Working Paper

Set-Valued Analysis, Viability Theory and Partial Differential Inclusions

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FOREWORD

Systems of first-order partial differential inclusions — solutions of which are feedbacks governing viable trajectories of control systems — are derived. A variational principle and an existence theorem of a (single-valued contingent) solution to such partial differential inclusions are stated. To prove such theorems, tools of set-valued analysis and tricks taken from viability theory are surveyed.

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Set-Valued Analysis, Viability Theory and Partial Differential Inclusions

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1 Introduction

We explain how systems of first-order partial differential inclusions arise in the case of the simplest economic example one can think of: one consumer, one commodity.

Let $K := [0, b]$ the subset of a scarce commodity x. Assume that the consumption rate of a consumer is equal to $a > 0$, so that, without any further restriction, its exponential consumption will leave the viability subset $[0, b]$. Hence its consumption is slowed down by a nonnegative price which is regarded as a control. We assume that a bound c is set to inflation. In summary, the consumption and the price evolve according to the simple system of differential inclusions:

$$(1) \qquad \begin{cases} i) & \text{for almost all } t \geq 0, \;\; x'(t) = ax(t) - u(t) \\ ii) & \text{and } \;\; -c \leq u'(t) \leq c \end{cases}$$

subjected to the **viability constraint**

$$\forall\, t \geq 0, \;\; x(t) \in K \;\;\&\;\; u(t) \in \mathbf{R}_+$$

A **subregulation map** $R : [0, b] \rightsquigarrow \mathbf{R}_+$ for this problem is a set-valued map R satisfying the **viability property**: from any $x_0 \in \mathrm{Dom}(R)$, $u_0 \in R(x_0)$, starts a solution $(x(\cdot), u(\cdot))$ to the above control system satisfying

$$\forall\, t \geq 0, \;\; u(t) \in R(x(t))$$

By using tools of **set-valued analysis**, and specifically, the concept of contingent derivative of a set-valued map, we shall see that such subregulation maps are solutions to the **first-order partial differential inclusion**

$$(2) \qquad \forall\, (x, u) \in \mathrm{Graph}(R), \;\; 0 \in DR(x, u)(ax - u) + [-c, +c]$$

In particular, single-valued solutions r to

$$(3) \qquad 0 \in r'(x)(ax - r(x)) + [-c, +c]$$

regarded as **feedbacks** in control theory or **planning procedures** in economics, are of special interest. In this example, we can exhibit (at least) two of them.

For that purpose, let us introduce the functions $\rho_c^\sharp$ and $\rho_c^\flat$ defined on $[0,\infty[$ by

$$\begin{cases} i) & \rho_c^\flat(u) := \frac{c}{a^2}(e^{-au/c} - 1 + \frac{a}{c}u) \approx \frac{u^2}{2c} \\ ii) & \rho_c^\sharp(u) := -ce^{a(u-ab)/c}/a^2 + u/a + c/a^2 \end{cases}$$

and the functions $r_c^\sharp$ and $r_c^\flat$ defined on $[0,b]$ by

$$\begin{cases} i) & r_c^\flat(x) = u \text{ if and only if } x = \rho_c^\flat(u) \\ ii) & r_c^\sharp(x) = 0 \text{ if } x \in [0, \rho_c^\sharp(0)] \ \ (\rho_c^\sharp(0) = \frac{c}{a^2}(1 - e^{-a^2b/c})) \\ iii) & r_c^\sharp(x) = u \text{ if and only if } x = \rho_c^\sharp(u) \text{ when } x \in [\rho_c^\sharp(0), b] \end{cases}$$

Then one can show that both $r_c^\flat$ and $r_c^\sharp$ are such single-valued solutions to (3).

But, instead of looking for examples of solutions, one can look for the **largest**[1] **subregulation map**, which can be shown to exist and computed in this particular example: **The subregulation map R^c defined by**

$$(4) \qquad \forall\, x \in [0,b], \ \ R^c(x) = [r_c^\sharp(x), r_c^\flat(x)]$$

is the largest subregulation map.

Indeed, set $u^\sharp(t) := u_0 + ct$ and $u^\flat(t) := u_0 - ct$ and denote by $x^\sharp(\cdot)$ and $x^\flat(\cdot)$ the solutions starting at x_0 to differential equations $x' = ax - u^\sharp(t)$ and $x' = ax - u^\flat(t)$ respectively. Then any solution $(x(\cdot), u(\cdot))$ to the system (1) satisfies $u^\flat(\cdot) \leq u(\cdot) \leq u^\sharp(\cdot)$ and thus, $x^\sharp(\cdot) \leq x(\cdot) \leq x^\flat(\cdot)$ because

$$x(t) = e^{at}x_0 - \int_0^t e^{a(t-s)}u(s)ds$$

1.) We first observe that the equations of the curves $t \mapsto (x^\sharp(t), u^\sharp(t))$ and $t \mapsto (x^\flat(t), u^\flat(t))$ passing through (x_0, u_0) are solutions to the differential equations

$$d\rho_c^\sharp = \frac{1}{c}(a\rho_c^\sharp - u)du \ \ \& \ \ d\rho_c^\flat = -\frac{1}{c}(a\rho_c^\flat - u)du$$

[1]in the sense that every subregulation map R satisfies $\mathrm{Graph}(R) \subset \mathrm{Graph}(R^c)$.

the solutions of which are

$$\begin{cases} i) & \rho_c^\sharp(u) = e^{a(u-u_0)/c}(x_0 - u_0/a - c/a^2) + u/a + c/a^2 \\ ii) & \rho_c^\flat(u) = e^{a(u_0-u)/c}(x_0 - u_0/a + c/a^2) + u/a - c/a^2 \end{cases}$$

Let $\rho_c^\flat$ be the solution passing through $(0,0)$, which is equal to $\rho_c^\flat(u) = \frac{c}{a^2}(e^{-au/c} - 1 + \frac{a}{c}u)$ and $\rho_c^\sharp(u) = -ce^{a(u-ab)/c}/a^2 + u/a + c/a^2$ be the solution passing through the pair (ab, b).

2.) If $u_0 > r_c^\flat(x_0)$, then any solution $(x(\cdot), u(\cdot))$ starting from (x_0, u_0) satisfies

$$x(t) \leq x^\flat(t) = \rho_c^\flat(u^\flat(t)) \leq \rho_c^\flat(u(t))$$

because $\rho_c^\flat(\cdot)$ is nondecreasing. Hence, when $x(t_1) = 0$, we deduce that $u(t_1) > 0$, so that such solution is not viable.

If $0 \leq u_0 < r_c^\sharp(x_0)$, any solution $(x(\cdot), u(\cdot))$ satisfies inequalities

$$x(t) \geq x^\sharp(t) = \rho_c^\sharp(u^\sharp(t)) \geq \rho_c^\sharp(u(t))$$

Therefore, when $x(t_1) = b$ for some time t_1, its velocity $x'(t_1) = ab - u(t_1)$ is positive, so that the solution is not viable.

3.) It remains to show that from any initial pair (x_0, u_0) where $u_0 \in R^c(x_0)$ starts at least a solution. Actually, we shall construct a **heavy solution**, i.e., a solution for which the prices evolve with minimal velocity. Assume for instance that $u_0 < ax_0$.

Since we want to choose the price velocity with minimal norm, we take $u'(t) = 0$ as long as the solution $x(\cdot)$ to the differential equation $x' = ax - u_0$ yields a consumption $x(t) < \rho_c^\sharp(u_0)$. When for some time t_1, the consumption $x(t_1) = \rho_c^\sharp(u_0)$, it has to be slowed down. Otherwise $(x(t_1 + \varepsilon), u_0)$ will be below the curve $\rho_c^\sharp$ and we mentioned that in this case, any solution starting from this situation will eventually cease to be viable. Therefore, prices should increase to slow down the consumption growth. The idea is to take the smallest velocity u' such that the vector $(x'(t_1), u')$ takes the state inside the graph of R^c: they are the velocities $u' \geq x'(t_1)/\rho_c^{\sharp\prime}(u_0)$. By construction, it is achieved by the velocity of $x^\sharp(\cdot)$, which is the highest one allowed to increase prices. Therefore, by taking

$$x(t) := x^\sharp(t) := e^{a(t-t_1)}(x(t_1) - u_0/a - c/a^2) + c(t - t_1)/a + u_0/a + c/a^2$$

and $u(t) := u_0 + c(t - t_1)$ for $t \in [t_1, t_1 + (ab - u_0)/c]$, we get a solution which ranges over the curve $x^\sharp(t) = \rho_c^\sharp(u^\sharp(t))$. According to the above differential equation, we see that $x(t)$ increases to b where it arrives with velocity 0 and the price increases linearly until it arrives at the equilibrium price ab. Since (b, ab) is an equilibrium, the heavy solution stays there: we take $x(t) \equiv b$ and $u(t) \equiv ab$ when $t \geq t_1 + u_0/c$.

One last remark: Quincampoix proved in [38,41] the semipermeability property of the part of the boundary of the graph of R^c contained in the interior of $[0, b] \times \mathbf{R}_+$: The solutions which reach this boundary cannot come back to it, and have to remain on its boundary.

Looking for both single-valued and set-valued solutions to systems of first-order partial differential inclusions is then the topic of this paper. We present it in the framework of control of systems under state constraints, which provided the motivation for studying this class of problems in the first place.

We shall review

1. **The Tools** coming from Set-Valued Analysis
2. **The Tricks** taken from Viability Theory
3. **The Theorems** dealing with single-valued and set-valued solutions to systems of first-order partial differential inclusions

2 The Tools

2.1 Upper Limits of Sets

In this paper, X, Y, Z denote finite dimensional vector-spaces. The unit ball is denoted by B (or B_X if the space must be mentioned). Let $K \subset X$, we denote by

$$d_K(x) \;:=\; d(x, K) \;:=\; \inf_{y \in K} \|x - y\|$$

the **distance from x to K**, where we set $d(x, \emptyset) := +\infty$. Upper Limits of sets have been introduced by Painlevé and popularized by Kuratowski in his famous book **Topologie**, so that they are often called **Kuratowski upper limits** of sequences of sets.

Let $(K_n)_{n \in \mathbf{N}}$ be a sequence of subsets of X. We say that the subset

$$\mathrm{Limsup}_{n \to \infty} K_n \;:=\; \left\{ x \in X \;\mid\; \liminf_{n \to \infty} d(x, K_n) = 0 \right\}$$

is the upper limit of the sequence K_n[2] .

Upper limits are obviously closed and $\mathrm{Limsup}_{n\to\infty} K_n$ is the set of cluster points of sequences $x_n \in K_n$, i.e., of limits of subsequences $x_{n'} \in K_{n'}$.

2.2 Contingent Cones

Let $K \subset X$ be a subset of a normed vector space X and $x \in K$. The contingent[3] cone $T_K(x)$ is the upper limit of the subsets $(K-x)/h$

$$T_K(x) \ := \ \mathrm{Limsup}_{h\to 0+} \frac{K-x}{h}$$

so that $T_K(x)$ is always a closed cone of "tangent directions" (which is convex when K is convex or, more generally, when the contingent cone is lower semicontinuous, a vector space when K is a smooth manifold).

2.3 Graphical Convergence of Maps

Let us consider a sequence of set-valued maps $F_n : X \rightsquigarrow Y$. The set-valued map $F^\sharp := \mathrm{Lim}^\sharp{}_{n\to\infty} F_n$ from X to Y defined by

$$\mathrm{Graph}(\mathrm{Lim}^\sharp{}_{n\to\infty} F_n) := \mathrm{Limsup}_{n\to\infty} \mathrm{Graph}(F_n)$$

is called the *(graphical) upper limit* of the set-valued maps F_n. Even for single-valued maps, this is a weaker convergence than the pointwise convergence: if $f_n : X \mapsto Y$ converges pointwise to f, then, for every $x \in X$, $f(x) \in f^\sharp(x)$. If the sequence is equicontinuous, then $f^\sharp(x) = \{f(x)\}$.

The following result justifies the introduction of this concept of convergence:

Theorem 2.1 (Convergence Theorem) *Let F_n be a sequence of nontrivial set-valued maps from $K \subset X$ to Y with uniform linear growth: there exists $c > 0$ such that, for any $n \geq 0$,*

$$\forall\, x \in K, \ \ \|F_n(x)\| \ := \ \sup_{y\in F_n(x)} \|y\| \ \leq \ c(\|x\|+1)$$

[2]and that the subset

$$\mathrm{Liminf}_{n\to\infty} K_n \ := \ \{x \in X \mid \lim_{n\to\infty} d(x, K_n) = 0\}$$

is its lower limit. We shall use only upper limits in this paper, but symmetric definitions based on lower limits can be introduced as well.

[3]introduced by G. Bouligand in the 30's.

Let us consider measurable *functions x_m and y_m from Ω to X and Y respectively, satisfying $y_m(\omega) \in F_m(x_m(\omega))$ for almost all $\omega \in \Omega$.*

If

$$\begin{cases} i) & x_m(\cdot) \text{ converges almost everywhere to a function } x(\cdot) \\ ii) & y_m(\cdot) \in L^1(\Omega;Y) \text{ and converges } \textit{weakly} \text{ in } L^1(\Omega;Y) \\ & \textit{to a function } y(\cdot) \in L^1(\Omega;Y) \end{cases}$$

then for almost all $\omega \in \Omega$, $y(\omega) \in \overline{co}F^\sharp(x(\omega))$.

2.4 Contingent Derivatives of Maps

We introduce the **differential quotients**

$$u \rightsquigarrow \nabla_h F(x,y)(u) := \frac{F(x+hu)-y}{h}$$

of a set-valued map $F : X \rightsquigarrow Y$ at $(x,y) \in \text{Graph}(F)$.

The contingent derivative $DF(x,y)$ of F at $(x,y) \in \text{Graph}(F)$ is the graphical upper limit of differential quotients:

$$DF(x,y) = \text{Lim}^\sharp{}_{h\to 0+} \nabla_h F(x,y)$$

We deduce the formula

$$\text{Graph}(DF(x,y)) = T_{\text{Graph}(F)}(x,y)$$

Indeed, we know that the contingent cone

$$T_{\text{Graph}(F)}(x,y) = \text{Limsup}_{h\to 0+} \frac{\text{Graph}(F)-(x,y)}{h}$$

is the upper limit of the differential quotients $\frac{\text{Graph}(F)-(x,y)}{h}$ when $h \to 0+$. It is enough to observe that

$$\text{Graph}(\nabla_h F(x,y)) = \frac{\text{Graph}(F)-(x,y)}{h}$$

and to take the upper limit to conclude.

2.5 Weak Derivatives: Distributional and Contingent Derivatives

Let us consider a single-valued map $f : X \mapsto Y$ and its differential quotients $\nabla_h f(x)(v) := \frac{f(x+hv)-f(x)}{h}$. The function f is Gâteaux differentiable if these differential quotients converge for the pointwise convergence topology. This strong requirement can be weakened in (at least) two ways, each way sacrificing different groups of properties of the usual derivatives.

- The distributional derivative is the limit of the difference-quotients $x \mapsto \nabla_h f(x)(v) := \frac{f(x+hv)-f(x)}{h}$ (when $h \to 0$) in the space of distributions, and the limit is a vectorial distribution $D_v f \in \mathcal{D}'(X;Y)$ (and no longer necessarily a single-valued function).

  Furthermore, one can define differential quotients of any vectorial distribution $T \in \mathcal{D}'(X;Y)$ and define the derivative of a distribution as their limit (when $h \to 0$) in the space of distributions.

- The contingent derivative is the *upper graphical limit* of the difference-quotients $v \mapsto \nabla_h f(x)(v) := \frac{f(x+hv)-f(x)}{h}$ (when $h \to 0+$), and the limit is a set-valued map $Df(x) : X \rightsquigarrow Y$ (and no longer necessarily a single-valued function).

  Furthermore, we have defined differential quotients of any set-valued map $F : X \rightsquigarrow Y$ and defined the contingent derivative of a set-valued map as their limit the *upper graphical limit* (when $h \to 0+$).

In both cases, the approaches are similar: they use (different) convergences weaker than the pointwise convergence for increasing the possibility for the difference-quotients to converge, at the price of losing some properties by passing to these weaker limits (the pointwise character for distributional derivatives, the linearity of the differential operator for graphical derivatives).

2.6 Epilimits

For reasons motivated both by optimization theory and Lyapunov stability, we involve the order relation on $\mathbf{R}$ by characterizing extended functions $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ by their epigraphs instead of their graphs.

The epigraph of the lower epilimit

$$\lim^{\sharp}_{\uparrow n\to\infty} V_n$$

of a sequence of extended functions $V_n : X \mapsto \mathbf{R} \cup \{+\infty\}$ is the upper limit of the epigraphs:

$$\mathcal{E}p(\lim^{\sharp}_{\uparrow n\to\infty} V_n) \ := \ \mathrm{Limsup}_{n\to\infty}\mathcal{E}p(V_n)$$

One can check that

$$\lim^{\sharp}_{\uparrow n\to\infty} V_n(x_0) \ = \ \liminf_{n\to\infty, x\to x_0} V_n(x)$$

2.7 Contingent Epiderivatives

Let $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be a nontrivial extended function and x belong to its domain.

We associate with it the **differential quotients**

$$u \rightsquigarrow \nabla_h V(x)(u) \ := \ \frac{V(x+hu) - V(x)}{h}$$

The **contingent epiderivative** $D_{\uparrow}V(x)$ of V at $x \in \mathrm{Dom}(V)$ is the lower epilimit of its differential quotients:

$$D_{\uparrow}V(x) \ = \ \lim^{\sharp}_{\uparrow h\to 0+} \nabla_h V(x)$$

The contingent cone to the epigraph of V at $(x, V(x))$ is the epigraph of contingent epiderivative:

$$\mathcal{E}p(D_{\uparrow}V(x)) \ = \ T_{\mathcal{E}p(V)}(x, V(x))$$

Indeed, we know that the contingent cone

$$T_{\mathcal{E}p(V)}(x, V(x)) \ = \ \mathrm{Limsup}_{h\to 0+}\frac{\mathcal{E}p(V) - (x, V(x))}{h}$$

is the upper limit of the differential quotients $\frac{\mathcal{E}p(V) - (x, V(x))}{h}$ when $h \to 0+$. It is enough to observe that

$$\mathcal{E}p(D_{\uparrow}V(x)) := T_{\mathcal{E}p(V)}(x,y) \ \& \ \mathcal{E}p(\nabla_h F(x,y)) = \frac{\mathcal{E}p(V) - (x, V(x))}{h}$$

to conclude.

We refer to **Set-Valued Analysis** ([14]) for further details on these concepts and their properties.

3 The Tricks

Let us consider a *control system* (U, f) defined by

- a feedback set-valued map $U : X \rightsquigarrow Z$
- a map $f : \text{Graph}(U) \mapsto X$ describing the dynamics of the system

governing the evolution

$$(5) \qquad \begin{cases} i) & \text{for almost all } t, \;\; x'(t) = f(x(t), u(t)) \\ ii) & \text{where } \; u(t) \in U(x(t)) \end{cases}$$

Let us remark that when we take for controls the velocities, i.e., $U(x) := F(x)$ and $f(x, u) := u$, we find the usual differential inclusion $x' \in F(x)$. Conversely, the above system is the differential inclusion $x' \in F(x)$ in disguise where $F(x) := f(x, U(x))$.

We say that a closed subset $K \subset \text{Dom}(U)$ is **viable under** (U, f) if from any initial state $x_0 \in K$ starts at least one solution on $[0, \infty[$ to the control system (5) *viable in* K (in the sense that for all $t \geq 0$, $x(t) \in K$).

We associate with any subset $K \subset \text{Dom}(U)$ the **regulation map** $R_K : K \rightsquigarrow Z$ defined by

$$\forall\, x \in K, \;\; R_K(x) \; := \; \{u \in U(x) \mid f(x, u) \in T_K(x)\}$$

where $T_K(x)$ is the contingent cone to K at $x \in K$.

We say that K **is a viability domain of** (U, f) **if and only if the regulation map** R_K **is strict (has nonempty values).**

The Viability Theorem holds true for the class of **Marchaud systems**, which satisfy the following conditions:

$$(6) \qquad \begin{cases} i) & \text{Graph}(U) \; \text{is closed} \\ ii) & f \; \text{is continuous} \\ iii) & \text{the velocity subsets } \; F(x) := f(x, U(x)) \; \text{ are convex} \\ iv) & f \; \text{and} \; U \; \text{have linear growth} \end{cases}$$

Theorem 3.1 (Viability Theorem) *Let us consider a Marchaud control system* (U, f). *Then a closed subset* $K \subset \text{Dom}(U)$ *is viable under* (U, f) *if and only if it is a* viability domain *of* (U, f).

Furthermore, any "open loop" *control* $u(\cdot)$ *regulating a viable solution* $x(\cdot)$ *in the sense that*

$$\text{for almost all } \; t, \;\; x'(t) = f(x(t), u(t))$$

obeys the regulation law

$$\text{(7)} \qquad \text{for almost all } t, \; u(t) \in R_K(x(t))$$

Otherwise, if K is not a viability domain of the control system (U, f), **there exists a largest closed viability domain** *of (U, f) contained in K (possibly empty), denoted* Viab(K), *called the* **viability kernel** *of K, and equal to the set of states $x_0 \in K$ from which starts a solution of the control system viable in K.*

Finally, the upper limit of closed viability domains K_n of control systems (U_n, f_n) satisfying uniform linear growth is a viability domain of $\overline{co}F^\sharp(x(\cdot))$, where $F^\sharp$ is the graphical upper limit of the maps defined by $F_n(x) := f_n(x, U_n(x))$.

What we are aiming at, now, are **closed loop or feedback** controls r, which are single-valued selections of the regulation map $R_K : \forall\, x \in K, \; r(x) \in R_K(x)$.

One can naturally use selection procedures of the regulation map. (See Chapter 6 of **Viability Theory**, [5, Aubin]). This raises some problems because the graph of the regulation map is not closed whenever **inequality constraints** are involved in the definition of K.)

The idea we propose here is to find systems of first-order partial differential inclusions the solutions of which are such feedbacks.

The **trick is then to set a bound to the velocities of the controls**: we associate with the control system and with any nonnegative continuous function $(x, u) \to \varphi(x, u)$ with linear growth[4] the system of differential inclusions

$$\text{(8)} \qquad \left\{ \begin{array}{llll} i) & x'(t) & = & f(x(t), u(t)) \\ ii) & u'(t) & \in & \varphi(x(t), u(t))B \end{array} \right.$$

and we regard the condition $u(t) \in U(x(t))$ as a new viability constraint defined on the state-control pairs by:

$$\forall\, t \geq 0, \; (x(t), u(t)) \in \text{Graph}(U)$$

Observe that any solution $(x(\cdot), u(\cdot))$ to (8) viable in Graph(U) is a an absolutely continuous solution to the control system (5).

[4] which can be a constant ρ, or the function $(x, u) \to c\|u\|$, or the function $(x, u) \to c(\|u\| + \|x\| + 1)$. One could also take other dynamics $u' \in \Phi(x, u)$ where Φ is a Marchaud map.

From now on, we assume that $K := \mathrm{Dom}(U)$ (by setting $U(x) := \emptyset$ when $x \notin K$ if needed.)

We are looking for **closed set-valued feedback maps** R contained in R_K (and thus, in U), called **subregulation maps**, the graph of which is made of the initial state-control pairs yielding viable solutions to the control system. Among these subregulation maps, we shall be particularly interested by single-valued subregulation maps — which are closed loop controls we are looking for.

4 The Theorems

This is naturally possible thanks to The Viability Theorem.

Theorem 4.1 *Let us assume that the control system (5) satisfies*

$$(9) \quad \begin{cases} i) & \mathrm{Graph}(U) \textit{ is closed} \\ ii) & f \textit{ is continuous and has linear growth} \end{cases}$$

Let $(x,u) \to \varphi(x,u)$ be a nonnegative continuous function with linear growth and $R : Z \rightsquigarrow X$ a closed set-valued map contained in U. Then the two following conditions are equivalent:

a) — R is a **subregulation map**: *from any initial state $x_0 \in \mathrm{Dom}(R)$ and any initial control $u_0 \in R(x_0)$, there exists a state-control solution $(x(\cdot), u(\cdot))$ to the control system (5) starting at (x_0, u_0) and viable in the graph of R : $\forall\, t \geq 0, u(t) \in R(x(t))$*

b) — R is a solution to the system of first-order partial differential inclusions

$$(10) \qquad \forall\, (x,u) \in \mathrm{Graph}(R),\ 0 \in DR(x,u)(f(x,u)) - \varphi(x,u)B$$

satisfying the constraint: $\forall\, x \in K,\ \ R(x) \subset U(x)$.

Such a subregulation map R is actually contained in the regulation map R_K. The law *regulating the evolution of state-control solutions viable in the graph of R takes the form of the system of differential inclusions*

$$(11) \quad \begin{cases} i) & x'(t) = f(x(t), u(t)) \\ \\ ii) & u'(t) \in G_R(x(t), u(t)) \end{cases}$$

where the set-valued map G_R defined by

$$G_R(x,u) := DR(x,u)(f(x,u)) \cap \varphi(x,u)B$$

is called the **metaregulation map** *associated with R.*

Furthermore, there exists a **largest subregulation** *map denoted* R^{φ} *contained in U.*

In the case of single-valued regulation maps, the *system of first-order partial differential inclusions* (10) can be written in the form

$$\forall\, x \in K, \;\; 0 \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

If r is differentiable and if we set $B := [-1, +1]^m$, it boils down to

$$\forall\, j = 1, \ldots, m, \;\; \sum_{i=1}^{n} \frac{\partial r_j(x)}{\partial x_i} f_i(x, r(x)) \;\in\; [-\varphi(x, r(x)), +\varphi(x, r(x))]$$

In this case, it is a "viable manifold" of the characteristic system (8).

4.1 Heavy Viable Evolution

Assume that a subregulation map R is given. We introduce its **minimal selection** g_R° associating with each state-control pair (x, u) the element $g_R^{\circ}(x, u)$ of minimal norm of $DR(x, u)(f(x, u))$ (which also minimizes the norm of elements of $G_R(x, u)$).

We shall say that the solutions to the closed loop differential system

$$\begin{cases} i) & x'(t) \;=\; f(x(t), u(t)) \\ ii) & u'(t) \;=\; g_R^{\circ}(x(t), u(t)) \end{cases}$$

are *heavy viable solutions* to the control system (U, f) associated with R. This minimal selection can be regarded as an instance of **dynamical closed loop control**.

Theorem 4.2 (Heavy Viable Solutions) *Let us assume that U is closed and that f, φ are continuous and have linear growth. Let* $R(\cdot) \subset U(\cdot)$ *be a subregulation map such that the associated metaregulation map is lower semicontinuous with closed convex images. Then from any initial state-control pair* (x_0, u_0) *in Graph(R), there exists a heavy viable solution to the control system* (U, f) *associated with R.*

The case when the growth φ is equal to 0 is particularly interesting, because the inverse N^0 of the 0-growth regulation map R^0 determines the areas $N^0(u)$ regulated by the constant control u, called the *viability cell or niche* of u. A control u is called a *punctuated equilibrium* if and only if its viability cell is not empty. Naturally, *when the viability cell of a punctuated equilibrium is reduced to a point, this point is an equilibrium.* So, punctuated equilibria are constant controls which regulate the control systems (in its viability cell).

Any heavy viable solution $(x(\cdot), u(\cdot))$ to the control system (U, f) satisfies the inertia principle: **"keep the controls constant as long as they provide viable solutions"**.

Indeed, set

$$C_R(u) := \{x \in K \mid 0 \in DR(x,u)(f(x,u))\}$$

We observe that if for some time t_1, the solution enters the subset $C_R(u(t_1))$, the control $u(t)$ remains equal to $u(t_1)$ as long as $x(t)$ remains in $C_R(u(t_1))$. Since such a subset is not necessarily a viability domain, the solution may leave it.

If for some $t_f > 0$, $u(t_f)$ is a punctuated equilibrium, then $u(t) = u_{t_f}$ for all $t \geq t_f$ and thus, $x(t)$ *remains in the viability cell $N_1^0(u(t_f))$ for all $t \geq t_f$.* □

This approach has been used in the regulation of AUV (autonomous underwater vehicles) by Nicolas Seube, when neural networks are introduces to learn in an adaptive way the feedbacks regulating viable evolutions of a tracking problem. See [47, Seube] and [7, Aubin] for further details.

We refer to **Viabililty Theory** ([5]) for an exhaustive presentation of these concepts and their properties.

We shall derive the existence of a feedback control from a **Variational Principle.**

We denote by $\mathcal{C}(K, X)$ the space of continuous single-valued maps $r : K \mapsto Z$. A **closed (convex) process** is a set-valued map whose graph is a closed (resp. convex) cone. Closed convex processes share most of the properties of continuous linear operators, and in particular can be transposed (see Chapter 2 of **Set-Valued Analysis**, [14]). The transpose of a closed process $A : X \leadsto Y$ is the closed convex process $A^\star : Y^\star \leadsto X^\star$ defined by

$$p \in A^\star(q) \text{ if and only if } \forall\, u,\ \forall\, v \in A(u),\ \langle p, u\rangle \leq \langle q, v\rangle$$

Since the contingent derivative $Dr(x) : X \leadsto Z$ is a closed process, we can define and use its transpose $Dr(x)^\star : Z^\star \leadsto X^\star$. If r is differentiable, we obtain the usual transpose of the linear operator $r'(x)$: $Dr(x)^\star = r'(x)^\star$. We introduce the functional Φ defined by

$$\Phi(r) := \sup_{q\in B_\star} \sup_{x\in K} \sup_{p\in Dr(x)^\star(q)} (< p, f(x,r(x)) > -\varphi(x,r(x))\|q\|)$$

This functional is **lower semicontinuous** on $\mathcal{C}(K,X)$ (supplied with the compact convergence topology), even though this functional involves the "derivatives" of r. This basic nontrivial property of Φ implies the following existence theorem:

Theorem 4.3 *Let $\mathcal{R} \subset \mathcal{C}(K,Y)$ be a nonempty compact subset of selections of the set-valued map U (for the compact convergence topology.)*[5] *Suppose that the functions f and φ are continuous and that*

$$c := \inf_{r\in\mathcal{R}} \Phi(r) < +\infty$$

Then there exists a single-valued solution $r(\cdot)$ to the partial differential inclusion

$$\forall\, x \in K, \;\; 0 \in Dr(x)(f(x,r(x))) - (\varphi(x,r(x)) + c)B$$

which is a closed-loop control of the system (U,f), i.e., a continuous map satisfying $r(x) \in U(x)$ for every $x \in K$ such that from every $x_0 \in K$ starts a viable solution to the differential equation $x'(t) = f(x(t), r(x(t)))$ and

$$\|\frac{d}{dt}r(x(t))\| \leq c + \varphi(x(t), r(x(t)))$$

In the case of partial differential inclusions, this variational principle is related to the concept of viscosity solutions. Naturally, this may force us to change the initial bound on the growth of the velocity control by adding this constant c.

Another way to proceed is to modify the bound on the velocity of the controls by replacing (8)ii) by

$$(12) \quad \begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u'(t) - Au(t) \in \varphi(x(t), u(t))B \end{cases}$$

[5] Let us recall that the Michael Theorem implies that every lower semicontinuous map with closed convex values from a metric space to a Banach space has continuous selections.

where $A \in \mathcal{L}(Z,Z)$ is a linear operator with $\lambda := \inf_{\|x\|=1}\langle Ax, x\rangle > 0$ large enough.

Then the associated single-valued subregulation maps r are closed set-valued solutions to *system of first-order partial differential inclusions*

$$(13) \qquad \forall\, x \in \text{Dom}(r), \;\; Ar(x) \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

Theorem 4.4 *Assume that the map $f : X \times Y \mapsto X$ is Lipschitz, that $\varphi : X \times Y \mapsto Y$ is Lipschitz with nonempty convex compact values and that*

$$\forall\, x,\; u,\;\; \|\varphi(x,u)\| \le \gamma(1 + \|u\|)$$

Let $A \in \mathcal{L}(Z,Z)$ such that $\lambda > \max(\gamma, 4\|f\|_\Lambda\|\varphi\|_\Lambda)$ (where $\|f\|_\Lambda$ denotes the Lipschitz constant of f). Then there exists a bounded Lipschitz contingent solution to the partial differential inclusion (13), which is a closed-loop control of the system (U, f), i.e., a continuous map satisfying $r(x) \in U(x)$ for every $x \in K$ such that from every $x_0 \in K$ starts a viable solution to the differential equation $x'(t) = f(x(t), r(x(t)))$ and

$$\left\| \frac{d}{dt} r(x(t)) - Ar(x(t)) \right\| \;\le\; \varphi(x(t), r(x(t)))$$

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