Working Paper

Input Reconstructibility for Linear Dynamics. Ordinary Differential Equations

A. V. Kryazhimskii and Yu. S. Osipov

WP-93-65
November 1993

IIASA
International Institute for Applied Systems Analysis ◻ A-2361 Laxenburg ◻ Austria
Telephone: +43 2236 715210 ◻ Telex: 079 137 iiasa a ◻ Telefax: +43 2236 71313

Input Reconstructibility for Linear Dynamics. Ordinary Differential Equations

A. V. Kryazhimskii and Yu. S. Osipov

WP-93-65
November 1993

IIASA International Institute for Applied Systems Analysis ◻ A-2361 Laxenburg ◻ Austria
Telephone: +43 2236 715210 ◻ Telex: 079 137 iiasa a ◻ Telefax: +43 2236 71313

Foreword

The paper deals with the standard input-output observation scheme for a dynamic system governed by a linear ordinary differential equation. The initial problem is to reconstruct the actually working time-varying input, given a state observation result. Normally, the problem has no solution: observation is too poor to select the real input from the collection of "possible" ones. It is proposed to turn the problem as follows: what information of the real input is reconstructable precisely? The dual setting: what information of the real input is totally non-reconstructable? The question of aftereffect arises naturally: does accumulation of observation results lead to the informational jump – from non-reconstructibility to complete reconstructibility – in the past? Posing and answering these questions is the goal of the present study.

The results were announced at the IIASA Conference "Modeling of Environmental Dynamics", Sopron, Hungary, 30 August - 2 September, 1993.

Contents

Input Reconstructibility for Linear Dynamics. Ordinary Differential Equations

A. V. Kryazhimskii and Yu. S. Osipov***

1 Introduction

In the present study, we deal with an *inverse problem of dynamics.* The term appeared in mechanics. It denoted initially the problem of detecting a dynamic force that makes a mechanical system go along a prescribed trajectory (see Galliulin, 1986). In 1960-70-s, the mathematical inversion theory for systems governed by linear ordinary differential equations was developed (see e.g. Brockett and Mesarovich, 1965; Sain and Massey, 1969; Silverman, 1969; Willsky, 1974); the theory centered on unique solvability conditions and synthesis of inverse systems. Later, the above questions were studied for several classes of non-linear system (see e.g. Hirschorn, 1981).

Simultaniously, the linear observation theory was created; we refer here to Krasovskii, 1968, where the foundations of the theory are narrated systematically, Wohnam, 1979, where the geometrical approach to observation problems is developed, and Kurzhanskii, 1977, where linear observation problems for systems with uncertainties are treated.

An observation problem requires finding a system's state, or its projection to a chosen direction, on the basis of state observation results. In fact, observation and inversion problems are close to each other. In the present study, a way to combining these two approaches is discussed. Roughly speaking, we transit the observation problem to the field of inverse problems by replacing the state's projection (the sought object) by the input's projection. So far as inputs are, unlike states, functions of time, the inputs' projections are taken in an appropriate functional space. We can express this in the other way saying that we put the inversion problem into the observation pattern by replacing the whole input's history (the sought object) by its projection to a chosen "functional direction". In other words, we combine the two approaches by changing the sought object (an input instead of a state) in an observation problem, or, symmetrically, the desired information of the sought object (an input's projection instead of the whole input's history) in an inversion problem. For the resulting synthetic problem we use the term the *reconstruction problem.*

Note that a state observation problem, when posed for a system with unobservable inputs (see Nikol'skii, 1971; Aubin and Frankowska, 1986; Kryazhimskii and Osipov, 1993), can be looked at as that of calculating the value of a special (vector) functional of an input, or an input's "projection" (given by a system's state).

This paper provides the outline of an approach to posing and treating input reconstruction problems. The study is restricted to the case of the simplest linear dynamics

*IIASA, A-2361, Laxenburg, Austria

**Presidium of Acad. Sci. of Russia, Leninski Prospect 14, Moscow, 117901, Russia

(ordinary differential equations). We use appropriately modified tools of linear observation theory, with duality between the initial and adjoint systems playing the central role.

In the last Section, some unsolved questions are listed; we plan to tackle them in our nearest investigations.

We are grateful to Professor Gurii I. Marchuk for fruitful discussions.

2 Problem Settings

2.1 Informal Discussion

We will consider a dynamical system governed by a linear differential equation

$$\dot{x}(t) = A(t) + B(t)u(t) \tag{2.1}$$

in a finite-dimensional vector space; here $x(t)$ is a system's state at time t, and $u(t)$ is the (finite-dimensional) value of a time-varying input inducing the system's motion; $A(t)$ and $B(t)$ are matrice functions of appropriate dimensions; $\dot{x}(t)$ stands for $dx(t)/dt$. We put $t \geq 0$ and fix an initial state

$$x(0) = x_0 \tag{2.2}$$

Let the system be observed by an Observer. The Observer knows a priori the system equation (2.1) and the initial state (2.2). Besides, at every time t the vector

$$z(t) = Px(t) \tag{2.3}$$

carrying information of the system's state $x(t)$ is measured by the Observer. Real input values $u(t)$ are unknown to the Observer. The Observer's task is to reconstruct $u(t)$ on the basis of all available data, i.e. the system equation (2.1), the initial state (2.2), the observation matrice P and the observation results (2.3). In other words, at a current time s the Observer is supposed to reconstruct the history $u(\cdot)$ (i.e. all values $u(t)$, $0 \leq t \leq s$) of the real input using the a priori information of the system and the history $z(\cdot)$ ($z(t)$, $0 \leq t \leq s$) of observation results.

The problem, in this very severe setting, is rarely solvable. Typically, the structure of the observation matrice P is too poor to enable the Observer to reconstruct the whole input history $u(\cdot)$.

To arrive anyway to a certain solution, one should pass to a weaker problem formulation. We propose to turn the problem as follows: find, what information of the real input history is reconstructable precisely. The dual setting: find, what information of the real input history is totally non-reconstructable.

Taking these preliminary formulations for the basis, specify the types of information of the real $u(\cdot)$ we will be dealing with. Our (the Observer's) task will be to calculate the value

$$\rho = l(u(\cdot)) \tag{2.4}$$

where l is a given scalar function (a functional) defined on the space of all possible input hystories. Assuming this space (of functions of time) to be linear, we suppose that l is a linear functional. Thus, a linear functional l determines a type of information to be reconstructed. Those l for which the values (2.4) can be calculated by the Observer precisely will be called *reconstructable*, and those l for which the Observer is unable to specify (2.4) will be called *non-reconstructable.* Our task is to describe all reconstructable and all non-reconstructable functionals. (After solving the problem, we will immidiately find whether there are functionals not belonging to the above two classes.)

The case where all functionals l are reconstructable is of special interest. Since knowing the values (2.4) for all linear l is practically equivalent to knowing $u(\cdot)$, we will say that in this case *the input is reconstructable.*

Let us now specify what do we mean saying that the value (2.4) can be calculated by the Observer precisely (the functional l is reconstructable). We refer here to the set $U_s(z(\cdot))$ of the input hystories (briefly, inputs) *compatible* with the observation history (or the observation result) $z(\cdot)$. An input $v(\cdot)$ is considered as compatible with $z(\cdot)$ if, being sent (imaginary) onto the system (2.1) instead of $u(\cdot)$, it produces the observation result $z(\cdot)$ actually provided by $u(\cdot)$. The Observer has no tool to select the real $u(\cdot)$ among all $v(\cdot)$ compatible with $z(\cdot)$, and therefore is compelled to consider any such $v(\cdot)$ as a real candidate for $u(\cdot)$. Consequently, all values

$$\zeta = l(v(\cdot)) \tag{2.5}$$

with $v(\cdot)$ running through $U_s(z(\cdot))$, are admissible for being the real value (2.4). Therefore if there are several (or many) different values (2.5), we have no reason to saying that the real value (2.4) can be calculated by the Observer precisely. The Observer can do so (at least potencially) if all values (2.5) coinside or, equivalently, the set $R_s(z(\cdot))$ of all values (2.5) is one-element. This is exactly the case of l reconstructable.

The other extreme case is $R_s(z(\cdot))$ covering the whole real line, with no opportunity for the Observer to detect where the value (2.4) is located. This is the case of l non-reconstructable.

The other variant of the problem is concerned with reconstructibility in the past. In this setting the instant s in the past, and the history up to s is analysed at an $\xi \geq s$. Namely, the Observer's task is to reconstruct the input history $u(\cdot)$ at the time interval $[0, s]$ using the observation result $z(\cdot)$ at the longer time interval $[0, \xi]$. As above, the classes of reconstructable and non-reconstructable (in the past) functionals are introduced and the problem of describing these classes is formulated. This problem is clearly more general than the previous one (where $\xi = s$). Therefore we start our study with the simpler problem of recnstructibility at present.

2.2 Basic Notations

Let us fix our basic notations.

By $\mathbf{R}^k$ is denoted the Euclidean space of k-dimensional column vectors, $| x |$ and (x, y) standing, respectively, for the norm of a vector $x \in \mathbf{R}^k$ and the scalar product of vectors $x, y \in \mathbf{R}^k$. The superscript ^{T} means transposition. The i-th coordinate of a vector $x \in \mathbf{R}^k$ is denoted $x^{(i)}$. We write $[1 : k]$ instead of $\{1, ..., k\}$. The symbol $x(\cdot)$ is used for a function defined on a subset of the real line, with the value $x(t)$ at a point t; the restriction of $x(\cdot)$ to an interval $[s, \xi]$ (belonging to the set of definition of $x(\cdot)$) is denoted by $x(\cdot)_{s,\xi}$.

We use the standard notation $\mathbf{L}^2([s, \xi], \mathbf{R}^k)$ (see e.g. Warga, 1975) for the Hilbert space of all functions $u(\cdot) \in [s, \xi]$ Lebesgue integrable with $| u(\cdot) |^2$; recall that the scalar product of $u(\cdot)$ and $v(\cdot)$ in the above space has the form

$$(u(\cdot), v(\cdot)) = \int_s^\xi (u(t), v(t))dt$$

and the norm of an $u(\cdot)$ is $\| u(\cdot) \| = ((u(\cdot), u(\cdot))^{1/2}$. Integration is always understood in the sense of Lebesgue.

By Lin$E$ we denote the linear hull of a set E in the space $\mathbf{L}^2([s,\xi],\mathbf{R}^k)$, i.e. the closure in the above space of the set of all finite linear combinations of elements from E. If E is a linear subspace of $\mathbf{L}^2([s,\xi],\mathbf{R}^k)$, then $E^\perp$ stands for the subspace of the above space orthogonal to E; if E lies in a subspace X, then $E_X^\perp = E^\perp \cap X$. The kernel of a linear operator F is denoted by Ker$F$.

2.3 The Observed System

Fix natural n and r, and bounded and (Lebesgue) measurable matrice functions $A(\cdot)$ and $B(\cdot)$ of dimensions $n \times n$ and $n \times r$ defined on $[0,\infty[$. Thus, the system (2.1) with n-dimensional states and r-dimensional input values is determined. If the matrice functions $A(\cdot)$ and $B(\cdot)$ are constant, we will say that the system is *stationary*, and denote $A = A(t)$ and $B = B(t)$.

Further, we denote briefly $\mathbf{L}^2_{s,\xi} = \mathbf{L}^2([s,\xi],\mathbf{R}^r)$.

An *input on* $[0,s]$ $(s \geq 0)$ is identified with a function $u(\cdot)$ from $\mathbf{L}^2_{0,s}$. A *trajectory* corresponding to an input $u(\cdot)$ on $[0,s]$ is a (Caratheodory) solution $x(\cdot)$ of the differential equation (2.1) on $[0,s]$ with the initial condition (2.2); as it is known, the above trajectory is unique.

Fix a natural m and an $m \times n$ matrice P determining the observation equation (2.3). An m-dimensional function $z(\cdot)$ defined on $[0,s]$ will be called an *observation result* (on $[0,s]$) if there exists an input $u(\cdot)$ on $[0,s]$ such that the trajectory $x(\cdot)$ corresponding to $u(\cdot)$ satisfies (2.3) for all $t \in [0,s]$; every $u(\cdot)$ with the above property will be called *compatible* with the observation result $z(\cdot)$. The set of all $u(\cdot)$ compatible with $z(\cdot)$ will be denoted by $U_s(z(\cdot))$. For the set of all observation results on $[0,s]$ we will use the notation Z_s.

2.4 Reconstructibility Problems

A continuous linear functional l on the space $\mathbf{L}^2_{0,s}$ of all inputs on $[0,s]$ will be as usual identified with an element

$$l(\cdot) \in \mathbf{L}^2_{0,s} \tag{2.6}$$

determined by

$$l(u(\cdot) = (l(\cdot),u(\cdot))$$

For every above $l(\cdot)$ and every observation result $z(\cdot)$ on $[0,s]$, introduce the image of the set $U_s(z(\cdot))$ under $l(\cdot)$:

$$R_s(l(\cdot),z(\cdot)) = \{(l(\cdot),u(\cdot)) \ : \ u(\cdot) \in U_s(z(\cdot))\} \tag{2.7}$$

(clearly, this set is nonempty). A functional (2.6) will be called *reconstructable at* $z(\cdot) \in Z_s$ if the set (2.7) is one-element, and *non-reconstructable at* $z(\cdot)$ if this set coincides with the whole real line. A functional (2.6) reconstructable (respectively, non-reconstructable) at every $z(\cdot) \in Z_s$ will be called *reconstructable* (respectively, *non-reconstructable*) *on* $[0,s]$.

Our basic problem is: given an observation result $z(\cdot)$ on $[0,s]$, find all functionals (2.6) reconstructable at $z(\cdot)$ and all functionals (2.6) non-reconstructable at $z(\cdot)$.

For a functional $l(\cdot)$ reconstructable at a $z(\cdot) \in Z_s$, denote by $\rho_s(l(\cdot),z(\cdot))$ the single element of the set (2.7). The problem of calculating this value will also be of interest for us.

We will say that *the input is reconstructable at* $z(\cdot) \in Z_s$ if every functional (2.6) is reconstructable at $z(\cdot)$. If it is so for every $z(\cdot) \in Z_s$, we will say that *the input is reconstructable on* $[0,s]$.

Proposition 2.1 *The input is reconstructable at a $z(\cdot) \in Z_s$ if and only if the set $U_s(z(\cdot))$ is one-element (in the sense that every two elements of this set coinside almast all, with respect to the Lebesgue measure).*

The proposition is evident. Below, some sufficient input reconstructibility conditions will be considered.

2.5 Reconstructibility in the Past

For any $\xi \geq s \quad (s \geq 0)$ and $z(\cdot) \in Z_s$, denote by $U_\xi(z(\cdot))_s$ the set of the restrictions to $[0, s]$ of all $u(\cdot) \in U_\xi(z(\cdot))$, and for any functional (2.6), put

$$R_\xi(l(\cdot), z(\cdot))_s = \{(l(\cdot), u(\cdot)) \; : \; u(\cdot) \in U_\xi(z(\cdot))_s\} \tag{2.8}$$

A functional (2.6) will be called *reconstructable at* $z(\cdot) \in Z_\xi$ where $\xi \geq s$ if the set (2.8) is one-element, and *non-reconstructable at* $z(\cdot)$ if this set coincides with the whole real line. A functional (2.6) reconstructable (respectively, non-reconstructable) at every $z(\cdot) \in Z_\xi$ will be called *reconstructable* (respectively, *non-reconstructable*) *on* $[0, \xi]$.

The problem of reconstructibility in the past is formulated as follows: given a $\xi \geq s$ and an observation result $z(\cdot)$ on $[0, \xi]$, find all functionals (2.6) reconstructable at $z(\cdot)$ and all functionas (2.6) non-reconstructable at $z(\cdot)$.

We will say that *the input on* $[0, s]$ *is reconstructable at* $z(\cdot) \in Z_\xi$ if every functional (2.6) is reconstructable at $z(\cdot)$. If it is so for every $z(\cdot) \in Z_\xi$, we will say that *the input on* $[0, s]$ *is reconstructable on* $[0, \xi]$.

3 Criteria of Reconstructibility

3.1 A Compatibility Criterion

In this Section, the problems of reconstructibility at present posed in Subsection 2.4 are analysed.

We start with a description of the set $U_s(z(\cdot))$ of all inputs compatible with an observation result $z(\cdot) \in Z_s$.

Let p_i be the transposed i-th line of the matrice P (so, $p_i \in \mathbf{R}^n$). For any $i \in [1 : m]$ and $\sigma \geq 0$, define the n-dimensional function $w_i(\cdot, \sigma)$ to be the solution of the Cauchy problem

$$\dot{w}(t) = -A^T w(t) \tag{3.1}$$

$$w(\sigma) = p_i \tag{3.2}$$

on $]-\infty, \sigma]$ and zero on $]\sigma, \infty[$, and assume the notations

$$\phi_i(\cdot, \sigma) = B^T w_i(\cdot, \sigma) \tag{3.3}$$

$$g_i(a, \sigma) = a^{(i)}(w_i(0, \sigma, x_0)) \quad (a \in \mathbf{R}^m) \tag{3.4}$$

Theorem 3.1 *An input $u(\cdot)$ is compatible with an observation result $z(\cdot)$ on $[0, s]$ (or $u(\cdot) \in U_s(z(\cdot))$) if and only if*

$$(\phi_i(\cdot, \sigma)_{0,s}, u(\cdot)) = g_i(z(\sigma), \sigma) \tag{3.5}$$

for all $\sigma \in [0, s]$ and $i \in [1 : m]$.

Proof. Let $u(\cdot)$ be compatible with $z(\cdot)$, and $x(\cdot)$ be the trajectory corresponding to $u(\cdot)$. Then for all $t \in [0, s]$ we have (2.3) or, equivalently,

$$z^{(i)}(t) = (p_i, x(t)) \tag{3.6}$$

with every $i \in [1 : m]$. Take arbitrary $\sigma \in [0, s]$ and $i \in [1 : m]$. Let

$$w(\cdot) = w_i(\cdot, \sigma) \tag{3.7}$$

Multiply scalarly (2.1) by $w(t)$ and (3.1) by $x(t)$, distract and integrate from 0 to σ (in fact we perform multiplication at those t where both (2.1) and (3.1) are fulfilled; the set of such t has the full measure on $[0, \sigma]$, and the integration is possible). We get

$$\int_0^\sigma [(w(t), \dot{x}(t)) - (\dot{w}(t), x(t))]dt =$$

$$\int_0^\sigma [(w(t), A(t)x(t)) - (A^T(t)w(t), x(t))]dt +$$

$$\int_0^\sigma (w(t), B(t)u(t))dt$$

The left hand side is integrated explicitly, and the first integrand in the right hand side is zero. Therefore the above equality can be rewritten as

$$\int_0^\sigma (B^T(t)w(t), u(t))dt = (w(\sigma), x(\sigma)) - (w(0), x(0))$$

This equality is equivalent to (3.5): see (3.7) and (3.3) to compare the left hand sides, and (3.2), (3.6), (3.7), (2.2) and (3.4) to compare the right hand sides.

Conversely, let $u(\cdot)$ satisfy (3.5) for all $\sigma \in [0, s]$ and $i \in [1 : m]$. Suppose that $u(\cdot)$ is not compatible with $z(\cdot)$. Then there exists a $\sigma \in [0, s]$ and an $i \in [1 : m]$ such that

$$z^{(i)}(\sigma) \neq (p_i, x(\sigma)) \tag{3.8}$$

where $x(\cdot)$ is the trajectory corresponding to $u(\cdot)$. As above, we come to the equality copying (3.5) with $z(\sigma)$ replaced by $Px(\sigma)$. Distract this equality from (3.5). The result contradicts (3.8).

□

Differentiation of the equality (3.5) in σ leads to a necessary compatibility condition. We formulate the result only for a stationary system (we will refer to it in Subsection 3.4). Put

$$g_i^0(a, \sigma) = a^{(i)} - (A^T w_i(-\sigma, 0), x_0) \quad (a \in \mathbf{R}^m) \tag{3.9}$$

Corollary 3.1 *Let the system be stationary. If an input $u(\cdot)$ on $[0, s]$ is compatible with an observation result $z(\cdot)$ on $[0, s]$, then*

$$(B^T p_i, u(\sigma)) = -\int_0^\sigma (B^T A^T w_i(t - \sigma, 0), u(t))dt + g_i^0(\dot{z}(\sigma), \sigma) \tag{3.10}$$

for all $i \in [1 : m]$ and almost all $\sigma \in [0, s]$.

Proof. Differentiate (3.5) at an σ where $z(\cdot)$ is differentiable (the set of such σ's has the full measure on $[0,s]$), with noting previously that

$$\frac{\partial\phi_i(t,\sigma)}{\partial\sigma} = B^T\frac{\partial w_i(t,\sigma)}{\partial\sigma} = B^T\frac{\partial w_i(t-\sigma,0)}{\partial\sigma} = B^T A^T w_i(t-\sigma)$$

(see (3.3) and (3.1)). We get

$$(\phi_i(\sigma,\sigma),u(\sigma)) + \int_0^\sigma (B^T A^T w_i(t-\sigma),u(t))dt = g_i^0(\dot z(\sigma),\sigma)$$

(see also (3.4) and (3.9)). This is equivalent to (3.10) since $\phi_i(\sigma,\sigma) = B^T w_i(\sigma,\sigma) = B^T p_i$.

□

3.2 The Reconstructibility Alternative

¿From Theorem 3.1 follows that for $l(\cdot) = \phi_i(\cdot,\sigma)_{0,s}$ where $\sigma \in [0,s]$ and $i \in [1:m]$, the value $(l(\cdot),u(\cdot))$ does not depend on an $u(\cdot) \in U_s(z(\cdot))$; therefore the above $l(\cdot)$ is reconstructable at $z(\cdot)$. Note that this is so for an arbitrary $z(\cdot) \in Z_s$ meaning that $l(\cdot)$ is reconstructable on $[0,s]$ (see Subsection 2.4). The next theorem states that this holds for every functional from the linear hull of all above $l(\cdot)$'s, and all other functionals are non-reconstructable on $[0,s]$.

Let

$$K_s = \{\phi_i(\cdot,\sigma)_{0,s} \ : \ \sigma \in [0,s], \ i \in [1:m]\} \tag{3.11}$$

$$L_s = \mathrm{Lin} K_s \tag{3.12}$$

Theorem 3.2 *Every $l(\cdot) \in L_s$ is reconstructable on $[0,s]$, and every $l(\cdot) \in \mathbf{L}^2_{0,s} \setminus L_s$ is non-reconstructable on $[0,s]$.*

Proof. Theorem 3.1 implies evidently that every $l(\cdot) \in K_s$ is reconstructable on $[0,s]$. Suppose that there is an $l(\cdot) \in L_s$ non-reconstructable at a certain $z(\cdot) \in Z_s$. Then one can find $u_1(\cdot), u_2(\cdot) \in U_s(\cdot)$ such that

$$(l(\cdot),u_1(\cdot)) \neq (l(\cdot),u_2(\cdot)) \tag{3.13}$$

By the definition of L_s there exists a sequence $(l_k(\cdot))$ from K_s converging to $l(\cdot)$ in $\mathbf{L}^2_{0,s}$. By (3.13)

$$(l_k(\cdot),u_1(\cdot)) \neq (l_k(\cdot),u_2(\cdot))$$

for large k implying that $l_k(\cdot)$ is not reconstructable at $z(\cdot)$. This is wrong since $z(\cdot) \in K_s$. The contradiction proves that $l(\cdot)$ is reconstructable on $[0,s]$.

Let now $l(\cdot) \in \mathbf{L}^2_{0,s} \setminus L_s$. Then $l(\cdot) = l_1(\cdot) + l_2(\cdot)$ where $l_1(\cdot) \in L_s$, $l_2(\cdot) \in L_s^\perp$, and $l_2(\cdot) \neq 0$. Let $z(\cdot) \in Z_s$. Take an $u(\cdot) \in U_s(\cdot)$ and put

$$v_\alpha(\cdot) = u(\cdot) + \alpha l_2(\cdot)$$

For every $\sigma \in [0,s]$ and $i \in [1:m]$,

$$(\phi_i\cdot,\sigma)_{0,s}, v_\alpha(\cdot)) = (\phi_i(\cdot,\sigma)_{0,s}, u(\cdot)) = g_i(z(\sigma),\sigma)$$

the last equality following from Theorem 3.1. Hence by Theorem 3.1 $v_\alpha(\cdot) \in U_s(z(\cdot))$. However

$$(l(\cdot),v_\alpha(\cdot)) = (l_1(\cdot),u(\cdot)) + \alpha \parallel l_2(\cdot) \parallel$$

covers the whole real line as α runs it through. Therefore $l(\cdot)$ is non-reconstructable at $z(\cdot)$. Due to arbitrarity of $z(\cdot)$, we conclude that $l(\cdot)$ is non-reconstructable on $[0,s]$.

□

3.3 Values of Reconstructable Functionals

In this Subsection, an $l(\cdot) \in L_s$ and a $z(\cdot) \in Z_s$ are fixed. Consider several examples showing how one can calculate the value $\rho_s(l(\cdot, z(\cdot))$ which is taken by $l(\cdot)$ at every input compatible with $z(\cdot)$.

For every $l_0(\cdot) \in K_s$ (see (3.11)) denote

$$\kappa_s(l_0(\cdot), z(\cdot)) = g_i(z(\sigma), \sigma) \tag{3.14}$$

where $\sigma \in [0, s]$ and $i \in [1 : m]$ are such that

$$l_0(\cdot) = \phi_i(\cdot, \sigma)_{0,s} \tag{3.15}$$

According to Theorem 3.1 (3.14) is the value of the functional $l_0(\cdot)$ at every $u(\cdot) \in U_s(z(\cdot)$. Hence if (3.15) holds for several σ and i, the value (3.14) does not depend on them; therefore the definition (3.14), (3.15) is correct.

Example 3.1 Let

$$l(\cdot) = \sum_{i=1}^{k} \alpha_i l_i(\cdot), \quad l_i(\cdot) \in K_s$$

¿From Theorem 3.1 follows immidiately

$$\rho_s(l(\cdot), z(\cdot)) = \sum_{i=1}^{k} \alpha_i \kappa_s(l_i(\cdot), z(\cdot))$$

Example 3.2 Let

$$\| l_j(\cdot) - l(\cdot) \| \to 0 \tag{3.16}$$

where

$$l_j(\cdot) = \sum_{i=1}^{k_j} \alpha_{ji}(\cdot) l_{ji}(\cdot), \quad l_{ji}(\cdot) \in K_s \tag{3.17}$$

Then

$$\rho_s(l(\cdot), z(\cdot)) = \lim \sum_{i=1}^{k_j} \alpha_{ji} \kappa_s(l_{ji}(\cdot), z(\cdot) \tag{3.18}$$

This follows obviously from Theorem 3.2 and Example 3.1.

Example 3.3 Let $[a, b]$ be a nonempty interval $\mu(\cdot)$ be a bounded measurable scalar function on $[a, b]$, and a measurable mapping $q(\cdot, \cdot) : (t, \nu) \mapsto q(t, \nu)$ from $[0, s] \times [a, b]$ to $\mathbf{R}^m$ satisfy $q(\cdot, \nu) \in K_s$ for all $\nu \in [a, b]$. Then for the $l(\cdot)$ defined by

$$l(t) = \int_a^b \mu(\nu) q(t, \nu) d\nu \tag{3.19}$$

we have

$$\rho_s(l(\cdot), z(\cdot)) = \int_a^b \mu(\nu) \kappa_s(q(\cdot, \nu), z(\cdot)) d\nu \tag{3.20}$$

(3.20) means in particular that $l(\cdot)$ is reconstructable on $[0, s]$ and the integral in the right hand side exists.

Prove (3.20). Since the function $\nu \mapsto q(\cdot, \nu) : [a, b] \mapsto \mathbf{L}^2_{0,s}$ is integrable (see e.g. Warga, 1975), there exists a sequence of functions $\nu \mapsto q_j(\cdot, \nu) : [a, b] \mapsto \mathbf{L}^2_{0,s}$ taking finite number of values (step functions) such that

$$\int_a^b \| q_j(\cdot, \nu) - q(\cdot, \nu) \|^2 d\nu \to 0 \tag{3.21}$$

Similarly, $\mu(\cdot)$ is mean-square approximated by a sequence of scalar step functions $\mu(\cdot)$:

$$\int_a^b \mid \mu_j(\nu) - \mu(\nu) \mid^2 d\nu \to 0 \tag{3.22}$$

Without loss of generality assume that for every j functions $\nu \mapsto q_j(\cdot,\nu)$ and $\nu \mapsto \mu_j(\nu)$ are constant at the same sets E_{ji} $(i \in [1:k])$:

$$q_j(\cdot,\nu) = l_{ji}(\cdot), \quad \mu_j(\cdot) = \mu_{ji} \quad (\nu \in E_{ji}) \tag{3.23}$$

Let

$$l_j(t) = \int_a^b \mu_j(\nu) q_j(t,\nu) d\nu$$

By (3.21) and (3.22) we have (3.16) where $l_j(\cdot)$ has the form (3.16) with $\alpha_{ji} = \mu_{ji}\mathrm{mes}_{ji}$ (mes stands for the Lebesgue measure). Therefore like in Example 3.4, we get the equality (3.18) (implying in particular that $l(\cdot)$ is reconstructable on $[0,s]$). Let us show that the limit on the right of (3.18) equals the integral from (3.20). Write this integral as

$$\int_a^b \gamma(\nu) d\nu$$

where

$$\gamma(\nu) = \mu(\nu)\kappa_s(q(\cdot, z(\cdot))$$

The sum under the limit sign in (3.18) equals

$$c_j = \int_a^b \gamma(\nu) d\nu$$

where

$$\gamma_j(\nu) = \mu_j \kappa_s(q_j(\cdot, z(\cdot))$$

Due to (3.23), $\gamma_j(\cdot)$ is a step function, and the integral c_j exists. Since $q(\cdot,\nu), q_j(\cdot,\nu) \in K_s$, (see also (3.14), (3.15)) we have by Theorem 3.1

$$(\mu(\nu)q(\cdot,\nu), u(\cdot)) = \gamma(\nu), \quad (\mu_j q_j(\cdot,\nu), u(\cdot)) = \gamma_j(\nu)$$

for an arbitrary $u(\cdot) \in U_s(z(\cdot))$. So far as $(t,\nu) \mapsto \mu(\nu)q(t,\nu)$ is integrable, $\gamma(\cdot)$ is integrable too. Thus the integral c exists. From (3.21), (3.22) follows

$$\int_a^b \mid \gamma_j(\nu) - \gamma(\nu) \mid^2 \to 0$$

yielding the desired convergence $c_j \to c$.

Example 3.4 Let $[a,b] = [0,s]$, $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_k = s$, and $q(t,\sigma) = \phi_{i_j}(t,\sigma)$ for $t \in [0,s]$ and $\sigma \in [\sigma_j, \sigma_{j+1}$. Then the mapping $q(\cdot,\cdot)$ satisfies the conditions of Example 3.3. Hence the formula (3.20) is true. Recall that $l(\cdot)$ is given by (3.19) with $\mu(\cdot)$ defined like in Example 3.3.

3.4 Input Reconstructibility Conditions

¿From Theorem 3.2 and the definition of input reconstructibility (Subsection 2.4) follows

Corollary 3.2 *The following assertions are equivalent:*
(i) the input is reconstructable on $[0, s]$,
(ii) the input is reconstructable at a certain $z(\cdot)$,
(iii) $L_s = \mathbf{L}^2_{0,s}$.

Let us provide a sufficient input reconstructibility condition based on the necessary compatibility condition of Corollary 3.1.

Theorem 3.3 *Let the system be stationary, and* $(B^T p_1, ..., B^T p_m)$ *be a basis in* $\mathbf{R}^r$. *Then the input is reconstructable on* $[0, s]$.

Proof. Let M_* be the $m \times r$-matrice whose i-th line is $(B^T p_i)^T$, $Q(\zeta)$ be the $m \times r$-matrice whose i-th line is $(-B^T A^T w_i(\zeta, 0))^T$, and $g0_*(\sigma)$ be the m-dimensional vector whose $i-th$ coordinate is $g^0(\dot{z}(\sigma), \sigma)$. For every $u(\cdot) \in U_s(z(\cdot))$ where $z(\cdot) \in Z_s$, we unite all conditions (3.10) Corollary 3.1) in

$$M_* u(\sigma) = \int_0^\sigma Q_*(t-\sigma)u(t)dt + g^0_*(\sigma) \tag{3.24}$$

holding for almost all $\sigma \in [0, s]$. By the assumption M_* containes a nondegenerate $r \times r$-submatrice M. Let the $r \times r$-matrice $Q(\zeta)$ be formed by the elements of $Q_*(\zeta)$ placed at the positions the elements of M have in M_*. Then (3.24) implies

$$Mu(\sigma) = \int_0^\sigma Q(t-\sigma)u(t)dt + g^0(\sigma)$$

the second item on the right being formed of the appropriate coordinates of $g^0_*(\sigma)$. Multiplying by M^{-1}, we get

$$u(\sigma) = \int_0^\sigma \gamma(t-\sigma)u(t)dt + \gamma(\sigma) \tag{3.25}$$

where

$$\Gamma(\zeta) = M^{-1}Q(\zeta), \ \gamma(\sigma) = M^{-1}g^0(\sigma)$$

This integral equation (with respect to $u(\cdot)$ has the unique solution. Indeed, the difference $v(\cdot)$ of two arbitrary solutions satisfies

$$v(\sigma) = \int_0^\sigma \Gamma(t-\sigma)v(t)dt$$

for almost all $\sigma \in [0, s]$, yielding

$$| v(\sigma) | \leq \int_0^\sigma c \mid v(t) \mid dt$$

where

$$c = \max\{ | \Gamma(\zeta) | : \ \zeta \in [0, s]\}$$

Hence $v(\cdot) = 0$ by the Gronwall's lemma (see Warga, 1975). Thus $U_s(z(\cdot))$ is one-element. By Proposition 3.1 the input is reconstructable at $z(\cdot)$. By Corollary 3.2 it is reconstructable on $[0, s]$.

□

Up to the end of this Subsection, the assumptions of Theorem 3.3 are fullfilled, a $z(\cdot) \in Z_s$ is chosen, and the integral equation (3.25) on $[0, s]$ (whose solution is the single input on $[0, s]$ compatible with $z(\cdot)$) is fixed. Consider briefly some numerical approximations to the solution $u(\cdot)$ of (3.25).

If $\gamma(\cdot)$ is continuous or piece-wise continuous, then the analogue of the Euler method is applicable. Namely, for a small time step δ, define the Euler approximation $u_\delta(\cdot)$ to $u(\cdot)$ by

$$u_\delta(\sigma) = \sum_{i=0}^{k} \Gamma((i-k)\delta)u_\delta(i\delta)\delta + \gamma(i\delta) \quad (\sigma \in [k\delta, (k+1)\delta[,\ \sigma \leq s)$$

Proposition 3.1 *Let $\gamma(\cdot)$ be continuous. Then $u_\delta(\cdot) \to u(\cdot)$ uniformly as $\delta \to 0$.*

Proposition 3.2 *Let $\gamma(\cdot)$ have a finite number of points of discontinuity. Then $u_\delta(\cdot) \to u(\cdot)$ in $\mathbf{L}^2_{0,s}$ as $\delta \to 0$.*

The proofs follow the standard Euler pattern, with using the Granwall's lemma.

In the general case where $\gamma(\cdot)$ is measurable, the values $\gamma(i\delta)$ are not defined, and the Euler method does not work. Here the modified extremal shift method from Krasovskii and Subbotin, 1974 is applicable. This method forms the approximation $v_\delta(\cdot)$ to $u(\cdot)$ by

$$v_\delta(\sigma) = - \mid c_k \mid b_k \quad (\sigma \in [k\delta, (k+1)\delta[,\ \sigma \leq s)$$

$$c_k = \sum_{j=0}^{k} \sum_{i=0}^{j} \Gamma((i-j)\delta)v_\delta(i\delta)\delta^2 + \int_0^{k\delta} \gamma(t)dt$$

$$b_k = \sum_{j=0}^{k} v_\delta(j\delta)\delta - c_k$$

Proposition 3.3 *We have $v_\delta(\cdot) \to u(\cdot)$ weakly in $\mathbf{L}^2_{0,s}$ as $\delta \to 0$.*

The proof follows the standard extremal shift scheme, with using the Gronwall's lemma to state uniform boundedness of functions $v_\delta(\cdot)$.

To build a strong $\mathbf{L}^2_{0,s}$-approximation to $u(\cdot)$, one can use regularized extremal shift methods following Kryazhimskii and Osipov, 1983, 1993. We do not emphasize here computational aspects and therefore not go into further details.

4 Criteria of Reconstructibility in the Past

4.1 Preliminary Estimates

In this Section, an $s \geq 0$ and a $\xi \geq s$ are fixed. We consider the problems of finding all functionals $l^*(\cdot) \in \mathbf{L}^2_{0,s}$ reconstructable and, respectively, non-reconstructable at an observation result $z(\cdot) \in Z_\xi$ (see Subsection 2.5).

In this Subsection, we select functionals whose reconstructibility or non-reconstructibility is easily verified. Thus we provide lower estimates for the sets of all reconstructable and, respectively, non-reconstructable (at $z(\cdot)$) functionals. In the next Subsections we will concentrate on the functionals lying between these estimates.

We start with the following observation (see notations in Subsection 2.5).

Lemma 4.1 *For any* $z(\cdot) \in Z_\xi$,

$$U_\xi(z(\cdot))_s \subset U_s(z(\cdot)_{0,s} \tag{4.1}$$

The Lemma follows immidiately from the definitions of the above sets.

Lemma 4.2 *Every* $l^*(\cdot) \in L_s$ *is reconstructable on* $[0, \xi]$.

Proof. For any $z(\cdot) \in Z_\xi$, the imbedding (4.1) implies

$$R_\xi(l^*(\cdot), z(\cdot))_{0,s} \subset R_s(l^*(\cdot), z(\cdot)_{0,s})$$

Now it is sufficient to note that the last set is one-element since $l^*(\cdot)$ is reconstructable on $[0, s]$ (Theorem 3.2).

□

The next lemma describes a set of functionals non-reconstructable on $[0, \xi]$.

Let $L_{\xi|s}$ denote the set of the restrictions to $[0, s]$ of all functionals from L_ξ (i.e. all functionals reconstructable on $[0, \xi]$; see Theorem 3.2 where s is replaced by ξ).

Lemma 4.3 *Every* $l^*(\cdot) \in \mathbf{L}^2_{0,s} \setminus L_{\xi|s}$ *is non-reconstructable on* $[0, \xi]$.

Proof. Take an above $l^*(\cdot)$. We have $l^*(\cdot) = l_1(\cdot) + l_2(\cdot)$ where $l_1(\cdot) \in L_{\xi|s}$, $l_2(\cdot) \in (L_{\xi|s})^\perp$, and $l_2(\cdot) \neq 0$. Define $v(\cdot) \in \mathbf{L}^2_{0,\xi}$ by $v(\cdot)_{0,s} = l_2(\cdot)$, $(\cdot)_{s,\xi} = 0$. For every $\lambda(\cdot) \in L_\xi$ it holds $\lambda(\cdot) \in L_{\xi|s}$ yielding

$$(v(\cdot), \lambda(\cdot)) = (v(\cdot)_{0,s}, \lambda(\cdot)_{0,s}) = 0$$

Consequently

$$v(\cdot) \in (L_\xi)^\perp \tag{4.2}$$

Take an arbitrary $z(\cdot) \in Z_\xi$ and an $u(\cdot) \in U_\xi(z(\cdot))$. By Theorem 3.1 (where s is replaced by ξ) we have

$$(\phi_i(\cdot, \sigma)_{0,\xi}, u(\cdot)) = g_i(z(\sigma), \sigma)$$

for all $\sigma \in [0, \xi]$ and $i \in [1 : m]$. Due to (4.2) this is true for $u(\cdot)$ replaced by $u_\alpha(\cdot) = u(\cdot) + \alpha v(\cdot)$. Hence by Theorem 3.1 (with s replaced by ξ) we have $u_\alpha(\cdot) \in U_\xi(z(\cdot))$ for every real α. Therefore by (4.1) and the definition of $v(\cdot)$ it holds

$$u_\alpha(\cdot)_{0,s} = u(\cdot)_{0,s} + \alpha l_2(\cdot) \in U_s(z(\cdot)_{0,s})$$

But

$$(l(\cdot), u_\alpha(\cdot)) = (l^*(\cdot), u(\cdot)_{0,s}) + \alpha \parallel l_2(\cdot) \parallel^2$$

covers the real line whenever α runs it through. Thus $l^*(\cdot)$ is non-reconstructable at $z(\cdot)$. Due to arbitrarity of $z(\cdot)$ it is non-reconstructable on $[0, \xi]$.

□

4.2 Mutant Functionals

In view of Theorem 3.2, Lemmas 4.2 and 4.3 can be summarized as follows. First, a functional $l(\cdot) \in \mathbf{L}^2_{0,s}$ is reconstructable on the longer time interval $[0,\xi]$ provided it is reconstructable on the shorter interval $[0,s]$ Second, $l(\cdot)$ is non-reconstructable on $[0,\xi]$ provided all its continuations to $[0,\xi]$ are non-reconstructable on $[0,\xi]$. Functionals not covered by these two classes are those that, first, are non-reconstructable at $[0,s]$ and, second, admit continuations to $[0,\xi]$ reconstructable on $[0,\xi]$. We will call them *mutant on* $[s,\xi]$. Formally, the set of all functionals mutant on $[s,\xi]$ is

$$M_{s,\xi} = L_{\xi|s} \setminus L_s \tag{4.3}$$

Note that by Lemmas 4.2 and 4.3

$$\mathbf{L}^2_{0,s} = L_s \cup (\mathbf{L}^2_{0,s} \setminus L_{\xi|s}) \cup M_{s,\xi}$$

Our goal now is to find among functionals mutant on $[s,\xi]$ those reconstructable and, respectively, non-reconstructable at a $z(\cdot) \in Z_\xi$.

It is convenient for us to consider, instead of $M_{s,\xi}$, the broader set $L_{\xi|s}$.

The next lemma shows that for an $l(\cdot) \in L_{\xi|s}$, its projection to $L_s^\perp$ is an indicator of $l(\cdot)$'s reconstructibility (or non-reconstructibility) at a $z(\cdot) \in Z_\xi$.

Let $\Pi_{\xi|s} : L_{\xi|s} \mapsto L_s^\perp$ be the projection operator: for any $l^*(\cdot) \in M_{\xi|s}$, the element

$$l(\cdot) = \Pi_{\xi|s} l^*(\cdot) \tag{4.4}$$

from $L_s^\perp$ is determined by

$$l^*(\cdot) = l(\cdot) + l_*(\cdot), \quad l_*(\cdot) \in L_s \tag{4.5}$$

Note that

$$M_{s,\xi} = L_{\xi|s} \setminus \mathrm{Ker}\Pi_{\xi|s} \tag{4.6}$$

Set

$$L^\Pi_{\xi|s} = \Pi_{\xi|s} L_{\xi|s} \tag{4.7}$$

Lemma 4.4 *Let $l(\cdot) \in L^\Pi_{\xi|s}$ and $z(\cdot) \in Z_\xi$. The following assertions hold:*

(i) if $l(\cdot)$ is reconstructable at $z(\cdot)$, then every $l^*(\cdot) \in L_{\xi|s}$ satisfying (4.4) is reconstructable at $z(\cdot)$,

(ii) if $l(\cdot)$ is non-reconstructable at $z(\cdot)$, then every $l^*(\cdot) \in L_{\xi|s}$ satisfying (4.4) is non-reconstructable at $z(\cdot)$.

Proof. Prove *(i)*. Let $l(\cdot)$ be reconstructable at $z(\cdot)$ and $l^*(\cdot)$ satisfy (4.4). We have (4.5). By Lemma 4.2 $l_*(\cdot)$ is reconstructable at $z(\cdot)$. Reconstructibility at $z(\cdot)$ of $l(\cdot)$ and $l_*(\cdot)$ and (4.5) imply obviously reconstructibility of $l^*(\cdot)$ at $z(\cdot)$.

Prove *(ii)*. Let $l(\cdot)$ be non-reconstructable at $z(\cdot)$, and $l^*(\cdot)$ satisfy (4.4). Again we have (4.5) with $l*(\cdot)$ reconstructable at $z(\cdot)$. Take an arbitrary real a. Since $l(\cdot)$ is non-reconstructable at $z(\cdot)$, there is an $u(\cdot) \in U_\xi(z(\cdot))_s$ such that $(l(\cdot), u(\cdot)) = a$. Hence $(l^*(\cdot), u(\cdot)) = a + \rho^*$ where ρ^* is the single element of the set $R_\xi(l_*(\cdot), z(\cdot))_s$. Due to arbitrarity of a the set $R_\xi(l^*(\cdot), z(\cdot))_s$ covers the whole real line.

□

4.3 Degenerate Continuability as a Reconstructibility Criterion

Define the operator $D_{\xi|s}: L_\xi \mapsto L^\Pi_{\xi|s}$ by

$$D_{\xi|s}\lambda(\cdot) = \Pi_{\xi|s}\lambda(\cdot)_{0,s} \tag{4.8}$$

Call an $l(\cdot) \in L^\Pi_{\xi|s}$ *degenerately continuable to* $[s,\xi]$ if there exists a $\lambda(\cdot) \in L_\xi$ such that

$$D_{\xi|s}\lambda(\cdot) = l(\cdot) \tag{4.9}$$

and

$$\lambda(\cdot)_{s,\xi} = 0 \tag{4.10}$$

In the opposite case call $l(\cdot)$ *non-degenerately continuable to* $[s,\xi]$. Our main technical result in this Section is

Theorem 4.1 *The following assertions hold:*

(i) every $l(\cdot) \in L^\Pi_{\xi|s}$ *degenerately continuable to* $[s,\xi]$ *is reconstructable on* $[0,\xi]$,

(ii) every $l(\cdot) \in L^\Pi_{\xi|s}$ *non-degenerately continuable to* $[s,\xi]$ *is non-reconstructable on* $[0,\xi]$.

Remark. Assertion *(i)* implies the conjecture of Lemma 4.1 for $l^*(\cdot) \in L_{\xi|s} \cap L_s$. Indeed for an above $l^*(\cdot)$, its projection $\Pi_{\xi|s}l^*(\cdot)$ to $L_s^\perp$ is zero and therefore degenerately continuable to $[s,\xi]$ ((4.9) holds for $\lambda(\cdot) = 0$). By Lemma 4.4 $l^*(\cdot)$ is reconstructable on $[0,\xi]$.

Proof of Theorem 4.1. Prove *(i)*. Let $l(\cdot)$ be degenerately continuable to $[s,\xi]$. Take a $\lambda(\cdot) \in L_\xi$ satisfying (4.9) and (4.10). By Theorem 3.2 (where s is replaced by ξ) $\lambda(\cdot)$ is reconstructable on $[0,\xi]$. Hence for an arbitrary $z(\cdot) \in Z_\xi$ and all

$$u(\cdot) \in U_\xi(z(\cdot)) \tag{4.11}$$

we have

$$(\lambda(\cdot), u(\cdot)) = \rho \tag{4.12}$$

where ρ does not depend on $u(\cdot)$. By (4.9) and (4.8) $l(\cdot)$ is the projection of $\lambda(\cdot)_{0,s}$ to $L_s^\perp$, i.e.

$$\lambda(\cdot)_{0,s} = l(\cdot) + l_*(\cdot)$$

where $l_*(\cdot) \in L_s$. By Theorem 3.2 $l_*(\cdot)$ is reconstructable on $[0,s]$. Consequently in view of Lemma 4.1 we have

$$(l_*(\cdot), u(\cdot)_{0,s}) = \zeta \tag{4.13}$$

for all inputs (4.11), with ζ not depending on $u(\cdot)$. Now for every input (4.11) we get

$$(l(\cdot), u(\cdot)_{0,s} = (\lambda(\cdot)_{0,s}, u(\cdot)_{0,s}) - (l_*(\cdot), u(\cdot)) = (\lambda(\cdot), u(\cdot)) - \zeta = \rho - \zeta$$

here (4.10), (4.13) and (4.12) have been exploited. Since $\rho - \zeta$ does not depend on an input (4.11), $l(\cdot)$ is reconstructable at $z(\cdot)$. Due to arbitrarity of $z(\cdot)$ it is reconstructable on $[0,\xi]$.

The rest of the Subsection is devoted to proving *(ii)*. Let $l(\cdot)$ be non-degenerately continuable to $[s,\xi]$. Then $i(\cdot) \neq 0$ (otherwise we have (4.9) with $\lambda(\cdot) = 0$, and $l(\cdot)$ is degenerately continuable to $[s,\xi]$. The pattern of our proof is as follows. First we point out an $l_*(\cdot) \in \mathbf{L}^2_{0,s}$ such that its non-reconstructibility on $[0,\xi]$ implies that of $l(\cdot)$. Then we prove that $l_*(\cdot)$ is indeed non-reconstructable on $[0,\xi]$. To come to $l_*(\cdot)$ we use an auxilliary element $l_0(\cdot)$. Several lemmas are built into the proof.

Let L^0_ξ be the space of all $\lambda^0(\cdot) \in L_\xi$ such that $\lambda^0(\cdot)_{s,\xi} = 0$.

Lemma 4.5 *Every functional from $D_{\xi|s} L^0_\xi$ is reconstructable on $[0,\xi]$.*

This is a reformulation of assertion *(i)*.

Lemma 4.6 *If $l_1^*(\cdot)$ and $l_2^*(\cdot)$ from $\mathbf{L}^2_{0,s}$ are, respectively, reconstructable and non-reconstructable on $[0,\xi]$, then $l_1^*(\cdot) + l_2^*(\cdot)$ is non-reconstructable on $[0,\xi]$.*

The proof of Lemma 4.6 is similar to that of assertion *(ii)* of Lemma 4.4.

Consider the representation

$$\lambda(\cdot) = \lambda_0(\cdot) + \lambda^0(\cdot) \tag{4.14}$$

where

$$\lambda^0(\cdot) \in L^0_\xi$$
$$\lambda_0(\cdot) \in (L^0_\xi)^\perp_{L_\xi} \tag{4.15}$$

Let

$$l_0(\cdot) = D_{\xi|s} \lambda_0(\cdot) \tag{4.16}$$

Lemma 4.7 *It holds*

$$\lambda_0(\cdot) \neq 0 \tag{4.17}$$

Proof. Otherwise $l(\cdot) = D_{\xi|s}\lambda^0(\cdot)$ (see (4.9), (4.14)) meaning that $l(\cdot)$ is degenerately continuable to $[s,\xi]$. This contradicts the assumption.

□

Lemma 4.8 *Let $l_0(\cdot)$ be non-reconstructable on $[0,\xi]$. Then $l(\cdot)$ is non-reconstructable on $[0,\xi]$.*

Proof. We have $l(\cdot) = l_0(\cdot) + l^0(\cdot)$ where $l_0(\cdot) = D_{\xi|s}\lambda_0(\cdot)$. By Lemma 4.5 $l^0(\cdot)$ is reconstructable on $[0,\xi]$. The reference to Lemma 4.6 completes the proof.

□

Let $L^0_{\xi|s}$ be the space of the restrictions to $[0,s]$ of all functionals from L^0_ξ. Define the functional $l_*(\cdot)$ to be the projection of $l_0(\cdot)$ to $(L^0_{\xi|s})^\perp_{L_s^\perp}$ (see (4.7)); recall that $l_0(\cdot) \in L_s^\perp$ due to (4.16). Therefore we have

$$l_0(\cdot) = l_*(\cdot) + l^*(\cdot) \tag{4.18}$$

$$l_*(\cdot) \in (L^0_{\xi|s})^\perp_{L_s^\perp} = (L^0_{\xi|s} \cap L_s^\perp)^\perp_{L_s^\perp} \tag{4.19}$$

$$l^*(\cdot) \in L^0_{\xi|s} \cap L_s^\perp \tag{4.20}$$

Lemma 4.9 *It holds $l_*(\cdot) \neq 0$.*

Proof. Suppose that this is not so. Then $l_0(\cdot) \in L^0_{\xi|s}$. Let $l^0(\cdot) \in L^0_\xi$ be such that

$$l^0(\cdot)_{0,s} = l_0(\cdot) \tag{4.21}$$

By the definition of L^0_ξ we have

$$l^0(\cdot)_{s,\xi} = 0 \tag{4.22}$$

Now we come to a contradiction as follows:

$$0 = (\lambda_0(\cdot), l^0(\cdot)) = (\lambda_0(\cdot)_{0,s}, l^0(\cdot)_{0,s}) = (\lambda_0(\cdot), l_0(\cdot)) =$$
$$(\Pi_{\xi|s}\lambda_0(\cdot)_{0,s}, l_0(\cdot)) = (D_{\xi|s}\lambda_0(\cdot), l_0(\cdot)) = (l_0(\cdot), l_0(\cdot)) > 0$$

$(0 > 0)$; here we have used one by one: orthogonality of $\lambda_0(\cdot)$ to $l^0(\cdot)$ (see (4.15)), (4.22), (4.21), the fact that $l_0(\cdot) \in L_s^\perp$, (4.8), (4.16), (4.17).

□

Lemma 4.10 *Let $l_*(\cdot)$ be non-reconstructable on $[0,\xi]$. Then $l(\cdot)$ is non-reconstructable on $[0,\xi]$.*

Proof. Due to Lemma 4.8 it is sufficient to show that $l_0(\cdot)$ is non-reconstructable on $[0,\xi]$. To showing this, it is sufficient to prove that $l^*(\cdot)$ is reconstructable on $[0,\xi]$ (see (4.18) and Lemma 4.6). As it is seen from (4.21), for $l^{*0}(\cdot) \in L_\xi^0$ such that $l^{*0}(\cdot)_{0,s} = l^*(\cdot)$, it holds

$$D_{\xi|s} l^{*0}(\cdot) = \Pi_{\xi|s} l^*(\cdot) = l^*(\cdot)$$

By Lemma 4.5 $l^*(\cdot)$ is reconstructable on $[0,\xi]$.

□

The rest of our proof is devoted to showing that $l_*(\cdot)$ is non-reconstructable on $[0,\xi]$.

Lemma 4.11 *Let $v(\cdot) \in \mathbf{L}^2_{0,\xi}$ be such that*

$$v(\cdot)_{0,s} = l_*(\cdot) \tag{4.23}$$

Then

$$v(\cdot) \in (L_\xi^0)^\perp \tag{4.24}$$

Proof. For every $l^0(\cdot) \in L_\xi^0$, we have

$$(l^0(\cdot), v(\cdot)) = (l^0(\cdot)_{0,\xi}, v(\cdot)_{0,s}) = (l^0(\cdot), l_*(\cdot)) = 0$$

The last equality follows from (4.20), and the obvious inclusion $l^0(\cdot)_{0,\xi} \in L^0_{\xi|s}$.

□

Let $\Pi^0_{\xi|s} : L_s^\perp \mapsto (L^0_{\xi|s} \cap L_s^\perp)^\perp_{L_s^\perp}$ be the projection operator, and $D^0_{\xi|s} : L_\xi \mapsto (L^0_{\xi|s} \cap L_s^\perp)^\perp_{L_s^\perp}$ be defined by

$$D^0_{\xi|s}\lambda^*(\cdot) = \Pi^0_{\xi|s} D_{\xi|s}\lambda^*(\cdot) = \Pi^0_{\xi|s}\Pi_{\xi|s}\lambda^*(\cdot)_{0,s} \tag{4.25}$$

(see (4.8)). As it is seen from (4.18), (4.16),

$$l_*(\cdot) = D^0_{\xi|s}\lambda_0(\cdot) \tag{4.26}$$

Now we base on the following

Lemma 4.12 *There exists a basis*

$$(\lambda_1(\cdot), \lambda_2(\cdot), ...) \tag{4.27}$$

in $(L_\xi^0)^\perp_{L_\xi}$ such that

$$\lambda_1(\cdot) = \lambda_0(\cdot) \tag{4.28}$$

$$\lambda_1(\cdot)_{s,\xi} \notin \mathrm{Lin}\{\lambda_2(\cdot)_{s,\xi}, \lambda_3(\cdot)_{s,\xi}, ...\} \tag{4.29}$$

and for

$$l_i(\cdot) = D^0_{\xi|s}\lambda_i(\cdot) \tag{4.30}$$

it holds

$$(l_1(\cdot), l_i(\cdot)) = 0 \quad (i \geq 2) \tag{4.31}$$

The proof of Lemma 4.12 is given at the end of the present Subsection.

Note that in view of (4.28), (4.30), (4.26), we have

$$l_1(\cdot) = l_*(\cdot) \tag{4.32}$$

Define the input $v(\cdot)$ on $[0,\xi]$ by

$$v(\cdot)_{0,s} = l_1(\cdot) \tag{4.33}$$

$$v(\cdot)_{s,\xi} \in \mathrm{Lin}\{\lambda_2(\cdot)_{s,\xi}, \lambda_3(\cdot)_{s,\xi}, ...\}^\perp \tag{4.34}$$

$$(v(\cdot)_{s,\xi}, \lambda_1(\cdot)_{s,\xi}) = - \parallel l_1(\cdot) \parallel^2 < 0 \tag{4.35}$$

(recall that the functional (4.22) is nonzero by Lemma 4.9 and (4.32)); (4.23) and (4.24) can be ensured due to (4.29). Note that (4.30) and (4.33) imply

$$v(\cdot) \in (L^0_{\xi|s} \cap L_s^\perp)^\perp_{L_s^\perp} \tag{4.36}$$

and by (4.32) and Lemma 4.11 we have (4.24). For $i \geq 2$, taking sequentially into account (4.34), (4.36), (4.25), (4.30), (4.33), (4.31), we get

$$(\lambda_i(\cdot), v(\cdot)) = (\lambda_i(\cdot)0, s, v(\cdot)_{0,s}) + (\lambda_i(\cdot)_{s,\xi}, v(\cdot)_{s,\xi}) =$$

$$(\lambda_i(\cdot)_{0,s}, v(\cdot)_{0,s}) = (\Pi^0_{\xi|s}\Pi_{\xi|s}\lambda_i(\cdot)_{0,s}, v(\cdot)_{0,s}) = (D^0_{\xi|s}\lambda_i(\cdot), v(\cdot)_{0,s}) =$$

$$(l_i(\cdot), v(\cdot)_{0,s}) = (l_i(\cdot), l_1(\cdot)) = 0 \tag{4.37}$$

For $i = 1$

$$\lambda_1(\cdot), v(\cdot)) = (\lambda_1(\cdot)_{0,s}, v(\cdot)_{0,s}) + (\lambda_1(\cdot)_{s,\xi}, v(\cdot)_{s,\xi})$$

Transforming the first item as in (4.37), we have

$$(\lambda_1(\cdot), v(\cdot)) = (l_1(\cdot), l_1(\cdot)) + (\lambda_1(\cdot)_{s,\xi}, v(\cdot)_{s,\xi}) = 0 \tag{4.38}$$

where the last equality is ensured by (4.36). The equalities (4.37) and (4.38) show that $v(\cdot)$ is orthogonal to all elements of the basis (4.27) of the subspace $(L^0_\xi)^\perp_{L_\xi}$ of L_ξ. By (4.24) $v(\cdot)$ is orthogonal to all elements of L^0_ξ. Therefore

$$v(\cdot) \in L_\xi^\perp \tag{4.39}$$

Take now an arbitrary $z(\cdot) \in Z_\xi$ and fix an input (4.11). Let

$$u_\alpha = u(\cdot) + \alpha v(\cdot)$$

By (4.39)

$$(u_\alpha(\cdot), \psi(\cdot)) = (u(\cdot), \psi(\cdot))$$

for every $\psi(\cdot) \in L_\xi$ and in particular $\psi(\cdot) \in K_\xi$ see (3.12) and (3.11) where s is replaced by ξ). Referring to Theorem 3.1 (where s is replaced by ξ), we conclude that

$$u_\alpha(\cdot) \in U_\xi(z(\cdot))$$

Then by Lemma 4.1

$$u_\alpha(\cdot)_{0,s} = u(\cdot)_{0,s} + \alpha v(\cdot)_{0,s} \in U_\xi(z(\cdot))_s \tag{4.40}$$

Taking into account (4.32) and (4.33), we obtain

$$(l_*(\cdot), u_\alpha(\cdot)_{0,s}) = (l_*(\cdot), u(\cdot)_{0,s}) + \alpha(l_*(\cdot), v(\cdot)_{0,s}) = (l_*(\cdot), u(\cdot)_{0,s}) + \alpha \parallel l_1(\cdot) \parallel^2$$

These values cover the whole real line whenever α runs it through. This together with (4.40) prove that $l(\cdot)$ is non-reconstructable at $z(\cdot)$. Due to arbitrarity of $z(\cdot)$ it is non-reconstructable on $[0,\xi]$.

□

Proof of Lemma 4.12. Choose an arbitrary basis $(\mu_1^0(\cdot), \mu_2^0(\cdot), ...)$ in $(L_\xi^0)_{L_\xi}^\perp$ such that $\mu_1^0(\cdot) = \lambda_0(\cdot)$. Using the standard orthogonalization procedure, pass to a basis

$$(\mu_1(\cdot), \mu_2(\cdot), ...) \tag{4.41}$$

in $(L_\xi^0)_{L_\xi}^\perp$ such that

$$\mu_1(\cdot) = \lambda_0(\cdot) \tag{4.42}$$

$$\mu_i(\cdot) \neq 0 \tag{4.43}$$

$$(\mu_i(\cdot)_{s,\xi}, \mu_j(\cdot)_{s,\xi}) = 0 \quad (i \neq j) \tag{4.44}$$

Namely, put (4.31), and in case $\mu_1(\cdot), ..., \mu_k(\cdot)$ satisfying (4.32) and the condition

$$\text{Lin}\{\mu_1(\cdot), ..., \mu_k(\cdot)\} = \text{Lin}\{\mu_1^0(\cdot), ..., \mu_k^0(\cdot)\}$$

are built, define

$$\mu_{k+1}(\cdot) = \mu_{k+1}^0(\cdot) + \sum_{i=1}^{k} \alpha_i \mu_i(\cdot)$$

so as

$$(\mu_{k+1}(\cdot)_{s,\xi}, \mu_i(\cdot)_{s,\xi}) = 0$$

for $i \leq k$. The last inequality is, due to (4.32), equivalent to

$$(\mu_{k+1}^0(\cdot)_{s,\xi}, \mu_i(\cdot)_{s,\xi}) + \alpha_i \mid \mu_i(\cdot)_{s,\xi} \mid = 0 \tag{4.45}$$

We put $\alpha_i = 0$ if $\mu_i(\cdot)_{s,\xi} = 0$ and calculate α_i from (4.33) in the opposite case. Finally, throw away all zero elements of the obtained collection to ensure (4.43). Note that (4.43) implies

$$\mu_i(\cdot)_{s,\xi} \neq 0 \tag{4.46}$$

Indeed, if it is not so, then (since $\mu_i(\cdot) \in L_\xi$) we have $\mu_i(\cdot) \in L_\xi^0$; thus $\mu_i(\cdot) \in L_\xi^0 \cap (L_\xi^0)_{L_\xi}^\perp$ yielding $\mu_i(\cdot) = 0$ which contradicts (4.43).

Now we pass from the basis (4.41) to the desired basis (4.27). Set (4.28) and, for $i \geq 2$,

$$\lambda_i(\cdot) = \mu_i(\cdot) + a_i \mu_1(\cdot) \tag{4.47}$$

where a_i is such that,under the notation (4.30), the equality (4.31) is satisfied; assuming

$$b_i(\cdot) = D_{\xi|s}^0 \mu_i(\cdot)$$

write (4.31) in the form

$$0 = (b_1(\cdot), b_i(\cdot) + a_i b_1(\cdot)) = (b_1(\cdot), b_i(\cdot)) = a_i \parallel b_1(\cdot) \parallel^2 \tag{4.48}$$

By (4.42) and (4.26) $b_1(\cdot) = l_*(\cdot)$. This element is nonzero by the supposition of assertion (ii) of Theorem 4.1. Hence a_i satisfying (4.48) exists. Consequently, for the basis (4.27) the conditions (4.28) and (4.31) are fulfilled (note that by (4.28) and (4.42) $\lambda_1(\cdot) = \mu_1(\cdot)$ and by (4.47) $\mu_i(\cdot) = \lambda_i(\cdot) - a_i \lambda_1(\cdot)$ confirming that (4.27) is indeed a basis in $(L_\xi^0)_{L_\xi}^\perp$). To complete the proof, we must verify (4.29). Suppose that (4.29) violates. Then for certain real α_i,

$$\epsilon = \parallel \lambda_1(\cdot)_{s,\xi} - \sum_{i=2}^{\infty} \alpha_i \lambda_i(\cdot)_{s,\xi} \parallel^2 = 0 \tag{4.49}$$

Using the equality $\lambda_1(\cdot) = \mu_1(\cdot)$ and (4.47) represent

$$\epsilon =\| \mu_1(\cdot)_{s,\xi} - \beta\cdot)_{s,\xi} \|^2=$$

$$\| \mu_1(\cdot)_{s,\xi} \|^2 -2(\mu_1(\cdot),\beta(\cdot)_{s,\xi})+ \| \beta(\cdot)_{s,\xi} \|^2 \tag{4.50}$$

where

$$\beta(\cdot)_{s,\xi} = \sum_{i=2}^{\infty} \alpha_i(\mu_i(\cdot)_{s,\xi} + a_i\mu_i(\cdot)_{s,\xi}) \tag{4.51}$$

We have

$$(\mu_1(\cdot)_{s,\xi}, \beta(\cdot)_{s,\xi}) = \sum_{i=2}^{\infty}(\mu_1(\cdot)_{s,\xi}, \alpha_i(\mu_i(\cdot)_{s,\xi} + a_i\mu_1(\cdot)_{s,\xi})) =$$

$$\sum_{i=1}^{\infty} \alpha_i a_i \| \mu_1(\cdot)_{s,\xi} \|^2 \tag{4.52}$$

Here we have used (4.44). So far as (4.41) is a basis in $(L_\xi^0)_{L_\xi}^{\perp}$, we conclude that, first, the element $\beta(\cdot)_{s,\xi}$ lies in the space $L_{s,\xi}^0$ of the restrictions to $[s,\xi]$ of all functions from $(L_\xi^0)_{L_\xi}^{\perp}$, and, second, $(\mu_1(\cdot)_{s,\xi}, \mu_2(\cdot)_{s,\xi}, \ldots)$ is a basis in $L_{s,\xi}$; furtheremore, (4.44) means that this basis is orthogonal. Consequently (see (4.46))

$$\| \beta(\cdot)_{s,\xi} \|^2= \sum_{j=1}^{\infty} \frac{(\mu_j(\cdot)_{s,\xi}, \beta(\cdot)_{s,\xi})^2}{\| \mu_j(\cdot)_{s,\xi} \|^2} =$$

$$\sum_{j=1}^{\infty} \frac{1}{\| \mu_j(\cdot)_{s,\xi} \|^2}(\sum_{j=2}^{\infty} \alpha_i(\mu_j(\cdot)_{s,\xi}, \mu_i(\cdot)_{s,\xi} + a_i\mu_1(\cdot)_{s,\xi}))^2 =$$

$$(\sum_{i=2}^{\infty} \alpha_i a_i)^2 \| \mu_1(\cdot) \|^2 + \sum_{j=2}^{\infty} \alpha_j^2 \| \mu_j(\cdot)_{s,\xi} \|^2 \tag{4.53}$$

Combining (4.50) - (4.53) we obtain

$$\epsilon =\| \mu_1(\cdot)_{s,\xi} \|^2 (1-2y+y^2) + \delta =$$

$$\| \mu_1(\cdot)_{s,\xi} \|^2 (1-y)^2 + \delta$$

where

$$y = \sum_{i=2}^{\infty} \alpha_i a_i, \quad \delta = \sum_{j=2}^{\infty} \alpha_j \| \mu_j(\cdot)_{s,\xi} \|^2$$

Now (4.49) yields $\delta = 0$. Hence, due to (4.43), $\alpha_j = 0$ for $j \geq 2$. Consequently (see (4.51) and (4.52)) we have

$$\mu_1(\cdot)_{s,\xi} = \beta(\cdot)_{s,\xi} = 0$$

which contradicts (4.43).

□

4.4 The Reconstructibility Alternative

Combining Theorem 4.1 and Lemma 4.4, we come to the following alternative assertions for functionals mutant on $[s,\xi]$ (see the notation (4.3)).

Corollary 4.1 *Let $l^*(\cdot) \in M_{s,\xi}$.*

1) *If the projection $l(\cdot)$ of $l^*(\cdot)$ to $L_s^\perp$ ($l(\cdot) = \Pi_{\xi|s} l^*(\cdot)$) is degenerately continuable to $[s,\xi]$, then $l^*(\cdot)$ is reconstructable on $[0,\xi]$;*

2) *if $l(\cdot)$ is non-degenerately continuable to $[s,\xi]$, then $l^*(\cdot)$ is non-reconstructable on $[0,\xi]$.*

Denote by $M_{s,\xi}^0$ the set of all $l^*(\cdot) \in M_{s,\xi}$ such that $l(\cdot) = \Pi_{\xi|s} l^*(\cdot)$ is degenerately continuable to $[0,\xi]$, and put

$$L_{s,\xi} = L_s \cup M_{s,\xi}^* \tag{4.54}$$

Our main result is

Theorem 4.2 *Every $l^*(\cdot) \in L_{s,\xi}$ is reconstructable on $[0,\xi]$, and every $l^*(\cdot) \in \mathbf{L}^2_{0,s} \setminus L_{s,\xi}$ is non-reconstructable on $[0,\xi]$.*

Proof. Every $l^*(\cdot) \in L_{s,\xi}$ is reconstructable on $[0,\xi]$ by Lemma 4.3 and Corollary 4.1, 1). Let $l^*(\cdot) \in \mathbf{L}^2_{0,s} \setminus L_{s,\xi}$. Then

$$l(\cdot) \notin L_s \tag{4.55}$$

$$l^*(\cdot) \notin M_{s,\xi}^0 \tag{4.56}$$

Recall that (see Subsection 4.2)

$$\mathbf{L}^2_{0,s} = L_s \cup (\mathbf{L}^2_{0,s} \setminus L_{\xi|s}) \cup M_{s,\xi}$$

Thus, from (4.55) we have either

$$l^*(\cdot) \in \mathbf{L}^2_{0,s} \setminus L_{\xi|s} \tag{4.57}$$

or

$$l^*(\cdot) \in M_{s,\xi} \tag{4.58}$$

In the case (4.57) $l^*(\cdot)$ is non-reconstructable on $[0,\xi]$ by Lemma 4.3. In the case (4.58) this is so by Corollary 4.1, 2), and (4.56).

□

Corollary 4.2 *The set $L_{s,\xi}$ is a linear subspace in $\mathbf{L}^2_{0,s}$.*

Proof. Copying (with minor changes) the proof of the first part of Theorem 3.2, one can show that every functional from $\mathrm{Lin} L_{s,\xi}$ is reconstructable on $[0,\xi]$. On the other hand, any functional from the difference $(\mathrm{Lin} L_{s,\xi}) \setminus L_{s,\xi}$ is non-reconstructable on $[0,\xi]$ by Theorem 4.2. Hence the above difference is empty.

□

4.5 Values of Reconstructable Functionals and Input Reconstructibility Conditions

The values of some functionals from $\mathbf{L}^0_{s,\xi}$ reconstructable on $[0,\xi]$ can be calculated like in Subsection 3.3 (we do not go into details).

The next corollary following from Theorem 4.2 is analogous to Corollary 3.3 (see also Subsection 2.5).

Corollary 4.3 *The following assertions are equivalent:*
(i) the input on $[0,s]$ is reconstructable on $[0,s]$
(ii) the input on $[0,s]$ is reconstructable at a certain $z(\cdot) \in Z_\xi$,
(iii) $L_{s,\xi} = \mathbf{L}^2_{0,s}$.

The next theorem is analogous to Theorem 3.3 (see the notations (3.11) and (3.12) where s is replaced by ξ).

Theorem 4.3 *Let there exist $\lambda_1(\cdot),...\lambda_r(\cdot) \in L_\xi$ such that $(\lambda_1(s),...\lambda_r(s))$ is a basis in $\mathbf{R}^r$. Then the input on $[0,s]$ is reconstructable on $[0,\xi]$.*

The proof copies, with obvious modifications, that of Theorem 3.3.

4.6 Remarks on Reconstructable Mutant Functionals

Let us observe several properties of the set $M^0_{s,\xi}$ of functionals mutant on $[s,\xi]$, (i.e. non-reconstructable on $[0,s]$ and reconstructable on $[0,\xi]$; see (4.54), Theorem 4.2 and Theorem 3.2).

Theorem 4.4 *It holds*

$$M^0_{s,\xi} \cap L_s = \emptyset \tag{4.59}$$

$$M^0_{s,\xi} = M^0_{s,\xi} + L_s \tag{4.60}$$

$$\mathrm{Lin} M^0_{s,\xi} \subset L_s \cup M^0_{s,\xi} \tag{4.61}$$

Proof. The equality (4.59) follows from $M_{s,\xi} \cap L_s = \emptyset$ (see (4.3)). Prove (4.60). Since $0 \in L_s$, we have

$$M^0_{s,\xi} \subset M^0_{s,\xi} + L_s$$

Suppose that the reverse imbedding is wrong, i.e. there exists an

$$l^*(\cdot) = l_1(\cdot) + l_2(\cdot)$$

where

$$l_1(\cdot) \in M^0_{s,\xi}, \quad l_2(\cdot) \in L_s$$

such that

$$l^*(\cdot) \notin M^0_{s,\xi} \tag{4.62}$$

Since $l_1(\cdot)$ and $l_2(\cdot)$ are reconstructable on $[0,\xi]$ by Theorem 4.2, $L^*(\cdot)$ has this property too. Then by Theorem 4.2 and (4.62) (see also (4.54)) $l*(\cdot) \in L_s$. So far as L_s is a linear subspace, we have

$$l_1(\cdot) = l^*(\cdot) - l_2(\cdot) \in L_s$$

which contradicts (4.59). Finally, (4.61) follows from Corollary 4.2.

□

Recall that by (4.3)

$$M^0_{s,\xi} \subset L_{\xi|s} \setminus L_s \tag{4.63}$$

where $L_{\xi|s}$ is the set of the restrictions to $[0,s]$ of all functionals from L_ξ (i.e. reconstructable on $[0,\xi]$; see Subsection 4.1). Let K^L_ξ be the set of all finite linear combinations of functionals from K_ξ (see (3.11) where s is replaced by ξ), and $K^L_{\xi|s}$ be the set of all restrictions to $[0,s]$ of all functionals from K^L_ξ.

By definition L_ξ is the closure of K^L_ξ in $\mathbf{L}^2_{0,\xi}$. Hence $L_{\xi|s}$ is the closure of $K^L_{\xi|s}$ in $\mathbf{L}^2_{0,s}$.

Theorem 4.5 *Let the system be stationary. Then*

$$M^0_{s,\xi} \cap K^L_{\xi|s} = \emptyset$$

Proof. Suppose the statement is untrue, i.e. there is an

$$l^*(\cdot) \in M^0_{s,\xi} \tag{4.64}$$

such that

$$l^*(\cdot) \in K^L_{\xi|s} \tag{4.65}$$

Let

$$\lambda(\cdot) \in K^L_\xi \tag{4.66}$$

be such that

$$l^*(\cdot) = \lambda(\cdot)_{0,s} \tag{4.67}$$

Then

$$\lambda(\cdot)_{s,\xi} = 0 \tag{4.68}$$

Indeed, if this is not so, then

$$l(\cdot) = D_{\xi|s}\lambda(\cdot) = \Pi_{\xi|s} l^*(\cdot)$$

is non-degenerately continuable to $[s,\xi]$, which contradicts (4.64). Note that (4.64) implies $l^*(\cdot) \neq 0$ (see (4.59) and take into account that $0 \in L_s$), and therefore (see (4.67))

$$\lambda(\cdot) \neq 0 \tag{4.69}$$

Let us show that (4.68) and (4.69) can not be fulfilled simultaniously; this will complete the proof. By (3.11) (where s is replaced by ξ) and (4.66) we have

$$\lambda(\cdot) = \sum_{j=1}^{k} \phi_{i_j}(\cdot, \sigma_j)_{0,\xi} \tag{4.70}$$

for some $i_j \in [1:m]$ and $\sigma_j \in [0,\xi]$. Due to (4.67), (4.64) and (4.59) $\lambda(\cdot)_{0,s} \notin L_s$ yielding $\sigma_j > s$. The sum of all items in (4.70) with a same σ_j has the form $B^T w(\cdot, \sigma, q_j)$ where $w(\cdot, \sigma_j, q_j)$ is zero on $]\sigma_j, \infty[$ and coincides with the solution of the equation (3.1) satisfying $w(\sigma_j) = q_j$ on $]-\infty, \sigma_j]$, and $q_j \in \mathrm{Lin}\{p_1, ..., p_m\}$ (see Subsection 3.1). Thus, with no loss of generality, assume

$$\lambda(\cdot) = \sum_{j=1}^{k} B^T w_j(\cdot) \tag{4.71}$$

where

$$w_j(\cdot) = w(\cdot, \sigma, q_j), \quad s < \sigma_1 < ... < \sigma_k \leq \xi$$

Let us show that

$$B^T w_k(\cdot) = 0 \tag{4.72}$$

Obviously $\lambda(t) = w_k(t)$ for $t \in]\sigma_{k-1}, \sigma_k]$. Hence in view of (4.68)

$$B^T w_k(t) = 0 \tag{4.73}$$

for the above t. Using the representation

$$w_k(t) = (q_k, \exp(-A^T(t - \sigma_k)) \tag{4.74}$$

rewrite (4.73) with $t = \sigma_k$ as

$$B^T q_k = 0$$

Sequential differentiation of (4.73) at this t yields

$$(B^T q_k, (A^T)^i) = 0 \quad (i \geq 1)$$

The obtained equalities and (4.74) prove (4.72). Similarly, we verify that every function from the sum (4.71) is zero. Therefore $\lambda(\cdot) = 0$ contradicting (4.69).

□

Theorem 4.4 and (4.64) imply

Corollary 4.4 *Let the system be stationary. Then*

$$M^0_{s,\xi} \subset L_{\xi|s} \setminus K^L_{\xi|s}$$

In other words, if the system is stationary, then only the limit points of $K^L_{\xi|s}$ can be reconstructable on $[0, \xi]$ without being reconstructable on $[0, s]$.

5 Examples

5.1 Example

Consider the stationary two-dimensional system

$$\begin{aligned} \dot{x}^{(1)}(t) &= x^{(2)}(t) + u^{(1)}(t) \\ \dot{x}^{(2)}(t) &= u^{(2)}(t) \end{aligned}$$

with the initial condition

$$x^{(1)}(0) = 0, \quad x^{(2)}(0) = 0$$

The observed signal is

$$z(t) = x^{(1)}(t)$$

Thus the observation matrice P consists of the single line $(1, 0) = p_1^T$. The adjoint equation (3.1) is

$$\begin{aligned} \dot{w}^{(1)}(t) &= 0 \\ \dot{w}^{(2)}(t) &= -w^{(1)}(t) \end{aligned}$$

and the function $\phi_1(\cdot, \sigma)$ (3.3) has the form

$$\phi_1^{(1)}(t, \sigma) = 1, \quad \phi_1^{(2)}(t, \sigma) = -t + \sigma \quad (t \leq \sigma)$$

(and takes the zero value for $t > \sigma$).

Fix an $s \geq 0$. Describe the space L_s of all functionals reconstructable on $[0, s]$ (Theorem 3.2).

Find first all functionals $\psi(\cdot)$ orthogonal to K_s (see (3.11)). These $\psi(\cdot)$ form clearly the space $L_s^{\perp}$. Then we find L_s as the space orthogonal to $L_s^{\perp}$.

From the above form of functions $\phi_1(\cdot, \sigma)$, we deduce easily that a $\psi(\cdot)$ is orthogonal to all these functions if and only if

$$\int_0^{\sigma} [\psi^{(1)}(t) - (t - \sigma)\psi^{(2)}(t)]dt = 0$$

for all $\sigma \in [0, s]$. Differentiation in σ gives the equivalent condition

$$\psi^{(1)}(\sigma) + \int_0^{\sigma} \psi^{(1)}(t)dt = 0 \tag{5.1}$$

The condition of $l(\cdot)$'s orthogonality to such $\psi(\cdot)$

$$\int_0^{s} (l^{(1)}(t)\psi^{(1)}(t) + l^{(2)}(t)\psi^{(2)}(t))dt = 0$$

is equivalent to

$$\lambda^{(1)}(s)\psi^{(1)}(s) + \int_0^{s} (\lambda^{(1)}(t) + l^{(2)}(t))\psi^{(2)}(t)dt = 0 \tag{5.2}$$

where

$$\lambda^{(1)}(t) = \int_0^{t} l^{(1)}(\tau)d\tau \tag{5.3}$$

The requirement that this should be fulfilled for all above $\psi(\cdot)$ is equivalent to (5.3) with

$$\lambda^{(1)}(\cdot) - l^{(2)}(\cdot) = c, \quad \lambda^{(1)}(s) = c \tag{5.4}$$

where c is a constant. Indeed, if (5.3) – (5.5) are fulfilled, then for every $\psi(\cdot)$ satisfying (5.1) the left hand side of (5.2) equals

$$c\psi^{(1)}(s) + c\int_0^{s} \psi^{(2)}(t)dt = 0 \tag{5.5}$$

Conversly, suppose that conditions (5.4) do not hold simultaniously. Assume first that $\lambda^{(1)}(\cdot) - l^{(2)}(\cdot)$ be not constant. Then in the case $\lambda^{(1)}(s) \neq 0$, Taking $\psi^{(2)}(\cdot)$ orthogonal to 1 (with $\psi^{(2)}(\cdot)$ satisfying (5.1)), we get that the second term in (5.2) is zero and, due to (5.1), $\psi^{(1)}(s) \neq 0$; thus (5.2) violates since its left hand side equals $\lambda^{(1)}(s)\psi^{(1)}(s) \neq 0$. In the case $\lambda^{(1)}(s = 0$, we obtain a contradiction by taking $\psi^{(2)}(\cdot) = \lambda^{(1)}(\cdot) + \psi^{(2)}(\cdot)$.

Let now the first condition in (5.4) be true and the second one violates, i.e. $\lambda^{(1)}(s) = c_1 \neq c$. Then the left hand side of (5.2) equals that of (5.5) with c_1 replacing c in the first item; this value is nonzero for an arbitrary $\psi(\cdot)$ satisfying (5.1) with $\psi^{(1)}(s) \neq 0$. Using (5.3) rewrite (5.4):

$$l^{(2)}(t) = c - \int_0^{t} l^{(1)}(\tau)d\tau \quad (0 \leq t \leq s), \quad \int_0^{s} l^{(1)}(\tau)d\tau = c$$

or, equivalently,

$$l^{(2)}(t) = \int_t^{s} l^{(1)}(\tau)d\tau \quad (0 \leq t \leq s) \tag{5.6}$$

Thus we come to

Proposition 5.1 *A functional $l(\cdot) \in \mathbf{L}^2_{0,s}$ is reconstructable on $[0,s]$ (lies in L_s) if and only if the condition (5.6) is satisfied.*

Let us now fix a $\xi \geq s$ and find the sets $M_{s,\xi}$ of all functionals mutant on $[s,\xi]$ and $M^0_{s,\xi}$ of all such functionals reconstructable on $[0,\xi]$ (see Subsections 4.2 and 4.4).

Recall that $M_{s,\xi}$ is defined by (4.3). According to Proposition 5.1 (where s is replaced by ξ), the set L_ξ consists of all $\lambda(\cdot) \in \mathbf{L}^2_{0,\xi}$ such that

$$\lambda^{(2)}(t) = \int_t^\xi \lambda^{(1)}(\tau)d\tau \quad (0 \leq t \leq \xi) \tag{5.7}$$

Then $L_{\xi|s}$ consists of all $l^*(\cdot) \in \mathbf{L}^2_{0,s}$ such that

$$l^{*(2)}(t) = c + \int_t^s l^{*(2)}(\tau)d\tau \quad (0 \leq t \leq s) \tag{5.8}$$

where c is an arbitrary constant. Indeed, if $l^*(\cdot) \in L_{\xi|s}$, i.e. $l^*(\cdot) = \lambda(\cdot)_{0,s}$ for a certain $\lambda(\cdot)$ satisfting (5.7), then we have (5.8) where

$$c = \int_0^\xi \lambda^{(1)}(\tau)d\tau \tag{5.9}$$

Conversly, let $l^*(\cdot)$ satisfy (5.8). Take an $\lambda(\cdot) \in \mathbf{L}^2_{0,\xi}$ such that $\lambda^{(1)}(\cdot)_{0,s} = l^{*(1)}(\cdot)$, (5.9) holds, and $\lambda(2)(\cdot)$ is defined by (5.7). Then $\lambda(\cdot) \in L_\xi$ and obviously

$$\lambda^{(2)}(t) = c + \int_t^s l^{*(1)}(\tau)d\tau \quad (0 \leq t \leq s)$$

Therefore due to (5.8) $\lambda^{(2)}(\cdot)_{0,s} = l^{*(2)}(\cdot)$; hence $\lambda(\cdot)_{0,s} = l^*(\cdot)$. Comparing (5.8) (the relation describing the set $L_{\xi|s}$), Proposition 5.1 and the definition (4.3) of the set $M_{s,\xi}$, we get

Proposition 5.2 *A functional $l^*(\cdot) \in \mathbf{L}^2_{0,s}$ is mutant on $[s,\xi]$ (lies in $M_{s,\xi}$) if and only if the condition (5.8) is fulfilled.*

Let us pass to the set $M^0_{s,\xi}$. Suppose that $M^0_{s,\xi}$ is nonempty. Take an arbitrary $l^*_0(\cdot) \in M^0_{s,\xi}$. Let $l(\cdot)$ be its projection to $L^\perp_s$:

$$l(\cdot) = \Pi_{\xi|s} l^*_0(\cdot) \tag{5.10}$$

By the definition of $M^0_{s,\xi}$ (Subsection 4.4) $l(\cdot)$ is degenerately continuable to $[s,\xi]$, i.e. there exists a $\lambda(\cdot) \in L_\xi$ such that

$$\lambda(\cdot)_{s,\xi} = 0 \tag{5.11}$$

and

$$l(\cdot) = \Pi_{\xi|s} l^*(\cdot) \tag{5.12}$$

where

$$l^*(\cdot) = \lambda(\cdot)_{0,s} \tag{5.13}$$

From (5.10) and (5.12) follows $l^*(\cdot) - l^*_0(\cdot) \in L_s$. Then by (4.61)

$$l^*(\cdot) = l^*_0(\cdot) + (l^*(\cdot) - l^*_0(\cdot))$$

As it was shown above, (5.13) implies (5.8) where c is given by (5.9). In view of (5.11) $c = 0$. Hence by Proposition 5.1 $l^*(\cdot) \in L_s$ yielding $l(\cdot) = 0$ (see (5.12)). Now (5.10) gives $l^*_0(\cdot) \in L_s$. By (4.60) $l^*(\cdot) \notin M^0_{s,\xi}$ which contradicts the initial assumption.

Thus we have proved

Proposition 5.3 *The set $M^0_{s,\xi}$ of all functions mutant on $[s,\xi]$ and reconstructable on $[0,\xi]$ is empty.*

This Proposition and Theorem 4.2 imply

Proposition 5.4 *The set $L_{s,\xi}$ of all functionals from $\mathbf{L}^2_{0,s}$ reconstructable on $[0,\xi]$ coincides with L_s.*

5.2 Example

Consider the system

$$\begin{aligned} \dot{x}^{(1)}(t) &= u^{(1)}(t) \\ \dot{x}^{(2)}(t) &= x^{(2)}(t) + u^{(2)}(t) \end{aligned}$$

The initial state is zero:

$$x^1(0) = 0, \quad x^{(2)}(t) = 0$$

The observed signal is

$$z(t) = x^{(1)}(t) + x^{(2)}(t)$$

The adjoint equation has the form

$$\begin{aligned} \dot{w}^{(1)}(t) &= 0 \\ \dot{w}^{(2)}(t) &= w^{(2)}(t) \end{aligned}$$

and (see (3.3))

$$\phi_1^{(1)}(t,\sigma) = 1, \quad \phi_1^{(2)}(t,\sigma) = e^{\sigma - t}$$

Fix an $s \geq 0$. A $\psi(\cdot) \in \mathbf{L}^2_{0,s}$ is orthogonal to K_s (belongs to $L_s^{\perp}$) if and only if

$$\int_0^{\sigma} (\psi^{(1)}(t) + e^{\sigma - t}\psi^{(2)}(t))dt = 0$$

Differentiation in σ gives the equivalent condition

$$\psi^{(1)}(\sigma) + \psi^{(2)}(\sigma) + \int_0^{\sigma} e^{\sigma - t}\psi^{(2)}(t)dt = 0$$

An $l(\cdot)$ is orthogonal to such $\psi(\cdot)$ if and only if

$$0 = \int_0^s (l^{(1)}(\sigma)\psi^{(1)}(\sigma) + l^{(2)}(\sigma)\psi^{(2)}(\sigma))d\sigma =$$

$$\int_0^s l^{(2)}(\sigma)\psi^{(2)}(\sigma)d\sigma - \int_0^s l^{(1)}(\sigma)[\psi^{(2)}(\sigma) + \int_0^{\sigma} e^{\sigma - t}\psi(2)(t)dt]d\sigma =$$

$$\int_0^s l(2)(\sigma 0\psi^{(2)}(\sigma)d\sigma - \int_0^s (\int_t^s l^{(1)}(\sigma)e^{\sigma - t}d\sigma)\psi^{(2)}(\sigma)d\sigma$$

Thus L_s consists of all $l(\cdot)$ satisfiyng the above equality with an arbitrary $\psi^{(2)}(\cdot)$, i.e. all $l(\cdot)$ such that

$$l^{(2)}(\sigma) = l^{(1)}(\sigma) + \int_{\sigma}^{s} e^{t-\sigma} l^{(1)}(t)dt \quad (0 \leq \sigma \leq s) \tag{5.14}$$

Hence we come to

Proposition 5.5 *A functional $l(\cdot) \in \mathbf{L}^2_{0,s}$ is reconstructable on $[0,s]$ (lies in L_s if and only if (5.14) is satisfied.*

Using the pattern of Example 5.1, one can prove that Proposition 5.4 is also true.

5.3 Example

Consider the three-dimensional non-stationary system of the form:

$$\begin{aligned}
\dot{x}^{(1)}(t) &= 0 \\
\dot{x}^{(2)}(t) &= -x^{(3)}(t) \\
\dot{x}^{(3)}(t) &= x^{(2)}(t) + u(t)
\end{aligned}$$

for $t > \pi/2$, and

$$\begin{aligned}
\dot{x}^{(1)}(t) &= x^{(2)}(t) \\
\dot{x}^{(2)}(t) &= -x^{(1)}(t) \\
\dot{x}^{(3)}(t) &= u(t)
\end{aligned}$$

for $t \leq \pi/2$. The initial state is zero:

$$x^{(1)}(0) = 0, \quad x^{(2)}(0) = 0, \quad x^{(3)}(0) = 0$$

The observed signal is

$$z(t) = x^{(1)}(t)$$

Thus the observation matrice is

$$P = (1 \quad 0 \quad 0) = p_1^T$$

and

$$B^T = (0 \quad 0 \quad 1) \tag{5.15}$$

The adjoint equation has the form

$$\begin{aligned}
\dot{w}^{(1)}(t) &= 0 \\
\dot{w}^{(2)}(t) &= -w^{(3)}(t) \\
\dot{w}^{(3)}(t) &= w^{(2)}(t)
\end{aligned}$$

for $t \leq \pi/2$, and

$$\begin{aligned}
\dot{w}^{(1)}(t) &= w^{(2)}(t) \\
\dot{w}^{(2)}(t) &= -w^{(1)}(t) \\
\dot{w}^{(3)}(t) &= 0
\end{aligned}$$

for $t > \pi/2$. For $w(t) = w_1(t, \sigma)$ $(t \leq \sigma)$ we have

$$w^{(1)}(t) = 1, \quad w^{(2)}(t) = 0, \quad w^{(3)} = 0 \quad (t \leq \sigma \leq \pi/2)$$

and if $\sigma > \pi/2$

$$\begin{aligned}
w^{(1)}(t) &= \cos(t - \sigma) \\
w^{(2)}(t) &= -\sin(t - \sigma) \\
w^{(3)}(t) &= 0
\end{aligned}$$

$$(\pi/2 \leq t \leq \sigma)$$

and

$$\begin{array}{rcl} w^{(1)}(t) & = & \cos(\pi/2-\sigma) \\ w^{(2)}(t) & = & -\cos(t-\pi/2)\sin(\pi/2-\sigma) \\ w^{(3)}(t) & = & -\sin(t-\pi/2)\sin(\pi/2-\sigma) \\ (0 \le t \le \pi/2) & & \end{array}$$

Hence due to (5.15), for

$$\phi_1(t,\sigma) = B^T w(t) = w^{(3)}(t) \quad (t \le \sigma)$$

it holds

$$\phi_1(t,\sigma) = 0 \quad (0 \le t \le \sigma \le \pi/2) \tag{5.16}$$

$$\phi_1(t,\sigma) = 0 \quad (\pi/2 \le t \le \sigma, \quad \sigma > \pi/2)$$

$$\phi_1(t,\sigma) = -\sin(t-\pi/2)\sin(\pi/2-\sigma) \quad (0 \le t \le \pi/2, \quad \sigma > \pi/2) \tag{5.17}$$

Let

$$s = \pi/2, \quad \xi = \pi$$

From (5.16) we get $K_s = \{0\}$. Hence

$$L_s = \{0\} \tag{5.18}$$

Therefore every nonzero $l(\cdot) \in \mathbf{L}^2_{0,s} \cap L_{\xi|s}$ is mutant on $[s,\xi]$. Clearly $L_{\xi|s}$ is the linear hull of all functionals (5.17) with $\sigma \in [s,\xi]$. We sum it up as follows:

Proposition 5.6 *The set $L_{s,\xi}$ of all functionals reconstructable on $[0,\xi]$ is the linear hull of all functionals (5.17) with $\sigma \in [s,\xi]$. Every functional from $\mathbf{L}^2_{0,s} \setminus L_{s,\xi}$ is non-reconstructable on $[0,s]$.*

6 Open Questions

6.1 Stationary System

In Examples 5.1 and 5.2 we have $L_{s,\xi} = L_s$ (Proposition 5.4) for two stationary systems. It looks like that this is so for every stationary system; in other words, the following theorem seems true.

Theorem 6.1 *Let the system be stationary. Then $L_{s,\xi} = L_s$.*

Example 5.3 shows that this is generally false for a non-stationary system.

Theorem 4.4 is a step towards Theorem 6.1. In order to pass from Theorem 4.4 to Theorem 6.1, it is sufficient to state that the limit points of $K^L_{\xi|s}$ do not belong to $M^0_{s,\xi}$. We do not have yet a proof for that.

6.2 Constructive Description of the Space of Reconstructable Functionals

In Examples 5.1, 5.2 and 5.3 the space L_s of all functionals reconstructable on $[0,s]$ is described explicitly. It would be useful to have a description of this space in the general case. For this purpose, the method sketched in the above Examples can be developed.

6.3 Constrained Inputs

¿From the viewpoint of applications, it is interesting to treat the problem under the constraint $u(t) \in C$. It is rational to start with the case where the set C of admissible input values is a closed cone. Technically, this is not too far from the case considered above ($C = \mathbf{R}^r$); our tools, basically, should work. On the other hand, a conic C involves in particular the case where inputs have nonnegative coordinates (this is the case in many applied models).

If inputs are constrained, the reconstructibility alternative (saying that a functional is either reconstructable, or non-reconstructable; see Theorem 3.3) seems no longer be true; in particular, *partly reconstructable* functionals $l(\cdot)$, with $R_s(l(\cdot), z(\cdot))$ bounded but non-one-element, can appear.

6.4 Relaxed Initial State

The above results can easily be modified for the case where the initial state (see (2.2)) is restricted to a set X_0 (a subspace of $\mathbf{R}^n$, a cone, a convex set). In this situation partly reconstructable functionals can appear too.

6.5 Parabolic Systems: Reconstruction of Pollution Intensities

The proposed approach can be extrapolated to the systems governed by partial differential equations of parabolic type. These systems provide basic models for description of contamination processes in various media. Within the framework of these models, the practically important problem of reconstructing time-varying pollution intensities on the basis of available observations of concentrations, can be posed. The problem fits entirely the above pattern; we suppose to treat it in our forthcoming publications. In particular, the question of rational allocation of observation areas (allowing to reconstruct desired input information) will be considered. Examples showing that for a stationary parabolic system, Theorem 6.1 is false will also be provided.

6.6 Other Types of System Equations

The proposed method can be extrapolated to dynamic systems governed by parabolic differential inequalities, hyperbolic equations, differential equations with time lages, as well as to linear discrete-time dynamic systems. (Note that for all these systems Theorem 6.1 is, in general, untrue.)

References

Aubin, J.-P. et Frankowska, H., Observabilite des systemes sous incertitude, C.R. Acad. Sci., Paris, V. 306 (1988), Ser. 1, 413 - 416.

Brockett, R.W. and Mesarovich, M.P., The reproducibility of multivariable control systems, J. Math. Anal. Appl., V. 11 (1965), No 1 - 3, 548 - 563.

Galiullin, A.S., Inverse problems of dynamics, Nauka, Moscow, 1986 (Russian).

Hirschorn, R.M. Invertibility of nonlinear control system, SIAM J. Control and Optimiz., V. 17 (1981), 289 - 297.

Krasovskii, N.N., Theory of controlling motions, Nauka, Moscow, 1968 (Russian).

Krasovskii, N.N. and Subbotin, A.I., Positional differential games, Nauka, Moscow, 1974 (Russian); rev. English transl., Game-theoretical control problems, Springer, 1988.

Kryazhimskii, A.V. and Osipov, Yu.S., On control modelling in a dynamical system, Izv. Akad. Nauk SSSR, 1983, No 2, 51 - 60 (Russian).

Kryazhimskii, A.V. and Osipov, Yu.S, Inverse problems of ordinary differential equations: dynamical solutions, Gordon and Breach, 1993 (to appear).

Kurzhanskii, A.B., Control and observation under uncartainty conditions, Nauka, Moscow, 1977 (Russian).

Nikol'skii, M.S., On ideally observed systems, Differents. Uravnen., V. 7 (1971), No 4, 631 - 638 (Russian).

Sain, M.K. and Massey, J.L., Invertibility of linear time invariant dynamical systems, IEEE Trans. Automat. Control, V. 14 (1969), No 2, 141 - 149.

Silverman, L.W., Inversion of multivariable linear systems, IEEE Trans. Automat. Control, V. 14 (1969), No 3, 270 - 276.

Willsky, A.S., On the invertibility of linear systems, IEEE Trans. Automat. Control, V. 19 (1974), No 3, 272 - 274.

Warga, J., Optimal control of differential and functional equations, Acad. Press, New York, 1975.

Wonham, W.H., Linear multivariable control systems: a geometric approach, Springer, New York, 1979.