Working Paper

Quasiinversion, Regularization and the Observability Problem

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Foreword

This paper deals with effective techniques for estimating a distributed field in the basis of available measurements. One of the motivations for this study comes from problems of monitoring air pollution and other related environmental issues. This work continues an earlier investigation undertaken at the System and Decision Sciences Program.

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Quasiinversion, Regularization and the Observability Problem

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Abstract

This paper deals with the problem of estimating the initial state of a distributed field on the basis of measurements generated by sensors. The original ill-posed problem is regularized here through an auxiliary "guaranteed estimation" problem. This yields a stable numerical procedure and also allows to establish a unified "systems-theoretic" framework for treating regularizers in general. Particularly the important point is that for finite dimensional sensor outputs a necessary condition for the existence of a stable numerical solution is the observability property which ensures existence of solution in the absence of measurement noise.

1 The Guaranteed Estimation Problem

In a bounded domain Ω of the finite-dimensional space $\mathbf{R^n}$ consider a distributed field $u(x,t)$ described as the solution to the boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = Au(\cdot,t); \tag{1.1}$$

$$t \in T = (0,\theta); \quad x \in \Omega \subset \mathbf{R^n}, \quad \mathbf{Q} = \boldsymbol{\Omega} \times \mathbf{T},$$

$$u(x,0) = w(x),$$

$$u(x,t)|\Sigma = 0, \quad \Sigma = \partial Q = \partial\Omega \times T. \tag{1.2}$$

Here $\partial\Omega$ is a piecewise smooth boundary of Ω, A is a selfadjoint operator

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}),$$

with elements

$$a_{ij}(x) \in L_\infty(\Omega),$$

that satisfies the coercitivity property

$$\nu_1\|\xi\| \leq \langle A\xi, \xi\rangle \leq \nu_2\|\xi\|; \quad \nu_1 > 0, \nu_2 > 0.$$

Taking $w(\cdot) \in L_2(\Omega)$ we will treat $u(x,t)$ as a generalized solution [1] to system (1.1) that belongs to the Banach space $\overset{\circ}{V}{}_2^{1,0}(Q)$. The latter consists of elements $u(\cdot,\cdot)$ of $H_0^{1,0}(Q)$ where

$$H_0^{1,0}(Q) = \{\varphi|\varphi \in H^{1,0}(Q), \varphi|_\Omega = 0\},$$

$$H^{1,0}(Q) = \{\varphi|\varphi \in L_2(Q), \frac{\partial\varphi}{\partial x_i} \in L_2(Q)\}$$

with traces $u(\cdot,t) \in L_2(\Omega)$, continuous in t, and with the norm

$$|u| = \max_{0\leq t\leq\theta} \|u(\cdot,t)\|_{L_2(\Omega)} + \|u(\cdot,\cdot)\|_{H^{1,0}}.$$

Under the given assumptions the generalized solution to system (1.1) exists and is unique.

Taking the initial distribution $w(\cdot) \in L_2(\Omega)$ and the respective solution

$$u(\cdot,t) = u(\cdot,t;w(\cdot)), \quad u(\cdot,0) = w(\cdot),$$

to the boundary value problem (1.1), (1.2) as an element of $\overset{\circ}{V}{}_2^{1,0}(Q)$, consider a mapping $\mathbf{G}(\overset{\circ}{V}{}_2^{1,0} \to L)$ where $L(T)$ is a Hilbert space, so that

$$y = \mathbf{G}u(\cdot,\cdot|w(\cdot)).$$

The latter mapping could be interpreted as an equation for the sensor (the measurement device). Also consider an element $z \in L$.

The first problem to solve is as follows:

Given a cost functional

$$J(x^0) = f(z(\cdot) - y(\cdot)) = f(z(\cdot) - \mathbf{G}u(\cdot,\cdot|w^0(\cdot)))$$

$$(f(\cdot) \geq 0, \quad f(0) = 0, \quad f(z) \to \infty, \quad \|z\| \to \infty)$$

and a number $\gamma > 0$, solve

Problem I:

Find an element $w^0(\cdot) \in L_2(\Omega)$, such that

$$J(w(\cdot)) \leq \gamma$$

with $w(\cdot) = w^0(\cdot)$.

The latter is an *inverse problem* which was investigated by many authors. Particularly, for

$$\mathbf{G}u(\cdot,\cdot) = u(\cdot,\theta),$$

it was studied by J.L. Lions and R. Lattes [2] who have proved solvability for a quadratic functional J and any $\gamma > 0$ and have introduced a numerically stable procedure of *quasiinversion* for its solution.

Let us now present the problem in a somewhat different way, namely, consider

Problem II:

Among the initial distributions $w(\cdot) \in L_2(\Omega)$ find the set $W^* = \{w^*(\cdot)\}$ of all those distributions $w^*(\cdot)$ that ensure the inequality

$$J(w(\cdot)) \leq \gamma, \text{ with } w(\cdot) = w^*(\cdot).$$

We preassume that the problem data ensures the solvability of the latter problem ($W^* \neq \emptyset$). Here one requires to find the set of *all solutions* W^* to the inequality (1.3) rather than to single out a special solution $w^0(\cdot)$ of this inequality.

Let us reformulate the two previous problems once again. This will be done in terms of systems theory, namely the *guaranteed estimation* concept, [3-5].

Consider the *measurement* equation

$$y = \mathbf{G}u(\cdot,\cdot) + \eta \tag{1.4}$$

where y is the measurement, $u(\cdot,\cdot)$ - the solution generated by an unknown initial distribution $w(\cdot)$, $\mathbf{G}u$ - the mapping for the sensor, η - the unknown but bounded measurement "noise" all the information on which is restricted to the inequality

$$\langle \eta, \eta \rangle \leq \gamma, \quad \eta \in L \tag{1.5}$$

$\gamma > 0$ given. Here $\langle \eta, \eta \rangle$ stands for a scalar product in the Hilbert space L. We further assume $J(z) = \langle z, z \rangle$ leaving the more general case for special treatment which follows from this paper.

The set W^* will then consist of all the initial distributions $w(\cdot)$ consistent with the system (1.1), the given measurement y and the restrictions (1.4), (1.5). In other words, W^* will be the set of all those functions $w(\cdot) \in L_2(\Omega)$ for which the solution $u(\cdot, \cdot; w(\cdot))$ to system (1.1) would satisfy relations (1.4), (1.5) with y given, namely the inequality

$$\langle y - \mathbf{G}u(\cdot, \cdot; w(\cdot)), y - \mathbf{G}u(\cdot, \cdot; w(\cdot)) \rangle \leq \gamma$$

In the general case the calculation of either W^* or of one of its elements may turn to be numerically unstable. The aim of this paper is therefore to indicate numerically stable regularizing procedures for the ill-posed problems of the above for a rather general variety of sensors $\mathbf{G}u$. (Other problems of this kind are indicated in [6]).

2 Measurement Maps (Sensors)

The present paper deals with the following types of measurement.

1. Distributed Measurement

$$\mathbf{G}u(\cdot, \cdot) = u(\cdot, \theta)$$

which gives *the state* of the process at time $t = \theta$.

2. Distributed Zone Measurement

$$\mathbf{G}u(\cdot, \cdot) = u(x, t) \quad (\text{ a.e. in } \Omega \times T, \Omega' \leq \Omega)$$

where we can take $L = L_2(Q)$.

3. Combined Distributed and Zone Distributed Measurement

Here

$$\mathbf{G}u(\cdot, \cdot) = (u(\cdot, \cdot), u(\cdot, \theta)) \in L_2(Q) \times L_2(\Omega)$$

and the error $\eta(\cdot, \cdot)$ of the measurement $y(x, t), z(x)$ due to

$$y(x, t) = u(x, t) + \eta(x, t),$$

$$z(x) = u(x, \theta) + \eta(x, \theta)$$

is restricted by the inequality

$$\alpha \int_0^\theta \int_\Omega \eta^2(x,t)dxdt + \beta \int_\Omega \eta^2(x,\theta)dx \leq \gamma^2$$

with $\alpha \geq 0, \beta \geq 0, \alpha^2 + \beta^2 \neq 0$, or otherwise, by

$$\alpha \|\eta(\cdot,\cdot)\|^2_{L_2(Q)} + \beta \|\eta(\cdot,\theta)\|^2_{L_2(\Omega)} \leq \gamma^2$$

Remark 2.1 Although sensors 2, 3 imply that we may observe the exact solution, the actual measurement y is corrupted by "noise" η that may be "worse" than the exact value $u(x,t)$.

4. Finite-dimensional sensors

Here the measurement trajectory

$$y(t) = \mathbf{G}(t)u(\cdot,\cdot) + \eta(t)$$

is a finite-dimensional function $y(t) \in L_2(\tau,\theta), 0 < \tau \leq \theta, L = L_2(\tau,\theta)$ and $\mathbf{G}(t)$ is a continuous linear mapping: $\overset{\circ}{V}{}_2^{1,2}(Q_t) \rightarrow \mathbf{R}^m$ $(Q_t = \Omega \times [0,t])$.

Particularly this includes pointwise measurements

$$y(t) = u(x^*,t) + \eta(t)$$

and pointwise scanning sensors

$$y(t) = u(x^*(t),t) + \eta(t)$$

along a preassigned or a controlled trajectory $x^*(t)$. The restriction on $\eta(t), t \in [0,\theta]$ may be of any conventional norm [3]. The solution $u(x,t)$ should then be restricted to a class of smooth functions (see for example, [7]).

Either of the types of sensors of the above may yield an ill-posed problem that requires regularization. However the existence of such a procedure may strongly depend on the invertibility of the map $\mathbf{T}$ where $y = \mathbf{G}u(\cdot,\cdot|w(\cdot) = \mathbf{T}w(\cdot)$ which reflects the *observability* property for system (1.1), (1.2) in the absence of any measurement noise η.

For measurements of type 1-3 the investigation of the invertibility of $\mathbf{T}$ does not lead to a major problem. Type 4 however requires additional consideration, since for finite- dimensional sensors the observability issue may turn to be a necessary condition for the existence of a regularizer.

We will now proceed with the formulation of a regularizing problem - the "guaranteed estimation" procedure. This will also allow to formulate the overall solution in terms of systems theory.

3 "Guaranteed Estimation" as a Regularizing Problem

Let us start with a simple auxiliary problem of estimating the *initial distribution* $w(\cdot)$ that generates the solution $u(\cdot,\cdot) = u(\cdot,\cdot|w(\cdot))$ of system (1.1), on the basis of an available measurement

$$y = \mathbf{G}u(\cdot,\cdot) + \eta, \quad y \in L \tag{3.1}$$

corrupted by unknown "noise" $\eta \in L$. The present problem presumes that given is a restriction

$$\langle w(\cdot), N_\epsilon w(\cdot)\rangle_{L_2(\Omega)} + \langle \eta, K_\epsilon \eta\rangle \leq \gamma + \mu_\epsilon, \quad \mu_\epsilon > 0, \tag{3.2}$$

with linear bounded selfadjoint mappings $N_\epsilon > 0, K_\epsilon \geq 0$, $(L_2(\Omega) \to L)$ also given.

The mapping N_ϵ is assumed to be invertible.

Problem III (Guaranteed Estimation)

The problem (1.1), (1.2), (3.1), (3.2) of guaranteed estimation will consist in finding the *informational set* $W_\epsilon(y) = \{w(\cdot)\}$ of all functions $w(\cdot) \in L_2(\Omega)$ such that for each of these there exists an element $\eta \in L$, so that the pair $\{w(\cdot), \eta\}$ would satisfy (3.1), (3.2) due to system (1.1), (1.2).

We further presume $W_\epsilon(y) \neq \emptyset$. This could be reached for any pair N_ϵ, K_ϵ by selecting an appropriate $\mu_\epsilon > 0$. Direct calculation gives us

Lemma 3.1 *An element $w(\cdot) \in L_2(\Omega)$ satisfies $w(\cdot) \in W_\epsilon(y)$ if and only if*

$$\langle w(\cdot) - w_\epsilon^0(\cdot), B_\epsilon(w(\cdot) - w_\epsilon^0(\cdot))\rangle \leq \kappa_\epsilon^2 \tag{3.3}$$

where

$$w_\epsilon^0(\cdot) = B_\epsilon - U^*G^*K_\epsilon y(\cdot),$$

$$B_\epsilon = N_\epsilon + U^*G^*K_\epsilon GU \tag{3.4}$$

$$\kappa_\epsilon^2 = \gamma + \mu_\epsilon - \langle y, K_\epsilon y\rangle + \langle w_\epsilon^0(\cdot), B_\epsilon w_\epsilon^0(\cdot)\rangle_{L_2(\Omega)}.$$

Here operator U maps the variety of initial distributions $w(\cdot) \in L_2(\Omega)$ into the set of solutions $u(\cdot,\cdot) \in \overset{\circ}{V}{}^{1,0}$ to (1.1), (1.2) ($u(\cdot, 0) = w(\cdot)$). The subindexes in the last scalar product or further in the symbols for the norm emphasize the respective Hilbert spaces. It is clear that W_ϵ is a nondegenerate ellipsoid in $L_2(\Omega)$, whose center $w^0_\epsilon(\cdot)$ is also its *Tchebycheff* center, namely

$$\min_{v(\cdot)\in W_\epsilon(y)} \max_{w(\cdot)\in W_\epsilon(y)} \|w(\cdot) - v(\cdot)\|_{L_2(\Omega)} = \max_{w(\cdot)\in W_\epsilon(y)} \|w(\cdot) - w^0_\epsilon(\cdot)\|_{L_2(\Omega)}$$

Therefore $w^0_\epsilon(\cdot)$ is also the ***minmax estimate*** for the initial distribution $w(\cdot)$ in the sense of the previous relation.

Let us now introduce the definition for *a (variational) regularizer.*

Definition 1 *A variety of functions $w_\epsilon(\cdot)$ will be a regularizer for Problem II (with respect to the functional J) if*

$$J(w_\epsilon(\cdot)) \to J_0 \text{ with } \epsilon \to 0 \tag{3.5}$$

where

$$J_0 = \inf\{J(w(\cdot)) : w(\cdot) \in L_2(\Omega)\}$$

We will now demonstrate that depending on the selection of N_ϵ, K_ϵ, the guaranteed (minmax) estimator $w^0_\epsilon(\cdot)$ can serve as a regularizer for Problem II. Moreover the latter "systems theoretic" viewpoint produces a unified framework both for the known regularizing schemes, of [2, 8, 9, 10], and also for schemes that are new.

(i) Assume

$$K_\epsilon = I, \quad N_\epsilon = \epsilon N$$

and

$$< w(\cdot),\ Nw(\cdot) > \alpha < w(\cdot), w(\cdot) >$$

for some $\alpha > 0$. Then particularly from [2, 3] it follows that function $w^0_\epsilon(\cdot)$ of Lemma 3.1 (see (3.4)) is a minimizer for

$$J_\epsilon(w(\cdot)) = < w(\cdot),\ \epsilon N w(\cdot) >_{L_2(\Omega)} +$$

$$+ < y - \mathbf{G}u(\cdot,\cdot),\ y - \mathbf{G}u(\cdot,\cdot) >$$

Assumption 3-A: The map $\mathbf{T} = \mathbf{G}U$ is invertible with a bounded inverse $\mathbf{T}^{-1}$.

In terms of paper [7] this reflects the property of *strong observability.*

Lemma 3.2 *The following assertions are true*

(a) *The variety $\{w_\epsilon^0(\cdot)\}$ given by (3.4) satisfies relation (3.5) and is therefore a regularizer for Problem II.*

(b) *Under assumption 3-A take $w^*(\cdot) \in L_2(Q)$,*

$$y_\gamma = \mathbf{G}Uw^*(\cdot) + \eta$$

and find the ellipsoid $W_\epsilon[y_\gamma]$ with center $w_\epsilon^0(\cdot)$ of initial sates consistent with (1.1), (1.2), (3.1), (3.2), then with $\gamma^2/\epsilon \to 0$, $\epsilon \to 0, \mu_\epsilon = 0$ we have

$$\| w_\epsilon^0(\cdot) - w^*(\cdot) \|_{\mathcal{L}_2(Q)} \to 0$$

The given variety $\{w_\epsilon^0(\cdot)\}$ is therefore a Tikhonov regularizer [9].

(ii) A developed method for resolving the operator equation $A_\delta w = y_\delta$ is the method of *quasisolutions* (V. Ivanov, [10]), where A_δ, y_δ are the approximate values for the parameters of the equation $Aw = y$, whose solution is presumed to exist within a compact set M.

The *quasisolution* is defined as

$$w_\delta(\cdot) = \arg\inf \{\| A_\delta w(\cdot) - y_\delta \| \; | w(\cdot) \in \mathrm{M}\}$$

For the specific problem (P) of this paper we come to its following version

$$w_\epsilon(\cdot) = \arg\inf \{\| \mathbf{G}u(\cdot,\cdot) - y \| \, L : w(\cdot) \in \mathrm{M}_\epsilon\} \tag{3.6}$$

$$M_\epsilon = \{w(\cdot) \in L_2(\Omega) : \| w(\cdot) \| \leq \epsilon\} \tag{3.7}$$

Solving problem (3.6), (3.7) by Lagrangian techniques of nonlinear analysis we come to the saddle-point problem

$$\chi^0 = \sup_{p\geq 0} \inf_{w(\cdot)\in L_2(\Omega)} \{p(< w(\cdot), w(\cdot) >_{L_2(\Omega)} - \epsilon^2) +$$

$$+ \| \mathbf{G}u(\cdot,\cdot) - y \|_{\mathcal{L}}^2\}$$

whose convexity properties imply that there exists a saddle-point $\{\bar{p}, \bar{w}(\cdot)\}$ that satisfies the *complementarity condition*

$$\bar{p}(< \bar{w}(\cdot), \bar{w}(\cdot) >_{L_2(\Omega)} - \epsilon^2) = 0$$

Presuming

$$\chi^0 > J_0 = \inf \{J(w(\cdot)) | w(\cdot) \in L_2(\Omega)\}$$

we observe that $\bar{p} \neq 0$, (otherwise one would have $\chi^0 = J_o$), and that therefore $\bar{p} > 0$.

The solution $w_\epsilon(\cdot) = \bar{w}(\cdot)$ to (3.6), (3.7) gives

$$w_\epsilon(\cdot) = (\bar{p}I + U^*\mathbf{G}^*\mathbf{G}^*U)^{-1} U^*\mathbf{G}^*y \tag{3.8}$$

where $\bar{p}$ is selected from the isoperimetric condition

$$< w_\epsilon(\cdot), w_\epsilon(\cdot) > = \epsilon^2.$$

Relation (3.8) obviously coincides with (3.4).

Therefore

$$J(w_{\epsilon *}^0(\cdot)) \to J_0, \quad \epsilon \to 0$$

and the variety $\{w_{\epsilon *}^0(\cdot)\}$ of (3.8) is a regularizer to Problem II.

If, however $\chi^0 = J_0 = J(w^0_*(\cdot))$, then depending on the value ϵ we have either the same solution $w^0_{\epsilon*}(\cdot) = w_{\epsilon*}$ (when $J_0 < \chi^0$) or $w^{(\cdot)}_{\epsilon*} = w^0_*$ for ($\chi^0 \leq J_0$) which is also a solution to the nonconstrained problem (3.6).

We will now indicate that the *quasisolution* $w^0_{\epsilon*}(\cdot)$ of (3.8) can be obtained through Problem III of *guaranteed estimation* in the form of an element $w^0_\epsilon(\cdot)$ (3.4) obtained by appropriate selection of N_ϵ, K_ϵ.

An obvious answer is given by

Lemma 3.3 *For the Problem III due to (3.1), (3.2), (1.1), (1.2) select*

$$K_\epsilon = I, \quad N_\epsilon = \alpha(\epsilon)I$$

where $\alpha(\epsilon) > 0$ is given through the relation

$$\| (\alpha(\epsilon)I + U^*\mathbf{G}^*GU)^{-1} U^*G^*y \|_{L_2(\Omega)} = \epsilon$$

(I is the identity map in $L_2(\Omega)$). Then the quasisolution $w^0_{\epsilon*}(\cdot)$ coincides with the guaranteed estimate $w^0_\epsilon(\cdot)$, namely $w^0_{\epsilon*}(\cdot) = w^0_\epsilon(\cdot)$.

(iii) A third conventional regularizing scheme is given through the *"bias method"*. This implies the solution of a constrained extremal problem

$$w_\epsilon(\cdot) = \arg\inf\{\| w(\cdot) \| : \| \mathbf{G}U(\cdot,\cdot) - y \|_L \leq \epsilon\} \tag{3.9}$$

which is *reciprocal* to (3.6), (3.7). It can therefore be handled similarly to the previous case.

The guaranteed estimator is thus shown to be a conventional regularizer of the *Tikhonov* type or a *quasisolution* or a *solution to the "bias method"*. Let us now indicate in more detail that the *quasiinversion* technique of [2] could also be treated in terms of guaranteed estimates.

4 The Quasiinvertibility Method of J-L. Lions and R. Lattes. (Distributed measurements)

A technique for regularizing Problem II for noninvertible evolutionary systems was suggested in [2]. This technique, which is known as the *quasiinvertibility method*, ensures numerical robustness for the respective class of ill-posed problems. We will now treat this technique in terms of *systems theory*.

Suppose an element $y(\cdot) \in L_2(\Omega)$ is given. Is it always possible to select an initial distribution $w(\cdot) \in L_2(\Omega)$ that we would have $u(\cdot, \theta | w(\cdot)) = y(\cdot)$ for a given instant $\theta > 0$? The answer is obviously negative. Therefore one comes to a particular case of Problem I (with $L = L_2(\Omega)$, $\mathrm{G}u(\cdot,\cdot) = u(\cdot, \theta)$) which is to find a distribution $w(\cdot) \in L_2(\Omega)$ such that

$$J(w(\cdot)) \equiv \int_{\Omega} (u(x, \theta | w(\cdot)) - y(x))^2 \, dx \leq \gamma. \tag{4.1}$$

with $\gamma > 0$ given.

According to the method of quasiinversion [2], one considers the following boundary value problem (in reversed time)

$$\frac{\partial v_\epsilon(x,t)}{\partial t} = A v_\epsilon(\cdot, t) + \epsilon A^* A v(\cdot, t)$$

$$x \in \Omega, \;\; t \in T, \;\; (\epsilon > 0)$$

$$v_\epsilon(x, \theta) = y(x), \tag{4.2}$$

$$v_\epsilon(x,t)\Big|_{\Sigma} = A v_\epsilon(x,t)\Big|_{\Sigma} = 0$$

This problem is well-posed, so that if

$$w_\epsilon(\cdot) = v_\epsilon(\cdot, 0),$$

then

$$\lim J(w_\epsilon(\cdot)) \rightarrow 0 \text{ with } \epsilon \rightarrow 0$$

Let us also pose the following problem: Does there exist a guaranteed estimation Problem III of finding W_ϵ due to (3.1), (3.2), (4.1) such that by appropriate selection of N_ϵ, K_ϵ we would achieve $v_\epsilon(\cdot, 0) = w_\epsilon^0(\cdot)$? (here v_ϵ and w_ϵ^0 are given through (4.2) and (3.4) respectively).

Some further notations.

Denote $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ to be the eigenvalues, $\{\varphi_i, (\cdot)\}_{i=1}$ – a complete orthonormal system of eigenfunctions for the problem

$$A\varphi(\cdot) = -\lambda\varphi(\cdot), \quad x \in \Omega$$

$$\varphi(x)\Big|_{\partial\Omega} = 0 \tag{4.3}$$

Also denote

$$N_\epsilon w(\cdot) = \sum_{i=1}^{\infty} (1 - \exp(-\epsilon\lambda_i^2\theta))w_i\varphi_i(\cdot) \tag{4.4}$$

$$K_\epsilon\eta(\cdot) = \sum_{i=1}^{\infty} \exp(-(\epsilon\lambda_i^2 - 2\lambda_i)\theta)\eta_i\varphi_i(\cdot)$$

where w_i, η_i are the Fourier coefficients for $w(\cdot)$, $\eta(\cdot)$ along the system $\{\varphi_i(\cdot)\}_{i=1}^{\infty}$, namely

$$w_i = \int_\Omega w(x)\varphi_i(x)dx,$$

$$\eta_i = \int_\Omega \eta(x)\varphi_i(x)dx.$$

We now come to a particular version of Problem II:

Problem 4.1

Find the guaranteed (minmax) estimate $w^0_\epsilon(\cdot)$ for the system (1.1), (1.2), (3.1) ($\mathbf{G}u(\cdot,\cdot) = u(\cdot,\theta)$), (3.2), (4.4).

What now follows is

Theorem 4.1 *Let the maps N_ϵ, K_ϵ in (3.2) be given by relations (4.4). Then the solution $w_\epsilon(\cdot) = v_\epsilon(\cdot,0|y(\cdot))$ to the quasiinversion problem (4.2) satisfies equality: $w^0_\epsilon = w_\epsilon(\cdot)$ (!) The explicit representation for $w_\epsilon(\cdot)$ is given by*

$$w_\epsilon(\cdot) = \sum_{i=1}^{\infty} \exp(-(\epsilon\lambda_i^2 - \lambda_i))y_i\varphi_i(\cdot) \tag{4.5}$$

$$y_i = \int_\Omega y(x)\varphi_i(x)dx$$

The proof of this theorem follows directly from relations (3.4) and the expansion (4.5) for $w_\epsilon(\cdot)$. We note that the properties of the eigenvalues ($\lambda_i \geq \lambda_1 > 0$) imply the invertibility of N_ϵ (4.4), since

$$(1 - \exp(-\epsilon\lambda_i^2\theta)) \geq (1 - \exp(-\epsilon\lambda_i^2\theta))) > 0$$

and therefore

$$\| N_\epsilon^{-1} w(\cdot) \| \leq \| w(\cdot) \| \ (1 - \exp(-\epsilon\lambda_i^2\theta)^{-2})$$

The next issue is how to ensure a robust procedure for calculating w^* – the solution set to Problem II. This will be also achieved by appropriate selection of N_ϵ, K_ϵ in (3.2) and by calculating the respective set $W_\epsilon(y)$.

Theorem 4.2 *Assume $\epsilon > 0$, $\nu > 0$ and take the inequality (3.2) with $\mu_\epsilon = 0$,*

$$N_\epsilon w(\cdot) \equiv N_{\epsilon,\nu} w(\cdot) =$$

$$= \sum_{i=1}^{\infty} (\exp(-2\lambda_i\theta(1+\nu\lambda_i)^{-1}) - \exp(-(\epsilon\lambda_i^2 + 2\lambda_i)\,\theta)w_i\varphi_i, \tag{4.6}$$

$$K_\epsilon \, \eta(\cdot) = \sum_{i=1}^{\infty} \exp(-\epsilon\lambda_i^2\theta)\eta_i\varphi_i,$$

Provided ϵ, ν are sufficiently small, the respective informational domains $W_\epsilon(y) \equiv W_{\epsilon,\nu}(y)$ of initial states $w(\cdot)$ consistent with (1.1), (1.2), (3.1), (3.2), (4.6) are nonvoid nondegenerate ellipsoids in $L_2(\Omega)$ with centers $w_\epsilon^0(\cdot) = w_{\epsilon,y}^0(\cdot)$ that satisfy the relations

$$\lim_{\nu\to 0} w_{\epsilon,\nu}^0(\cdot) = w_\epsilon(\cdot) \tag{4.7}$$

$$\lim_{\epsilon,\nu\to 0} W_{\epsilon,\nu}(\cdot) = W^* \tag{4.8}$$

The limits are taken in the $L_2(\Omega)$ metric ((4.7)) and in the sense of Kuratowski ((4.8)).

Recall that a sequence of closed sets $\{C_n, n \in N\}$ in $L_2(\Omega)$ converges to set N in the sense of Kuratowski if

$$\limsup_{h\to\infty} C_n = \liminf_{n\to\infty} C_n = C,$$

where

$$\limsup_{n\to\infty} C_n = \{x = \lim x_m, \ x_m \in C_m, \ m \in M \subset N\}$$

$$\liminf_{n\to\infty} C_n = \{x = \lim x_n, \ x_n \in C_n, \ n \in N\}$$

Here N is an ordered set of integers, M is its countable subset.

In this case one writes

$$\lim_{n\to\infty} C_n = C$$

The proof of this theorem will appear in a separate publication.

5 Other Regularization Methods

The problem of solving inequality (4.1) can be associated with a whole class of quasiinvertibility maps if, for example, we substitute (4.2) by the system

$$\frac{\partial v_\epsilon(x,t)}{\partial t} = Av_\epsilon(\cdot,t) + \epsilon(-1)^m (B^*B)^m v_\epsilon(\cdot,t)$$

where B is equivalent to A in the sense that

$$D(B) \subset D(A)$$

$$\| Bv \| \leq k \| Av \|, \quad \forall v(\cdot) \in D(B), \quad k > 0,$$

We further assume $B = A$.

Taking the quasiinvertibility equation

$$\frac{\partial v_\epsilon(x,t)}{\partial t} = Av_\epsilon(\cdot,t) + \epsilon(-1)^m A^m v_\epsilon(\cdot,t)$$

$$x \in \Omega, \ \ t \in T$$

$$v_\epsilon(x,\theta) = y(x)$$

$$v_\epsilon(x,t)\Big|_\Sigma = Av_\epsilon(\cdot,t)\Big|_\Sigma = \ldots = A^{m-1}v_\epsilon(\cdot,t)\Big|_\Sigma = 0 \tag{5.1}$$

consider the element $w_\epsilon(\cdot) = v_\epsilon(\cdot,0)$. The explicit relation for $w_\epsilon(\cdot)$ is

$$w_\epsilon(\cdot) = \sum_{i=1}^{\infty} \exp(-(\epsilon\lambda_i^m - \lambda_i)\theta)y_i\varphi_i,$$

and $w_\epsilon(\cdot) = v_\epsilon(\cdot,0)$ is the center or the informational ellipsoid $W_\epsilon(y)$ (3.3) if

$$N_\epsilon w(\cdot) \;=\; \sum_{i=1}^{\infty} (1 - \exp(-\epsilon\lambda_i^m\theta))w_i\varphi_i,$$

$$K_\epsilon\eta(\cdot) \;=\; \sum_{i=1}^{\infty} \exp(-\epsilon\lambda_i^m\theta \;+\; 2\lambda_i\theta)\eta_i\varphi_i,$$

The relation (4.3) is also true.

Papers [13, 14] are devoted to the regularization of problem (4.1) through the *Sobolev equation*

$$\frac{\partial v_\epsilon(x,t)}{\partial t} \;=\; Av_\epsilon(\cdot,t) \;+\; \epsilon\frac{\partial}{\partial t}Av_\epsilon(\cdot,t),$$

$$x \in \Omega, \quad t \in T, \tag{5.2}$$

$$v_\epsilon(x,\theta) \;=\; y(x), \quad y(\cdot) \in D(A),$$

$$v_\epsilon(x,t)\Big|_\Sigma \;=\; 0.$$

Taking $w_\epsilon(\cdot) = v_\epsilon(\cdot,\cdot)$ we may again observe that (4.3) is true. Further on, assume

$$N_\epsilon w(\cdot) \;=\; \sum_{i=1}^{\infty} \exp(-\lambda_i\theta)\,(1 - \exp((-\epsilon\lambda_i^2\theta)\,(1+\epsilon\lambda_i)^{-1}))w_i\varphi_i \tag{5.3}$$

$$K_\epsilon\eta(\cdot) \;=\; \sum_{i=1}^{\infty} \exp(-\lambda_i\theta(1+\epsilon\lambda_i)^{-1})\eta_i\varphi_i;$$

and $W_\epsilon(y) \neq \emptyset$ (through appropriate selection of $\mu_\epsilon > 0$).

Lemma 5.1 *The ellipsoid $W_\epsilon(y)$ need not be bounded for N_ϵ, K_ϵ (3.3), (5.3). It is given by the inequality*

$$< (w(\cdot) \;-\; w_\epsilon^0(\cdot),\; \mathbf{U}_\theta(w(\cdot) - w_\epsilon^0(\cdot)) > \;\leq$$

$$\leq \gamma^2 + \mu_\epsilon - h_\epsilon^2(y) \tag{5.4}$$

where

$$\mathbf{U}_\theta w(\cdot) = u(\cdot, \theta; w(\cdot)),$$

$$w_\epsilon^0(\cdot) = \sum_{i=1}^{\infty} \exp(\lambda_i \theta (1 + \epsilon \lambda_i)^{-1}) y_i \varphi_i, \tag{5.5}$$

$$h_\epsilon^2(y) = < y, K_\epsilon y > - < w_\epsilon^0, \mathbf{U}_\theta w_\epsilon^0 >$$

and where $w_\epsilon^0(\cdot)$ *is the only center of symmetry for* $w_\epsilon(y)$.

The map N_ϵ (when applied to $L_2(\Omega)$) does not have a bounded inverse. This implies that $W_\epsilon(y)$ need not be bounded. However, the properties of map $\mathbf{U}w(\cdot)$ also imply that set $W_\epsilon(y) \neq \phi$ does not contain any affine varieties, so that $w_\epsilon^0(\cdot)$ is the only center of symmetry for $W_\epsilon(y)$. Therefore inequality (5.4) gives a unique representation for $W_\epsilon(y)$.

According to [13] the solution to (4.9) could be represented as

$$v_\epsilon(x,t) = \sum_{i=1}^{\infty} \exp(\lambda_i (\theta - t)\,(1 + \epsilon \lambda_i)^{-1}) y_i \varphi_i(x)$$

$$x \in \Omega, \quad t \in T$$

This yields

$$v_\epsilon(x,0) = w_\epsilon(\cdot) = \sum_{i=1}^{\infty} \exp(\lambda_i (\theta - t)\,(1 + \epsilon \lambda_i)^{-1} y_i \varphi_i(x)$$

$$x \in \Omega, \quad t \in T$$

or after comparing with (5.4), that

$$w_\epsilon(\cdot) \;=\; w_\epsilon^0(\cdot) \tag{5.5}$$

Finally, taking (5.1) for $m \geq 2$ and considering the respective problem (3.1), (3.2) with

$$N_\epsilon w(\cdot) \;=\; \sum_{i=1}^{\infty} \exp(-\lambda_i\theta)\,(1 - \exp(-\epsilon\lambda_i^{m+1}(1+\epsilon\lambda_i^m)^{-1})w_i\varphi_i(\cdot) \tag{5.6}$$

$$K_\epsilon\eta(\cdot) \;=\; \sum_{i=1}^{\infty} \exp(\lambda_i\theta(1+\epsilon\lambda_i^m)^{-1}\eta_i\phi_i(\cdot),$$

we again come to property (4.3), (5.5) where $w_\epsilon(\cdot) = v_\epsilon(\cdot, 0)$ is taken due to (5.1), $m \geq 2$ and N_ϵ, K_ϵ – due to (5.6). For $m = 2$ the quasiinvertibility problem was treated in [15].

6 Zone measurement

Assuming the observation to be a zone measurement ($\Omega' = \Omega$)

$$y(x,t) \;=\; u(x,t) \;+\; \eta(x,t)$$

with

$$\|\eta\|^2 \;=\; \int_0^\theta\int_\Omega \eta^2(x,t)dtdx \;\leq\; \gamma$$

and treating the respective version of Problems I, II we again come to the minimization of $J(w(\cdot))$ which is now

$$J(w(\cdot)) \;=\; \int_0^\theta\int_\Omega (u(x,t;w(\cdot)) \;-\; y(x,t))^2 dxdt, \;\; w(\cdot) \in L_2(\Omega)$$

With $y(\cdot,\cdot) \in L_2(Q)$ we have in general

$$J_0 = \inf\{J(w(\cdot))|w(\cdot) \in L_2(\Omega)\} \;> 0$$

We further give some techniques related to the quasiinvertibility idea, that allow to handle the problem of this section.

Theorem 6.1 *The value*

$$J_0 \equiv \inf\{J(w(\cdot)) : w(\cdot) \in L_2(\Omega)\} =$$

$$= \|y(\cdot,\cdot)\|^2_{L_2} - 2\sum_{i=1}^{\infty} p_i^2 \lambda_i (1 - \exp(-2\lambda_i\theta))^{-1}, \tag{6.1}$$

where

$$p_i = \int_{\Omega} p(x)\, \varphi_i(x) dx,$$

$$p(x) = \int_0^{\theta} u(x, t; y(\cdot, t)) dt,$$

and

$$J(w_\epsilon(\cdot)) \to J_0, \quad \epsilon \to 0,$$

where

$$w_\epsilon(\cdot) = 2\sum_{i=1}^{\infty} \lambda_i p_i \exp(-\epsilon\lambda_i\theta)\, (1 - \exp(-2\lambda_i\theta))\varphi_i.$$

Let us introduce a function

$$y_\epsilon(x,t) = U(x, \epsilon\theta; y(\cdot, t))$$

and a functional

$$J_\epsilon(w(\cdot)) = \int_0^{\theta}\int_{\Omega} (u(x,t; w(\cdot)) - y_\epsilon(x,t))^2 dx dt$$

Then

$$J_\epsilon(w(\cdot)) = \int_0^\theta \int_\Omega (\sum_{i=1}^{\infty} \exp(-\lambda_i t) w_i \varphi_i(x) -$$

$$- \sum_{i=1}^{\infty} \exp(-\lambda_i \epsilon\theta) y_i(t) \varphi_i(x))^2 dx dt$$

where

$$y_i(t) = \int_\Omega y(x,t)\varphi_i(x)dx,$$

Its minimizer $w_\epsilon(\cdot)$ exists and is given by

$$w_\epsilon(\cdot) = \arg\ \min\{J_\epsilon(w(\cdot)) : w(\cdot) \in L_2(\Omega)\} =$$

$$= \sum_{i=1}^{\infty} 2(1 - \exp -2\lambda_i\theta)^{-1} \lambda_i p_i \exp(-\epsilon\lambda_i\theta)\varphi_i(\cdot)$$

The element $w_\epsilon(\cdot) \in L_2(\Omega)$ since

$$\sum_{i=1}^{\infty} (1 - \exp(-2\lambda_i\theta))^2 \lambda_i^2 p_i^2 \exp(-2\epsilon\lambda_i\theta) \leq$$

$$\leq \|y(\cdot,\cdot)\|^2_{L_2(Q)} \sum_{i=1}^{\infty} \lambda_i^2 \exp(-2\epsilon\lambda_i\theta) < \infty$$

The property

$$\lim_{\epsilon\to 0} J(w_\epsilon(\cdot)) = \inf\{J(w(\cdot)) : w(\cdot) \in L_2(\Omega)\}, \tag{6.3}$$

now follows from the obvious relation

$$\|y(\cdot,\cdot) - y_\epsilon(\cdot,\cdot)\|_{L_2(\theta)} \to 0, \quad \epsilon \to 0$$

and the estimate

$$|J^{1/2}(w(\cdot)) - J_\epsilon^{1/2}(w(\cdot))| \leq \|y(\cdot,\cdot) - y_\epsilon(\cdot,\cdot)\|_{L_2(Q)}$$

Finaly, by direct calculation of $J(w_\epsilon)$ we have

$$J(w_\epsilon) = \|y(\cdot,\cdot)\|^2_{L(Q)} +$$

$$+2\sum_{i=1}^{\infty}(\lambda_i p_i^2(1-\exp(-2\lambda_i\theta))^{-1} \times$$

$$\times(\exp(-2\epsilon\lambda_i\theta) - 2\exp(-\epsilon\lambda_i\theta))),$$

which, after a limit transition ($\epsilon \to 0$), does yield the relation (6.1) of Theorem 6.1.

Remark 6.1 The suggested approach is actually not very different from the quasiinvertibility techniques. Indeed, taking the problem of Section 4 which is to find a numerically stable solution to inequality

$$\mathcal{R}(w(\cdot), y(\cdot)) \equiv \|u(\cdot,\theta; w(\cdot)) - y(\cdot)\|^2_{L_2(\Omega)} \leq \gamma, \quad w(\cdot) \in L(\Omega)$$

and solving it through the techniques of this section we introduce a function

$$y_\epsilon(\cdot) = \sum_{i=1}^{\infty} y_i \exp(-\lambda_i^2\epsilon\theta)\varphi_i(\cdot)$$

$$y_i = \int_\Omega y(x)\varphi_i(x)dx$$

constructed through function $y(\cdot)$.

The respective functional $\mathcal{R}(w(\cdot); y(\cdot))$ has a minimizer which is

$$w_\epsilon(\cdot) = \sum_{i=1}^{\infty} \exp(-\epsilon\lambda_i^2\theta + \lambda_i\theta)y_i\varphi_i(\cdot) \tag{6.4}$$

and which coincides precisely with the Lions-Lattes solution.

Let us now solve problem (6.1) through guaranteed estimation with measurement

$$y(x,t) = u(x,t) + \eta(x,t)$$

under restriction (3.2) with N_ϵ, K_ϵ given.

Theorem 6.2 *Taking N_ϵ as in Theorem 4.1 and*

$$K_\epsilon \eta = 2\sum_{i=1}^{\infty} \lambda_i \exp(-\epsilon\lambda_i\theta)(1-\exp(-2\lambda_i\theta))^{-1}\eta_i(t)\varphi_i(x)$$

with

$$\eta_i(t) = \int_\Omega \eta(x,t)\,\varphi_i(x)dx$$

we observe that the center $w_\epsilon^0(\cdot)$ for the respective ellipsoid W_ϵ does coincide with w_ϵ of (6.4)

7 Finite-Dimensional Sensors and the Observability Property

Let $\mathbf{G}u$ be a finite-dimensional sensor of type (iv) in Section 2. Taking Problem II for this case, we will solve it through the guaranteed estimation procedure for Problem III with

$$N_\epsilon = \epsilon I, \quad K_\epsilon = I \tag{7.1}$$

Assumption 7.1 Taking Problem II, assume that

$$\gamma > \inf\{J(w(\cdot))|w(\cdot) \in L_2(\Omega)\}$$

and therefore $W_\epsilon \neq \emptyset$.

Denote

$$U(\cdot,\theta|W) = \cup\{u(\cdot,\theta|w(\cdot))|w(\cdot) \in W\}$$

Assumption 7.2 The system (7.1), (7.2)

$$y(t) = \mathbf{G}(t)\, u(\cdot, t)$$

is strongly observable (set $U(\cdot, \theta | W(\cdot))$ is bounded for $\gamma > 0$).

The following assertions are true:

Lemma 7.1 *Under the assumptions 7.1, 7.2 with* $\mathbf{G}u$ *- finite dimensional and* N_ϵ, K_ϵ *taken according to (7.1), the solution* W_ϵ *to Problem III satisfies the relation*

$$\lim_{\epsilon \to 0} W_\epsilon = W^*$$

Lemma 7.2 *Under the assumptions 7.1, 7.2, the centers* $u_\epsilon^0(\cdot, \theta) = u(\cdot, \theta | w_\epsilon^0(\cdot))$ *of set* $U(\cdot, \theta | W_\epsilon)$ *for the* W_ϵ *of Lemma 7.1 do have a weak convergence*

$$\lim_{\epsilon \to 0} u_\epsilon^0(\cdot, \theta) = u^0(\cdot, \theta), \quad u^0(\cdot, \theta) \in L_2(\Omega).$$

Take the boundary-value problem

$$\frac{\partial v_\delta(x,t)}{\partial t} = A v_\delta(\cdot, t) + \delta A^2 v_\delta(\cdot, t),$$

$$x \in \Omega, \quad 0 \le t \le \theta, \tag{7.2}$$

$$v_\delta(\cdot, \theta) = u^0(\cdot, \theta)$$

$$v_\delta(x,t)\Big|_{\Sigma_\theta} = A v_\delta(x,t)\Big|_{\Sigma_\theta} = 0$$

Then the following proposition is true.

Theorem 7.1 *Assume* $\mathbf{G}u(\cdot,\cdot)$ *to be a compact map* $(\overset{\circ}{V}{}_2^{1,0}(Q) \to L_2[0,\theta])$. *Then under the assumptions 7.1, 7.2 with* $u^0(\cdot,\theta)$ *of (7.2) taken as in Lemma 7.2 we have*

$$h(x_\delta(\cdot,0),\, W^*(\theta)) \;\to\; 0 \;\; (\delta \to 0)$$

where $h(v, W^*)$ *is the Hausdorff semidistance*

$$d(v, W^*) = \inf\{\epsilon : W^* + \epsilon S \supseteq v\},$$

$$S \;=\; \{s :< s, s > \leq 1\}$$

The observability property is therefore crucial for the existence of a regularizer.

Remark 7.1 With the interval $[0,\theta]$ being variable, the techniques of this paper could be presented in terms of solutions to the guaranteed filtering problem as described, for example in [4, 7].

Remark 7.2 An important class of inverse problems however, are those that require *on-line estimation (or "reconstruction")* of the unknown distributions or parameters on the basis of observations taken on a variable time interval $[0,\theta]$ with θ increasing. Effective numerical techniques for treating these were proposed in papers [18-20].

References

1. Ladyzhenskaya, O.S. *Boundary value problems of mathematical physics.* v. 1. Nauka, Moscow, 1973.

2. Lions, J-L and R. Lattes. *Méthode de quasi-reversibilité et applications.* Dunod, Paris, 1967.

3. Kurzhanski, A.B. *Control and Observation Under Uncertainty.* Nauka, Moscow, 1977.

4. Kurzhanski, A.B. and A.Yu. Khapalov. On the state estimation problem for distributed systems. Analysis and Optimization of Systems. *Lect. Notes on Control and Inf. Sciences.* v. 83. A. Bensoussan and J-L. Lions (eds). Springer-Verlag, 1986.

5. Kurzhanski, A.B. and I. Sivergina. On noninvertible evolutionary systems: guaranteed estimates and the regularization problem. *Soviet Math. Doklady.* v. 314, No. 2, 1990.

6. El Jai, A. and A. Pritchard. *Sensors and Controls in the Analsysis of Distributed Systems.* J. Wiley, 1988.

7. Kurzhanski, A.B. and A.Yu. Khapalov. Observers for distributed parameter systems. *Journal of Math. Systems, Estimation and Control.* v. 1, No. 4, 1991.

8. Alifanov, O., E. Arutiynov and S. Rumiantsev. *Extremal Methods for ill-posed problems and their applications to inverse problems of heat exchange.* Nauka, Moscow, 1988.

9. Tikhonov, A.N. and V. Arsenin. *Methods for solving ill-posed problems.*

10. Ivanov, V., V. Vasin and V. Tanana. *The theory of linear ill-posed problems and their applications.* Nauka, Moscow, 1978.

11. Ekeland, I. and R. Temam. *Convex analysis and variational problems.* Mir, Moscow, 1979.

12. Kuratowski, K. *Topology.* v. 1. Mir, Moscow, 1966.

13. Showalter, R.E. Final value problem for evolution equations. *J. Math. Anal. & Appl.* v. 47, 1974.

14. Gaevski, H. and K. Zacharias. Zur Regularisierung einer Klasse nichtkorrekter Probleme bei Evolutionsgleichungen. *J. Math. Anal. Appl.* v. 38, No. 4, 1972.

15. Samarski, A.A. and P.N. Vabischevich. *Difference schemes for unstable problems.* Moscow, 1990 (preprint N. 111 of the Keldysh Institute for Applied Mathematics).

16. Petrovski, I.G. *Lectures on the theory of integral equations.* Nauka, Moscow, 1985.

17. Beck, J., B. Blackwell and Ch. Saint-Claire. Ill-posed inverse problems of heat transfer., Wiley, 1985.

18. Kryazhimski, A.V., Yu.S. Osipov. Inverse problems of dynamics and control models. In: *Mechanics and Scientific Technical Progress*, vol. 1, Moscow, Nauka, 1987.

19. Osipov, Yu.S. Inverse Problems of Dynamics for systems described by parabolic inequality. In: *Modeling and Inverse Problems of Control for Distributed Parameter Systems*, WP-89-101, IIASA, Laxenburg, Austria, 1989.

20. Osipov, Yu.S. On the Reconstruction of a Parameter of a Hyperbolic System, WP-91-54, IIASA Laxenburg, Austria, 1991.