$$\iff (p, -1) \in \left[T_{\mathcal{H}yp(\varphi)}(x_0, \varphi(x_0))\right]^+ \quad \text{(the positive polar cone)}$$

4.1 Value Function of Mayer's Problem

Let an extended function $g : \mathbf{R}^d \mapsto \mathbf{R} \cup \{+\infty\}$ be given. Consider the minimization problem (called *Mayer's problem*):

$$\min \{g(x(T)) \mid x \in S_{[t_0,T]}(x_0)\} \tag{4.3}$$

The value function $\mathcal{V} : [0,T] \times \mathbf{R}^d \mapsto \mathbf{R} \cup \{\pm\infty\}$ is defined by:

$$\forall\, (t_0, x_0) \in [0,T] \times \mathbf{R}^d, \quad \mathcal{V}(t_0, x_0) = \inf\{g(x(T)) \mid x \in S_{[t_0,T]}(x_0)\} \tag{4.4}$$

We assume that

$$\begin{cases} F \text{ has nonempty convex compact images} \\ \forall\, x \in \mathbf{R}^d,\ F(\cdot, x) \text{ is measurable} \\ \exists\, \mu \in L^1(0,T) \text{ such that for almost all } t \in [0,T], \text{ we have} \\ \forall\, x \in \mathbf{R}^d,\ \|F(t,x)\| \leq \mu(t) \\ g \text{ is lower semicontinuous} \end{cases} \tag{4.5}$$

Proposition 4.3 *If (4.5) holds true and for almost all $t \in [0,T]$, $F(t,\cdot)$ is upper semicontinuous, then $\mathcal{V}$ is lower semicontinuous and*

$$\forall\,(t_0,x_0) \in [0,T] \times \mathbf{R}^d,\;\; \mathcal{V}(t_0,x_0) = \min\left\{g(x(T)) \mid x \in S_{[t_0,T]}(x_0)\right\} \tag{4.6}$$

Furthermore, the set-valued map

$$t \rightsquigarrow P(t) = \{(x,r) \in \mathbf{R}^d \times \mathbf{R} \mid r \geq \mathcal{V}(t,x)\} \text{ is absolutely continuous} \tag{4.7}$$

and

$$\left\{\begin{array}{l} \exists\, A \subset]0,T[\text{ of full measure such that } \forall\,(t,x) \in \mathrm{Dom}(\mathcal{V}) \cap A \times \mathbf{R}^d, \\ \inf_{v\in F(t,x)} D_\uparrow \mathcal{V}(t,x)(1,v) \leq 0,\;\; \sup_{v\in F(t,x)} D_\uparrow \mathcal{V}(t,x)(-1,-v) \leq 0 \end{array}\right.$$

Remark — We observe that Graph(P) is equal to the epigraph $\mathcal{E}p(\mathcal{V})$ of $\mathcal{V}$ and (4.7) yields the following relations: for every $\overline{x} \in \mathbf{R}^d$

$$g(\overline{x}) = \mathcal{V}(T,\overline{x}) = \liminf_{t\to T-,\, x\to\overline{x}} \mathcal{V}(t,x),\;\; \mathcal{V}(0,\overline{x}) = \liminf_{t\to 0+,\, x\to\overline{x}} \mathcal{V}(t,x) \;\; \square \tag{4.8}$$

The first two statements in Proposition 4.3 follow by exactly the same arguments as in the proof of [13, Proposition 2.1].

Assume that F has nonempty compact images and define the Hamiltonian $H : [0,T] \times \mathbf{R}^d \times \mathbf{R}^d \mapsto \mathbf{R}$ by

$$H(t,x,p) \;=\; \max_{v\in F(t,x)} < p, v > \tag{4.9}$$

Then $H(t,x,\cdot)$ is convex and positively homogeneous. Furthermore, if $F(t,\cdot)$ is upper semicontinuous (resp., lower semicontinuous), then so is $H(t,\cdot,p)$ and if $F(\cdot,x)$ is measurable, then $H(\cdot,x,p)$ is also measurable.

Consider an extended function $V : [0,T] \times \mathbf{R}^d \mapsto \mathbf{R} \cup \{+\infty\}$. We may always assume that V is defined on $\mathbf{R} \times \mathbf{R}^d$ by setting $V(t,x) = +\infty$, whenever $t \notin [0,T]$. In theorem below we use Definition 4.1 with such extension of V.

Theorem 4.4 *Assume (3.12), (4.5) and let $V : [0,T] \times \mathbf{R}^d \mapsto \mathbf{R} \cup \{+\infty\}$ be an extended lower semicontinuous function. Consider the set-valued map*

$$[0,T] \ni t \rightsquigarrow P(t) \;=\; \{(x,r) \in \mathbf{R}^d \times \mathbf{R} \mid r \geq V(t,x)\} \tag{4.10}$$

Then the following three statements are equivalent:

$$\begin{cases} i) & V \text{ is the value function, i.e., } V = \mathcal{V} \\ \\ ii) & \exists\, A \subset]0,T[\text{ of full measure such that } \forall\,(t,x) \in \mathrm{Dom}(V) \cap A \times \mathbf{R}^d, \\ & \inf_{v\in F(t,x)} D_\uparrow V(t,x)(1,v) \le 0, \ \ \sup_{v\in F(t,x)} D_\uparrow V(t,x)(-1,-v) \le 0 \\ & P(\cdot) \text{ is absolutely continuous and } V(T,\cdot) = g(\cdot) \\ \\ iii) & \exists\, C \subset]0,T[\text{ of full measure such that } \forall\,(t,x) \in \mathrm{Dom}(V) \cap C \times \mathbf{R}^d, \\ & \forall\,(p_t,p_x,q) \in \left[T_{\mathcal{E}p(V)}(t,x,V(t,x))\right]^-, \ \ -p_t + H(t,x,-p_x) = 0 \\ & P(\cdot) \text{ is absolutely continuous and } V(T,\cdot) = g(\cdot) \end{cases}$$

Remark — If a function $V : [0,T] \times \mathbf{R}^d \mapsto \mathbf{R}$ is locally uniformly absolutely continuous in the sense of [20], then the tube P given by (4.10) is always absolutely continuous. □

Theorem 4.5 *Under all assumptions of Theorem 4.4 suppose that* Dom(V) *is closed, the restriction of V to its domain is continuous and the maps*

$$t \rightsquigarrow \{(x,r) \in \mathbf{R}^d \times \mathbf{R} \mid r \ge V(t,x)\}$$

$$t \rightsquigarrow \{(x,r) \in \mathbf{R}^d \times \mathbf{R} \mid r \le V(t,x) \ne +\infty\}$$

are absolutely continuous. Then V is the value function if and only if

$$\begin{cases} V(T,\cdot) = g(\cdot), \ \exists\, D \subset [0,T] \text{ of full measure } \forall\,(t,x) \in D \times \mathbf{R}^d \\ \forall\,(p_t,p_x,q) \in \left[T_{\mathcal{E}p(V)}(t,x,V(t,x))\right]^-, \ \ -p_t + H(t,x,-p_x) \ge 0 \\ \forall\,(p_t,p_x,q) \in \left[T_{\mathcal{H}yp(V)}(t,x,V(t,x))\right]^+, \ \ -p_t + H(t,x,-p_x) \le 0 \end{cases} \tag{4.11}$$

4.2 Solutions of the Hamilton-Jacobi-Bellman Equation with the Hamiltonian Measurable in Time

Consider $H : [0,T] \times \mathbf{R}^d \times \mathbf{R}^d \mapsto \mathbf{R}$ and the Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V}{\partial t}(t,x) + H\left(t,\ x,\ -\frac{\partial V}{\partial x}(t,x)\right) = 0 \tag{4.12}$$

We assume:

$$\begin{cases} i) & \forall\, t \in [0,T], \;\; H(t,\cdot,\cdot) \;\text{ is continuous} \\ ii) & \forall\, (x,p) \in \mathbf{R}^d \times \mathbf{R}^d, \;\; H(\cdot,x,p) \;\text{ is measurable} \\ iii) & H(t,x,\cdot) \;\text{ is convex} \\ iv) & \exists\, \mu \in L^1(0,T), \; \forall\, p \in B, \;\; |H(t,x,p)| \;\leq\; \mu(t) \\ v) & \forall\, k > 0, \; \exists\, c_k \in L^1(0,T) \text{ such that for almost all } t \in [0,T], \\ & \forall\, p \in B, \; H(t,\cdot,p) \;\text{ is }\; c_k(t) - \text{Lipschitz on } kB \\ vi) & H(t,x,\cdot) \;\text{ is positively homogeneous} \end{cases} \tag{4.13}$$

where B denotes the closed unit ball in $\mathbf{R}^d$.

Remark — Assumption $vi)$ may be replaced by the Lipschitz continuity of $H(t,x,\cdot)$ together with modified with respect to p conditions $iv)$, $v)$. Then it is possible to study solutions of (4.12) via a Hamilton-Jacobi-Bellman equation with the new (conjugate) Hamiltonian meeting assumptions (4.13) (as it was done for instance in [5]). □

Define $F : [0,T] \times \mathbf{R}^d \leadsto \mathbf{R}^d$ by

$$F(t,x) = \bigcap_{\|p\|=1} \{\, v \in \mathbf{R}^d \mid \langle p,v\rangle \;\leq\; H(t,x,p) \,\} \tag{4.14}$$

Proposition 4.6 *If (4.13) holds true, then F verifies (3.12), (4.5) and*

$$\forall\, p \;\in\; \mathbf{R}^d, \quad \sup_{v\in F(t,x)} < p,v > \;=\; H(t,x,p)$$

Proof — Fix $x \in \mathbf{R}^d$ and consider a dense subset $\{p_i\}_{i\geq 1}$ of the unit sphere in $\mathbf{R}^d$. For every $i \geq 1$ define the set-valued map $\mathcal{P}_i : [0,T] \leadsto \mathbf{R}^d$ by

$$\mathcal{P}_i(t) \;=\; \{\, v \in \mathbf{R}^d \mid \langle p_i,v\rangle \;\leq\; H(t,x,p_i)\,\}$$

From the separation theorem and continuity of $H(t,x,\cdot)$ it follows that

$$F(t,x) \;=\; \bigcap_{i\geq 1} \mathcal{P}_i(t)$$

By [3, Theorem 8.2.9], $\mathcal{P}_i$ is measurable. Thus by [3, Theorem 8.2.4] the set-valued map

$$t \rightsquigarrow \bigcap_{i\geq 1} \mathcal{P}_i(t) \;=\; F(t,x)$$

is also measurable. The remaining properties of F were checked in the proof of Proposition 7.1 of [13].

Q.E.D.

Consider the differential inclusion

$$x'(t) \;\in\; F(t,x(t)) \text{ almost everywhere} \tag{4.15}$$

and let $S_{[t_0,T]}(x_0)$ have the same meaning as before. From Theorems 4.4, 4.5 we immediately deduce

Theorem 4.7 *Assume (4.13) and consider an extended lower semicontinuous function* $V : [0,T] \times \mathbf{R}^d \mapsto \mathbf{R} \cup \{+\infty\}$. *Set* $g(\cdot) = V(T,\cdot)$.

Then the following two statements are equivalent:
i) The set-valued map $t \rightsquigarrow \{(x,r) \mid r \geq V(t,x)\}$ *is absolutely continuous and there exists* $A \subset [0,T]$ *of full measure such that for all* $(t,x) \in A \times \mathbf{R}^d$

$$\forall\, (p_t,p_x,q) \in \left[T_{\mathcal{E}p(V)}(t,x,V(t,x))\right]^-, \quad -p_t + H(t,x,-p_x) = 0 \tag{4.16}$$

ii) For all $(t_0,x_0) \in [0,T] \times \mathbf{R}^d$,

$$V(t_0,x_0) \;=\; \inf\left\{g(x(T)) \mid x \in S_{[t_0,T]}(x_0)\right\} \tag{4.17}$$

Corollary 4.8 (Maximum Principle) *Assume (4.13) and let* V_1, V_2 *be extended lower semicontinuous functions from* $[0,T] \times \mathbf{R}^d$ *into* $\mathbf{R} \cup \{+\infty\}$ *satisfying i) of Theorem 4.7.*

If $V_1(T,\cdot) \geq V_2(T,\cdot)$, *then* $V_1 \;\geq\; V_2$.

Results of Section 4.1 imply different equivalent formulations of statement *ii)* of Theorem 4.7 linking V to viscosity solutions. For instance we have.

Corollary 4.9 *Assume (4.13) and consider a locally Lipschitz function* $V : [0,T] \times \mathbf{R}^d \mapsto \mathbf{R}$. *Set* $g(\cdot) = V(T, \cdot)$.

Then the following three statements are equivalent:

i) There exists a set $A \subset [0,T]$ *of full measure such that*

$$\forall\, (t,x) \in A \times \mathbf{R}^d,\ \ \forall\, (p_t, p_x) \in \partial_- V(t,x),\ \ -p_t + H(t,x,-p_x) = 0$$

ii) For all $(t_0, x_0) \in [0,T] \times \mathbf{R}^d$,

$$V(t_0, x_0) \ = \ \inf \left\{ g(x(T)) \mid x \in S_{[t_0,T]}(x_0) \right\}$$

iii) There exists $C \subset [0,T]$ *of full measure such that for all* $(t,x) \in C \times \mathbf{R}^d$

$$\left\{ \begin{array}{ll} \forall\, (p_t,p_x) \in \partial_- V(t,x), \ \ -p_t + H(t,x,-p_x) & \geq \ 0 \\ \forall\, (p_t,p_x) \in \partial_+ V(t,x), \ \ -p_t + H(t,x,-p_x) & \leq \ 0 \end{array} \right.$$

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