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Optimality and Characteristics of Hamilton-Jacobi-Bellman Equations

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Foreword

In this paper the authors study the Bolza problem arising in nonlinear optimal control and investigate under what circumstances the necessary conditions for optimality of Pontryagin's type are also sufficient. This leads to the question when shocks do not occur in the method of characteristics applied to the associated Hamilton-Jacobi-Bellman equation. In this case the value function is its (unique) continuously differentiable solution and can be obtained from the canonical equations. In optimal control this corresponds to the case when the optimal trajectory of the Bolza problem is unique for every initial state and the optimal feedback is an upper semicontinuous set-valued map with convex, compact images.

1 Introduction

This paper is concerned with the Hamilton-Jacobi equation

$$- \frac{\partial V}{\partial t} + H\left(x, -\frac{\partial V}{\partial x}\right) = 0, \quad V(T,\cdot) = \varphi(\cdot) \tag{1}$$

associated to the Bolza type problem in optimal control:

$$\text{minimize} \int_{t_0}^{T} L(x(t), u(t))dt + \varphi(x(T)) \tag{2}$$

over solution-control pairs (x, u) of control system

$$\begin{cases} x'(t) &= f(x(t)) + g(x(t))u(t), \quad u(t) \in U \\ x(t_0) &= x_0 \end{cases} \tag{3}$$

where U is a finite dimensional space and

$$H(x,p) = \sup_{u \in U} \langle p, f(x) + g(x)u \rangle - L(x,u)$$

In general H is not differentiable, but here we shall restrict our attention only to problems with smooth Hamiltonians.

The characteristics of the Hamilton-Jacobi-Bellman equation (1) are solutions to the *Hamiltonian system*

$$\begin{cases} x'(t) &= \dfrac{\partial H}{\partial p}(x(t), p(t)), \quad x(T) = x_T \\ \\ -p'(t) &= \dfrac{\partial H}{\partial x}(x(t), p(t)), \quad p(T) = -\nabla\varphi(x_T) \end{cases} \tag{4}$$

Such system is also called "canonical equations" or "equations of the extremals" in optimal control theory, since the Pontryagin maximum principle claims that if $x : [t_0, T] \to \mathbf{R}^n$ is optimal for problem (2), (3), then there exists $p : [t_0, T] \to \mathbf{R}^n$ such that (x, p) solves (4) with $x_T = x(T)$. This is not however a sufficient condition for optimality because it may happen that to a given $x_0 \in \mathbf{R}^n$ corresponds a solution (x, p) of (4) with $x(t_0) = x_0$ and x is not optimal. If such is the case and the optimal solution to (2), (3) does exist, then by the maximum principle, we can find another solution (x_1, p_1) of (4) with $x_1(t_0) = x_0$ and $p_1(t_0) \neq p(t_0)$. The situation when there are two solutions (x_i, p_i), $i = 1, 2$ of (4) satisfying $x_i(t_0) = x_0$ and $p_1(t_0) \neq p_2(t_0)$ is called *shock* arising in the method of characteristics.

If shocks never occur on the time interval $[0, T]$, then the solution of (1) can be constructed by considering all trajectories (x, p) of (4) and setting

$$V(t_0, x(t_0)) = \varphi(x(T)) + \int_{t_0}^{T} L(x(t), u(t))dt$$

where $u(t) \in U$ is such that

$$H(x(t), p(t)) = \langle p(t), f(x(t)) + g(x(t))u(t)\rangle - L(x(t), u(t)) \text{ a.e. in } [t_0, T]$$

Then, by [3], V is continuously differentiable,

$$\frac{\partial V}{\partial x}(t, x(t)) = -p(t) \ \& \ \frac{\partial V}{\partial t}(t, x(t)) = H(x(t), p(t))$$

Furthermore V is the so called value function of our optimal control problem. In summary if we can guarantee that on some time interval $[t_0, T]$ there is no shocks, then the value function would be the continuously differentiable on $[t_0, T] \times \mathbf{R}^n$ solution to (1).

It is well known that (unfortunately) shocks do happen. This is the very reason why the value function is nonsmooth and why one should not expect to have smooth solutions. Also it was shown in [5] and [3] that the value function is not regularly differentiable at some point (t_0, x_0) if and only if the optimal trajectory of the control problem (2), (3) is not unique.

Thus if we provide conditions that guarantee the absence of shocks in the same time we get the useful information about uniqueness of optimal solutions. Furthermore, under the same assumptions as in [3] we get the optimal feedback low on $[t_0, T] \times \mathbf{R}^n$:

$$U(t, x) = \{u \in U \mid H(x, -\frac{\partial V}{\partial x}(t, x)) = \langle -\frac{\partial V}{\partial x}(t, x), f(x) + g(x)u\rangle - L(x, u)\}$$

with the set-valued map $U(\cdot)$ being upper semicontinuous with convex compact images. In this case there exists also exactly one solution of

$$x' = f(x) + g(x)u(t, x), \ u(t, x) \in U(t, x), \quad x(t_0) = x_0$$

and it is optimal for problem (2), (3).

It was proved in [3] that the shocks would not occur till time t_0 if for every (x,p) solving (4) on $[t_0,T]$ the matrix Riccati equation

$$\left\{\begin{array}{rcl} P' & + & \dfrac{\partial^2 H}{\partial p \partial x}(x(t),p(t))P \ + \ P\dfrac{\partial^2 H}{\partial x \partial p}(x(t),p(t)) + \\ & + & P\dfrac{\partial^2 H}{\partial p^2}(x(t),p(t))P \ + \ \dfrac{\partial^2 H}{\partial x^2}(x(t),p(t)) \ = \ 0 \\ P(T) & = & -\varphi''(x(T)) \end{array}\right. \tag{5}$$

has a solution on $[t_0,T]$.

In this paper we provide some sufficient conditions for global solvability of the above Riccati equation for all (x,p) verifying (4).

In Section 2 we recall some results from [3]. Section 3 is devoted to few useful informations about the matrix Riccati equations. In particular (5) is reduced to the equation

$$S' + S^2 + D(t) \ = \ 0, \ \ S(T) \ = S_T$$

where $D(t)$, S_T are defined from the coefficients of (5) and which is much simpler to investigate. In Section 4 we provide some applications to the optimal control problem mentioned above.

2 Matrix Riccati Equations and Shocks

In this section we recall some results concerning differentiability of the value function and shocks of the Hamilton-Jacobi-Bellman equation (1).

Consider the Bolza problem in the nonlinear optimal control setting:

$$(P) \qquad \min \int_{t_0}^{T} L(x(t),u(t))dt \ + \ \varphi(x(T))$$

over solution-control pairs (x,u) of control system

$$\left\{\begin{array}{rcl} x'(t) & = & f(x(t)) + g(x(t))u(t), \quad u(\cdot) \in L^1(t_0,T;\mathbf{R}^m) \\ x(t_0) & = & x_0 \end{array}\right. \tag{6}$$

where $t_0 \in [0,T]$, $x_0 \in \mathbf{R}^n$, $f : \mathbf{R}^n \mapsto \mathbf{R}^n$, $g : \mathbf{R}^n \mapsto L(\mathbf{R}^m,\mathbf{R}^n)$, $L : \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}$, $\varphi : \mathbf{R}^n \mapsto \mathbf{R}$.

We associate to these data the *Hamiltonian* H defined on $\mathbf{R}^n \times \mathbf{R}^n$ by

$$H(x,p) = \sup_u \langle p, f(x) + g(x)u \rangle - L(x,u)$$

If H is differentiable, then the *Hamiltonian system*

$$\left\{ \begin{array}{rcl} x'(t) & = & \dfrac{\partial H}{\partial p}(x(t), p(t)) \\ \\ -p'(t) & = & \dfrac{\partial H}{\partial x}(x(t), p(t)) \end{array} \right. \tag{7}$$

is called *complete* if for all $t_0 \in [0, T]$, x_0, $p_0 \in \mathbf{R}^n$ it has a solution (x, p) defined on $[0, T]$ and satisfying $x(t_0) = x_0$, $p(t_0) = p_0$.

We impose the following assumptions:

H_1) f and g are differentiable, locally Lipschitz and have linear growth:

$$\exists\, M \geq 0,\ \forall\, x \in \mathbf{R}^n,\ \ \|f(x)\| + \|g(x)\| \leq M(\|x\| + 1)$$

H_2) $\varphi \in C^1$, $\liminf_{\|x\| \to \infty} \varphi(x) = +\infty$,

H_3) $L(x, \cdot)$ is continuous, convex, $\exists\, c > 0$, $\forall\, (x, u) \in \mathbf{R}^n \times \mathbf{R}^m$, $L(x, u) \geq c\,\|u\|^2$.

Furthermore for all $r > 0$, there exists $k_r \geq 0$ such that

$$\forall\, u \in \mathbf{R}^m,\ \ L(\cdot, u) \text{ is differentiable and } k_r - \text{Lipschitz on } B_r(0)$$

H_4) The Hamiltonian H is differentiable, its gradient $\nabla H(\cdot, \cdot)$ is locally Lipschitz

and the Hamiltonian system (7) is complete.

We denote by $x(\cdot; t_0, x_0, u)$ the solution to (6) starting at time t_0 from the initial state x_0 and corresponding to the control $u(\cdot)$.

The value function associated to this problem is given by

$$V(t_0, x_0) \ = \ \inf_{u \in L^1(t_0, T)} \int_{t_0}^{T} L(x(t; t_0, x_0, u), u(t))dt \ + \ \varphi(x(T; t_0, x_0, u))$$

where (t_0, x_0) range over $[0, T] \times \mathbf{R}^n$. It is well known that whenever V is differentiable, it satisfies the Hamilton-Jacobi-Bellman equation (1). The following result was proved in [3]:

Theorem 2.1 *Assume that $H_1) - H_4)$ hold true. Then the following three statements are equivalent:*

i) The value function V is continuously differentiable

ii) $\forall\, (t_0, x_0) \in [0, T] \times \mathbf{R}^n$ the optimal trajectory to problem (P) is unique *iii) The system (4) does not exhibit shocks on* $[0,T]$.

Furthermore, if one of the above (equivalent) statements holds true, then any solution (x,p) *to (4) satisfies:*
for all $t \in [0,T]$, $p(t) = -\frac{\partial V}{\partial x}(t, x(t))$ *and* x *restricted to* $[t_0, T]$ *is optimal for problem (P) with* $x_0 = x(t_0)$.

The above implies that whenever shocks do not occur on $[t_0, T]$, then the Pontryagin's necessary conditions for optimality of a solution $\overline{x}(\cdot)$ to (6): there exists $p : [t_0, T] \to \mathbf{R}^n$ such that $(\overline{x}, p)$ solves (4) on $[t_0, T]$ with $x_T = \overline{x}(T)$ are also sufficient.

It was observed in [3] that if φ, H are twice continuously differentiable and H'' is locally Lipschitz, then $V(t,\cdot) \in C^2$ for all $t \in [0,T]$ if and only if for every (x,p) solving (4) on $[0,T]$ the equation (5) has a solution on $[0,T]$. Since (5) describes the evolution of the tangent space to the set $\text{Graph}(-\frac{\partial V}{\partial x}(t,\cdot))$ at $(x(t), p(t))$ in the sense that $\text{Graph}(P(t))$ is tangent to this set at $(x(t), p(t))$, $-\frac{\partial^2 V}{\partial x^2}(t, x(t))$ solves the Riccati differential equation (5) on $[0,T]$.

3 Properties of Solutions to Riccati Equations

We investigate here the matrix differential equations of the following type

$$P' + A(t)^\star P + PA(t) + PE(t)P + D(t) = 0, \quad P(T) = P_T \tag{8}$$

By the classical theory of Riccati equations if for all $(x,p) \in \mathbf{R}^n \times \mathbf{R}^n$, $\frac{\partial^2 H}{\partial x^2}(x,p) \le 0$ and $\varphi'' \ge 0$ (i.e. φ is convex), then the solution $P(\cdot)$ to (8) exists on $[0,T]$ for every choice of continuous $(x(\cdot), p(\cdot))$.

3.1 Comparison Theorems

The aim of this section is to provide two comparison properties for solutions of Riccati equations. Results of a similar nature can be found in [2], [8], [6].

Theorem 3.1 *Let* A, E_i, $D_i : [0,T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$, $i = 1,2$ *be integrable. We assume that* $E_1(t)$ *and* $D_1(t)$ *are self-adjoint for almost every* $t \in [0,T]$ *and*

$$D_1(t) \le D_2(t), \;\; E_1(t) \le E_2(t) \;\; \text{a.e. in } [0,T] \tag{9}$$

Consider self-adjoint operators $P_{iT} \in L(\mathbf{R}^n, \mathbf{R}^n)$ such that $P_{1T} \leq P_{2T}$ and solutions $P_i(\cdot) : [t_0, T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$ to the matrix equations

$$P' + A(t)^\star P + PA(t) + PE_i(t)P + D_i(t) = 0, \quad P_i(T) = P_{iT} \tag{10}$$

for $i = 1, 2$. If P_2 is self-adjoint, then $P_1 \leq P_2$ on $[t_0, T]$.

Proof — From uniqueness of solution to (10), using that $E_1(t)$ and $D_1(t)$ are self-adjoint, it is not difficult to deduce that P_1 is self-adjoint. For all $t \in [t_0, T]$, set

$$Z = P_2 - P_1, \quad \mathcal{A}(t) = A(t) + \frac{1}{2}E_1(t)(P_1(t) + P_2(t))$$

Then

$$\mathcal{A}(t)^\star Z(t) + Z(t)\mathcal{A}(t) = A(t)^\star Z(t) + Z(t)A(t) - P_1(t)E_1(t)P_1(t) + P_2(t)E_1(t)P_2(t)$$

Therefore Z solves the Riccati equation

$$Z' + \mathcal{A}(t)^\star Z + Z\mathcal{A}(t) + P_2(t)(E_2(t) - E_1(t))P_2(t) + D_2(t) - D_1(t) = 0$$

Denote by $X(\cdot, t)$ the solution to

$$X' = -\mathcal{A}(s)^\star X, \quad X(t,t) = Id$$

A direct verification yields

$$Z(t) = X(t,T)(P_{2T} - P_{1T})X(t,T)^\star +$$

$$+\int_t^T X(t,s)(D_2(s) - D_1(s) + P_2(s)(E_2(s) - E_1(s))P_2(s))X(t,s)^\star ds$$

This and assumptions (9) imply $Z \geq 0$ on $[t_0, T]$. □

Theorem 3.2 *Let $A, E_i, D_i : [0,T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$, $i = 1, 2$ be integrable. We assume that $E_1(t)$, $D_1(t)$ are self-adjoint for almost all $t \in [0,T]$ and*

$$D_1(t) \leq D_2(t), \;\; 0 \leq E_1(t) \leq E_2(t) \;\text{ a.e. in }\; [0,T]$$

Consider self-adjoint operators $P_{iT} \in L(\mathbf{R}^n, \mathbf{R}^n)$ such that $P_{1T} \leq P_{2T}$ and solutions $P_i(\cdot) : [t_i, T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$, $i = 1, 2$ to the matrix equations

$$P' + A(t)^\star P + PA(t) + PE_i(t)P + D_i(t) = 0, \quad P_i(T) = P_{iT}$$

If P_2 is self-adjoint, then the solution P_1 is defined at least on $[t_2, T]$ and $P_1 \leq P_2$.

Proof — Consider the square root $B(t)$ of $E_1(t)$, i.e. for almost every $t \in [0,T]$, $E_1(t) = B(t)B(t)^\star$ and set

$$t_0 = \inf_{t\in[0,T]} \{P_1 \text{ is defined on } [t,T]\}$$

Thus either the solution P_1 exists on $[0,T]$ or $\|P_1(t)\| \to \infty$ when $t \to t_0+$. It is enough to check that if $t_2 \le t_0$, then P_1 is bounded on $]t_0,T]$. So let us assume that $t_2 \le t_0$. By Theorem 3.1 for every $t_0 < t \le T$, $P_1(t) \le P_2(t)$. Since $P_1 = P_1^\star$ for every $x \in \mathbf{R}^n$ of norm one and all $t_0 < t \le T$

$$\int_t^T \|B(s)^\star P_1(s)x\|^2\, ds \le$$

$$\le -\int_t^T < P_1'(s)x, x > +2\int_t^T \|A(s)\|\,\|P_1(s)\|\, ds + \|D_1\|_{L^1(t,T)}$$

$$\le \langle P_1(t)x, x\rangle + \|P_{1T}\| + 2\int_t^T \|A(s)\|\,\|P_1(s)\|\, ds + \|D_1\|_{L^1(t,T)}$$

$$\le \|P_2(t)\| + 2\int_t^T \|A(s)\|\,\|P_1(s)\|\, ds + \|P_{1T}\| + \|D_1\|_{L^1(t,T)}$$

$$\le c + 2\int_t^T \|A(s)\|\,\|P_1(s)\|\, ds$$

for some c independent from t, because P_2 is bounded on $[t_2,T]$.

On the other hand for any $y \in \mathbf{R}^n$ of norm one

$$-\langle P_1'(t)x, y\rangle = \langle P_1(t)B(t)B(t)^\star P_1(t)x, y\rangle + \langle A(t)^\star P_1(t)x, y\rangle + $$
$$+ \langle P_1(t)A(t)x, y\rangle + \langle D_1(t)x, y\rangle$$

Integrating on $[t,T]$ and using the latter inequality and the Hölder inequality, we obtain

$$\langle P_1(t)x, y\rangle \le \|P_{1T}\| + \|B^\star(\cdot)P_1(\cdot)x\|_{L^2(t,T)}\, \|B^\star(\cdot)P_1(\cdot)y\|_{L^2(t,T)} +$$

$$+2\int_t^T \|A(s)\|\,\|P_1(s)\|\, ds + \|D_1\|_{L^1(t,T)}$$

$$\le c_1 + 2\int_t^T \|A(s)\|\,\|P_1(s)\|\, ds + \left[\left(c + 2\int_t^T \|A(s)\|\,\|P_1(s)\|\, ds\right)^{1/2}\right]^2$$

for some c_1 independent from t. Since this holds true for all x, $y \in \mathbf{R}^n$ of norm one,

$$\forall\, t_0 < t \le T, \quad \|P_1(t)\| \le c + c_1 + 4\int_t^T \|A(s)\|\,\|P_1(s)\|\, ds$$

Applying the Gronwall lemma we deduce that $\|P_1(t)\|$ is bounded on $]t_0,T]$ by a constant independent from t. □

3.2 Reduction to a Simpler Form

Our next aim is to associate to the Riccati equation (8) a new equation

$$S' - \mathcal{A}(t)S + S\mathcal{A}(t) + S^2 + \mathcal{D}(t) \;=\; 0, \;\; S(T) = S_T \tag{11}$$

where $\mathcal{A}(t)^\star = -\mathcal{A}(t)$, in such way that the existence of solution to (11) on $[t_0,T]$ implies that of (8).

Theorem 3.3 *Consider $E : [0,T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$ such that for some $\omega > 0$ and a.e. $t \in [0,T]$, $E(t) \geq \omega I$ and is self-adjoint. We assume that the square root of $E(t)$, denoted by $B(t)$, is twice differentiable. Let $A : [0,T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$ be absolutely continuous, $D : [0,T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$ be integrable, $P_T \in L(\mathbf{R}^n, \mathbf{R}^n)$.*

Then the solution to (8) exists on $[t_0,T]$ if and only if so does the solution to

$$\begin{cases} S' - \mathcal{A}(t)S + S\mathcal{A}(t) + S^2 + \mathcal{D}(t) = 0 \\ S(T) = \frac{1}{2}(A_1(T) + A_1(T)^\star) + B(T)^\star P_T B(T) \end{cases} \tag{12}$$

where

$$A_1(t) = B(t)^{-1}A(t)B(t) - B(t)^{-1}B'(t) \quad \& \quad D_1(t) = B(t)^\star D(t)B(t)$$

$$\mathcal{A}(t) = \frac{1}{2}(A_1(t) - A_1(t)^\star) \quad \& \quad \overline{\mathcal{A}}(t) = \frac{1}{2}(A_1(t) + A_1(t)^\star)$$

$$\mathcal{D}(t) = D_1(t) + \mathcal{A}(t)\overline{\mathcal{A}}(t) - \overline{\mathcal{A}}(t)\mathcal{A}(t) - \overline{\mathcal{A}}'(t) - \overline{\mathcal{A}}(t)^2$$

Proof — Let P solves (8) on $[t_0,T]$. Set $R(t) = B(t)^\star P(t)B(t)$. Differentiating this relation we obtain

$$R'(t) = B'(t)^\star P(t)B(t) + B(t)^\star P(t)B'(t) -$$

$$-B(t)^\star \left(A(t)^\star P(t) + P(t)A(t) + P(t)E(t)P(t) + D(t)\right) B(t)$$

$$= B'(t)^\star B(t)^{\star^{-1}} R(t) + R(t)B(t)^{-1}B'(t) - B(t)^\star A(t)^\star B(t)^{\star^{-1}} R(t) -$$

$$-R(t)B(t)^{-1}A(t)B(t) - R(t)^2 - D_1(t)$$

and conclude that R is the solution to the Riccati equation

$$R' + A_1(t)^\star R + RA_1(t) + R^2 + D_1(t) = 0, \quad R(T) = B(T)^\star P_T B(T) \quad (13)$$

Conversely, if R solves (13), then $P(t) := B(t)^{\star^{-1}} R(t) B(t)^{-1}$ is the solution to (8). We rewrite the equation (13) in the following form

$$R' + (\overline{\mathcal{A}}(t) - \mathcal{A}(t))R + R(\overline{\mathcal{A}}(t) + \mathcal{A}(t)) + R^2 + D_1(t) = 0, \quad R(T) = B(T)^\star P_T B(T)$$

and define $S(t) = \overline{\mathcal{A}}(t) + R(t)$. Then,

$$S'(t) = \overline{\mathcal{A}}'(t) - (\overline{\mathcal{A}}(t) - \mathcal{A}(t))R(t) - R(t)(\overline{\mathcal{A}}(t) + \mathcal{A}(t)) - R(t)^2 - D_1(t)$$

$$= \overline{\mathcal{A}}'(t) - (\overline{\mathcal{A}}(t) + R(t))^2 + \overline{\mathcal{A}}(t)^2 + \mathcal{A}(t)R(t) - R(t)\mathcal{A}(t) - D_1(t)$$

$$= \overline{\mathcal{A}}'(t) - S(t)^2 - D_1(t) + \mathcal{A}(t)S(t) - S(t)\mathcal{A}(t) - \mathcal{A}(t)\overline{\mathcal{A}}(t) + \overline{\mathcal{A}}(t)\mathcal{A}(t) + \overline{\mathcal{A}}(t)^2 \quad \square$$

Under some additional assumptions Theorem 3.3 can be improved in the following way.

Theorem 3.4 *Let us consider an integrable* $D : [0,T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$, *an absolutely continuous* $A : [0,T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$, E, $P_T \in L(\mathbf{R}^n, \mathbf{R}^n)$ *and* $t_0 \in [0,T]$. *We assume that for for almost every* $t \in [0,T]$, $A(t)E$ *is self-adjoint. Then the solution to the matrix equation*

$$P' + A(t)^\star P + PA(t) + PEP + D(t) = 0, \quad P(T) = P_T \quad (14)$$

exists on $[t_0, T]$ *if and only if so does the solution to*

$$S' + S^2 + ED(t) - A'(t) - A(t)^2 = 0, \quad S(T) = A(T) + EP_T \quad (15)$$

Furthermore, solutions of (14) and (15) are related by $S(\cdot) = A(\cdot) + EP(\cdot)$. *If in addition* E *is invertible, then the solution to (14) exists on* $[t_0, T]$ *if and only if so does the solution to*

$$Q' + QE^{-1}Q + ED(t)E - A'(t)E - A(t)^2E = 0, \quad S(T) = A(T)E + EP_TE \quad (16)$$

and Q *is self-adjoint whenever* E, P_T *and* $D(t)$ *are self-adjoint for all* $t \in [0,T]$.

Proof — Let P solve (14) on $[t_0, T]$. Set $S(t) = A(t) + EP(t)$. Differentiating this expression we obtain

$$\begin{aligned} S'(t) &= A'(t) - E\left(A(t)^\star P(t) + P(t)A(t) + P(t)EP(t) + D(t)\right) \\ &= -(A(t) + EP(t))^2 + A(t)^2 + A'(t) - ED(t) \end{aligned}$$

Thus S solves (15). Conversely let

$$t_1 = \inf_{t\in[0,T]} \{ \text{ The solution } P \text{ to (14) is defined on } [t, T] \}$$

and S solves (15) on $[t_0, T]$. It is enough to prove that if $t_0 \le t_1$, then P is bounded on $]t_1, T]$, that is it can happen only if $t_0 = t_1 = 0$. So let $t_0 \le t_1$. From the first part of the proof and uniqueness of solution we know that for every $t \in]t_1, T]$, $S(t) = A(t) + EP(t)$. Hence $\sup_{t\in]t_1,T]} \|EP(t)\| < \infty$. Integrating (14) we deduce that for all $x \in \mathbf{R}^n$ with $\|x\| \le 1$ and $t_1 < t \le T$

$$\|P(t)x\| \le \|P_T\| + \int_t^T \|D(s)\|\, ds + \int_t^T (2\,\|A(t)\| + \|EP(t)\|)\, \|P(t)\|\, dt$$

Since x is an arbitrary element of the unit ball we proved that for some $c \ge 0$ independent from $t \in]t_1, T]$, $\|P(t)\| \le c + \int_t^T c\, \|P(t)\|\, dt$. This and the Gronwall lemma yield $\sup_{t\in]t_1,T]} \|P(t)\| < \infty$. To prove the last statement it is enough to multiply (15) by E from the right and to set $Q = SE$. □

Our next result is similar to Theorem 3.3.

Theorem 3.5 *Under all the assumptions of Theorem 3.3, the solution to (8) exists on* $[t_0, T]$ *if and only if so does the solution to*

$$S' + S^2 + \mathcal{D}(t) = 0,\ S(T) = \frac{1}{2}(A_1(T) + A_1(T)^\star) + B(T)^\star P_T B(T)$$

where A_1 *is defined as in Theorem 3.3,*

$$\mathcal{D}(t) = \frac{1}{4}X(t)^\star \left(A_1(t)A_1(t)^\star - A_1(t)^2 - A_1(t)^{\star^2}\right) X(t) +$$

$$+\, X(t)^\star \left(B(t)^\star D(t)B(t) - \tfrac{1}{2}A_1'(t) - \tfrac{1}{2}A_1'(t)^\star - \tfrac{3}{4}A_1(t)^\star A_1(t)\right) X(t)$$

and $X(\cdot)$ *denotes the matrix solution to*

$$X' = \frac{1}{2}(A_1(t) - A_1(t)^\star)X,\ \ X(T) = Id$$

Proof — Let P solve (8) on $[t_0, T]$. By the proof of Theorem 3.3, $R(\cdot) := B(\cdot)^\star P(\cdot) B(\cdot)$ solves (13). Define $\mathcal{A}$, $\overline{\mathcal{A}}$, D_1 as in Theorem 3.3 and observe that $\mathcal{A}(t)^\star = -\mathcal{A}(t)$. Therefore

$$X(t)^\star X(t) \;=\; Id \tag{17}$$

Set $S(t) = X(t)^\star(\overline{\mathcal{A}}(t) + R(t))X(t)$. Then, differentiating this equality, using (17) and the proof of Theorem 3.3, we obtain

$$S'(t) = X(t)^\star \mathcal{A}(t)^\star(\overline{\mathcal{A}}(t) + R(t))X(t) + X(t)^\star(\overline{\mathcal{A}}(t) + R(t))\mathcal{A}(t)X(t) +$$

$$+ X(t)^\star(\overline{\mathcal{A}}'(t) - (\overline{\mathcal{A}}(t) + R(t))^2 + \overline{\mathcal{A}}(t)^2 + \mathcal{A}(t)R(t) - R(t)\mathcal{A}(t) - D_1(t))X(t)$$

$$= -X(t)^\star \mathcal{A}(t)(\overline{\mathcal{A}}(t) + R(t))X(t) + X(t)^\star(\overline{\mathcal{A}}(t) + R(t))\mathcal{A}(t)X(t) - S(t)^2 +$$

$$+ X(t)^\star(\overline{\mathcal{A}}'(t) + \overline{\mathcal{A}}(t)^2 + \mathcal{A}(t)R(t) - R(t)\mathcal{A}(t) - D_1(t))X(t) =$$

$$-S(t)^2 + X(t)^\star\left(\overline{\mathcal{A}}'(t) - D_1(t) - \mathcal{A}(t)\overline{\mathcal{A}}(t) + \overline{\mathcal{A}}(t)\mathcal{A}(t) + \overline{\mathcal{A}}(t)^2\right)X(t) \quad \square$$

3.3 Existence of Solutions

We deduce from the previous section sufficient conditions for existence of solutions to the matrix Riccati equations.

Theorem 3.6 *Let $A, E, D : [0,T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$ be integrable. We assume that $E(t)$, $D(t)$ are self-adjoint and $E(t) \geq 0$ for almost every $t \in [0,T]$. Consider a self-adjoint operator $P_T \in L(\mathbf{R}^n, \mathbf{R}^n)$ and assume that there exists an absolutely continuous $P : [t_0, T] \mapsto L(\mathbf{R}^n, \mathbf{R}^n)$ such that for every $t \in [t_0, T]$, $P(t)$ is self-adjoint, $P_T \leq P(T)$ and*

$$P'(t) + A(t)^\star P(t) + P(t)A(t) + P(t)E(t)P(t) + D(t) \leq 0 \text{ a.e. in } [t_0, T]$$

Then the solution $\overline{P}$ to (8) is defined at least on $[t_0, T]$ and $\overline{P} \leq P$.

Proof — Set

$$\Gamma(t) = P'(t) + A(t)^\star P(t) + P(t)A(t) + P(t)E(t)P(t) + D(t)$$

Then $\Gamma(t) \leq 0$ is self-adjoint and P solves the Riccati equation

$$P' + A(t)^\star P + PA(t) + PE(t)P + D(t) - \Gamma(t) = 0$$

where $D(t) - \Gamma(t) \geq D(t)$. By Theorem 3.2, $\overline{P}$ is defined at least on $[t_0, T]$ and $\overline{P} \leq P$. $\square$

Corollary 3.7 *Under all assumptions on* A, E, D *of Theorem 3.6 consider a self-adjoint nonpositive* $P_T \in L(\mathbf{R}^n, \mathbf{R}^n)$. *If for almost all* $t \in [0,T]$, $D(t) \leq 0$, *then the solution* $\overline{P}$ *to the matrix Riccati equation (8) is well defined on* $[0,T]$ *and* $\overline{P} \leq 0$.

Theorem 3.8 *Under all the assumptions of Theorem 3.3, let* $\mathcal{D}$, $S(T)$ *be defined as in Theorem 3.3. Assume that for some* $\lambda \geq 0$ *and all* $t \in [0,T]$, $\mathcal{D}(t) \leq -\lambda^2 I$ *and* $S(T) \leq \lambda I$. *Then the solution to (8) is defined on* $[0,T]$.

Proof — By Theorem 3.3 we have to check that (12) has a solution on $[0,T]$. Set $\overline{S}(\cdot) \equiv \lambda I$. Then for every $t \in [0,T]$,

$$\overline{S}'(t) - \mathcal{A}(t)\overline{S}(t) + \overline{S}(t)\mathcal{A}(t) + \overline{S}(t)^2 + \mathcal{D}(t) \leq 0$$

Theorem 3.6 ends the proof. □

Theorem 3.9 *Under all the assumptions of Theorem 3.4, suppose that* E, $D(t)$, P_T *and* $A(T) + EP_T$ *are self-adjoint,* $E \geq 0$ *and*

$$A'(t) + A(t)^2 - ED(t) \text{ is self-adjoint for almost every } t \in [0,T]$$

If there exists $a \in \mathbf{R}$ *such that*

$$A'(t) + A(t)^2 - ED(t) \geq a^2 I \text{ for a.e. } t \in [0,T] \quad \& \quad A(T) + EP_T \leq aI$$

then the solution to the Riccati equation (14) is defined on $[0,T]$.

Proof — By Theorem 3.4 it is enough to show that the problem (15) has a solution on $[0,T]$. For all $t \in [0,T]$, set $S(t) = aI$. Then

$$S'(t) + S(t)^2 + ED(t) - A'(t) - A(t)^2 \leq 0, \quad S(T) = aI$$

By Theorem 3.6 the solution to

$$S' + S^2 + ED(t) - A'(t) - A(t)^2 = 0, \quad S(T) = A(T) + EP(T)$$

is defined on $[0,T]$. □

Theorem 3.10 *Under the assumptions of Theorem 3.4, suppose that* E, $D(t)$, P_T *are self-adjoint and* $E > 0$. *If there exists* $a \in \mathbf{R}$ *such that*

$$A'(t)E + A(t)^2E - ED(t)E \geq a^2 E \text{ for a.e. } t \in [0,T] \quad \& \quad A(T)E + EP_TE \leq aE$$

then the solution to the Riccati equation (14) is defined on $[0,T]$.

Proof — By Theorem 3.4 we have to verify that the problem (16) has a solution on $[0,T]$. For all $t \in [0,T]$, set $Q(t) = aE$. The proof ends by the same arguments as the one of Theorem 3.9. □

4 Applications to the Bolza Problem

We apply the previous results to the problem treated in Section 2.

4.1 Linear with Respect to Controls System

Consider the problem

$$V(t_0, x_0) = \min \int_{t_0}^{T} \left(l(x(t)) + \frac{1}{2}\langle Ru(t), u(t)\rangle \right) dt \;+\; \varphi(x(T)) \qquad (18)$$

over solution-control pairs (x, u) of the control system

$$x'(t) = f(x(t)) + Bu(t), \quad x(t_0) = x_0, \quad u(t) \in \mathbf{R}^m \qquad (19)$$

where $t_0 \in [0, T]$, $x_0 \in \mathbf{R}^n$,

$$f = (f_1, ..., f_n) : \mathbf{R}^n \mapsto \mathbf{R}^n, \quad l : \mathbf{R}^n \mapsto \mathbf{R}, \quad \varphi : \mathbf{R}^n \mapsto \mathbf{R}$$

$B \in L(\mathbf{R}^m, \mathbf{R}^n)$ and $R \in L(\mathbf{R}^m, \mathbf{R}^m)$ is a self-adjoint operator such that for some $\omega > 0$ and all $u \in \mathbf{R}^m$, $\langle Ru, u\rangle \geq \omega \|u\|^2$.

The associated Hamiltonian system is

$$\begin{cases} x' &= f(x) + BR^{-1}B^\star p, \quad x(T) = x_T \\ -p' &= f'(x)^\star p - \nabla l(x), \quad p(T) = -\nabla\varphi(x_T) \end{cases} \qquad (20)$$

We impose the following assumptions:

h$_1$) $\exists\, M \geq 0,\ \forall\, x \in \mathbf{R}^n,\ \|f(x)\| \leq M(\|x\| + 1)$
h$_2$) $\liminf_{\|x\| \to \infty} \varphi(x) = +\infty$
h$_3$) The functions $f, l, \varphi \in C^2$
h$_4$) The Hamiltonian system (20) is complete
h$_5$) $f'(x)BR^{-1}B^\star$ is self-adjoint

Observe that **h**$_5$) yields that

$$\forall\, j = 1, ..., n,\ f_j''(x)BR^{-1}B^\star = BR^{-1}B^\star f_j''(x).$$

Linear convex problems in general do not satisfy **h**$_5$), but we treat this case separately, in the next subsection.

Theorem 4.1 *Assume* $\mathbf{h}_1) - \mathbf{h}_5)$ *and that at least one of the following two assumptions is verified*

i) B is surjective and there exists $a \in \mathbf{R}$ such that for every $x \in \mathbf{R}^n$

$$BR^{-1}B^\star l''(x)BR^{-1}B^\star + \left(\frac{1}{2}\|f\|^2\right)''(x)BR^{-1}B^\star \geq a^2BR^{-1}B^\star$$

$$f'(x)BR^{-1}B^\star - BR^{-1}B^\star\varphi''(x)BR^{-1}B^\star \leq aBR^{-1}B^\star$$

ii) For every $x \in \mathbf{R}^n$, $l''(x)BR^{-1}B^\star$, $f'(x) - BR^{-1}B^\star\varphi''(x)$ are self-adjoint and there exists $a \in \mathbf{R}$ such that for every $x \in \mathbf{R}^n$

$$BR^{-1}B^\star l''(x) + \left(\frac{1}{2}\|f\|^2\right)''(x) \geq a^2 I \quad \& \quad f'(x) - BR^{-1}B^\star\varphi''(x) \leq aI$$

Then

a) V is continuously differentiable and $V(t,\cdot) \in C^2$

b) the optimal control problem (18), (19) has the unique optimal control for any initial condition $(t_0, x_0) \in [0,T] \times \mathbf{R}^n$

c) for every solution (x,p) to the system (20) and every $t_0 \in [0,T]$, $x(\cdot)$ restricted to $[t_0,T]$ is optimal for the problem (18), (19) with $x_0 = x(t_0)$ and $p(t) = -\frac{\partial V}{\partial x}(t,x(t))$

d) The map $t \mapsto f'(x(t)) - BR^{-1}B^\star\frac{\partial^2 V}{\partial x^2}(t,x(t))$ solves the equation

$$\begin{cases} P' + P^2 - \left(\frac{1}{2}\|f\|^2\right)''(x(t)) - BR^{-1}B^\star l''(x(t)) = 0 \\ P(T) = f'(x(T)) - BR^{-1}B^\star\varphi''(x(T)) \end{cases}$$

Furthermore the optimal feedback low $u : [0,T] \times \mathbf{R}^n \mapsto \mathbf{R}^n$ is given by

$$\forall\ (t,x) \in [0,T] \times \mathbf{R}^n,\ \ u(t,x) = -R^{-1}B^\star\frac{\partial V}{\partial x}(t,x)$$

Corollary 4.2 *Assume that $U = \mathbf{R}^n$, $R = B = Id$, that the map $x \mapsto l(x) + \frac{1}{2}\|f(x)\|^2$ is convex and*

$$\forall\ x \in \mathbf{R}^n,\ f'(x) - \varphi''(x) \leq 0$$

If $\mathbf{h}_1) - \mathbf{h}_4)$ hold true and $f'(x)$ is self-adjoint for all x, then all the conclusions of Theorem 4.1 are valid.

We observe first that the Hamiltonian corresponding to the problem (18), (19) is given by

$$\forall\, x,p \in \mathbf{R}^n, \quad H(x,p) \;=\; \langle p, f(x)\rangle - l(x) + \frac{1}{2}\left\langle BR^{-1}B^\star p, p\right\rangle$$

Thus,

$$\frac{\partial H}{\partial x}(x,p) \;=\; f'(x)^\star p - l'(x) \quad \& \quad \frac{\partial H}{\partial p}(x,p) \;=\; f(x) + BR^{-1}B^\star p$$

and

$$\frac{\partial^2 H}{\partial x \partial p}(x,p) = f'(x), \;\; \frac{\partial^2 H}{\partial p^2}(x,p) = BR^{-1}B^\star, \;\; \frac{\partial^2 H}{\partial x^2}(x,p) = \sum_{k=1}^n p_k\, f_k''(x) - l''(x)$$

Proof of Theorem 4.1 — It is not difficult to check, using $\mathbf{h}_1) - \mathbf{h}_4)$, that for all (t_0, x_0) there exists an optimal solution of our problem and the value function is locally Lipschitz (see [3]). From our assumptions we know that if for every solution (x,p) to (20) the matrix Riccati equation

$$\left\{ \begin{array}{l} P' + f'(x(t))^\star P + Pf'(x(t)) + PBR^{-1}B^\star P + \sum_{k=1}^n p_k\, f_k''(x) - l''(x(t)) = 0 \\ P(T) = -\varphi''(x(T)) \end{array} \right. \tag{21}$$

has a solution on $[0,T]$, then the conclusion $a)$ of our theorem is valid. On the other hand, if $(\overline{x}, \overline{u})$ is optimal and $p(\cdot)$ is the corresponding co-state, then

$$H(\overline{x}(t), p(t)) = \langle p(t), f(\overline{x}(t)) + B\overline{u}(t)\rangle - l(\overline{x}(t)) - \langle R\overline{u}(t), \overline{u}(t)\rangle \;\; \text{a.e.}$$

Thus

$$\overline{u}(t) = R^{-1}B^\star p(t) = -R^{-1}B^\star \frac{\partial V}{\partial x}(t, \overline{x}(t))$$

which yields $b)$ and $c)$. Set

$$A(t) = f'(x(t)) \quad \& \quad D(t) = \frac{\partial^2 H}{\partial x^2}(x(t), p(t))$$

Differentiating A we get

$$A'(t) \;=\; \left(\sum_{k=1}^n \frac{\partial^2 f_i}{\partial x_k \partial x_j}(x(t)) f_k(x(t)) + \left\langle \nabla \frac{\partial f_i}{\partial x_j}(x(t)),\; BR^{-1}B^\star p(t) \right\rangle \right)_{i,j}$$

Let e_{rs} denote the elements of the (symmetric) matrix $BR^{-1}B^{\star}$. By $\mathbf{h}_5$),

$$\forall\, i, r = 1, ..., n, \qquad \sum_{s=1}^{n} \frac{\partial f_i}{\partial x_s} e_{sr} = \sum_{s=1}^{n} \frac{\partial f_i}{\partial x_s} e_{rs} = \sum_{s=1}^{n} e_{is} \frac{\partial f_r}{\partial x_s}$$

and therefore

$$\sum_{s=1}^{n} e_{rs} \frac{\partial^2 f_i}{\partial x_j \partial x_s} = \sum_{s=1}^{n} e_{is} \frac{\partial^2 f_r}{\partial x_j \partial x_s}$$

Thus

$$\left\langle \nabla \frac{\partial f_i}{\partial x_j}(x), BR^{-1}B^{\star}p \right\rangle = \left\langle BR^{-1}B^{\star}\nabla \frac{\partial f_i}{\partial x_j}(x),\ p \right\rangle =$$

$$\sum_{r=1}^{n} p_r \sum_{s=1}^{n} \frac{\partial^2 f_i}{\partial x_s \partial x_j}(x) e_{rs} \ = \ \sum_{r=1}^{n} p_r \sum_{s=1}^{n} \frac{\partial^2 f_r}{\partial x_s \partial x_j}(x) e_{is} = \sum_{r=1}^{n} p_r BR^{-1}B^{\star} f_r''(x)$$

Consequently,

$$BR^{-1}B^{\star}D(t) - A'(t) - A(t)^2$$

$$= -BR^{-1}B^{\star}l''(x(t)) - \left(\sum_{k=1}^{n} \frac{\partial^2 f_i}{\partial x_k \partial x_j}(x(t)) f_k(x(t)) \right)_{i,j} - f'(x(t))^2 =$$

$$= \ -BR^{-1}B^{\star}l''(x(t)) - \left(\tfrac{1}{2}\, \|f(\cdot)\|^2\right)''(x(t))$$

Theorems 3.9 and 3.10 imply that the solution to the matrix Riccati equation (21) is defined on $[0, T]$. Finally, the conclusion *d*) follows from Theorem 3.4.

4.2 Linear Convex Bolza Problem

We consider the problem

$$\text{minimize} \int_{t_0}^{T} \left(l(t, x(t)) + \frac{1}{2}\langle R(t)u, u\rangle \right) dt \ + \ \varphi(x(T)) \tag{22}$$

over solution-control pairs (x, u) of the linear control system

$$x' = A(t)x + B(t)u(t), \quad x(t_0) = x_0, \quad u(t) \in \mathbf{R}^m \tag{23}$$

where $t_0 \in [0,T]$, $x_0 \in \mathbf{R}^n$,

$$l : [0,T] \times \mathbf{R}^n \mapsto \mathbf{R}_+, \quad \varphi : \mathbf{R}^n \mapsto \mathbf{R}$$

$A(t) \in L(\mathbf{R}^n, \mathbf{R}^n)$, $B(t) \in L(\mathbf{R}^m, \mathbf{R}^n)$ and $R(t) \in L(\mathbf{R}^m, \mathbf{R}^m)$ is a self-adjoint operator such that for some $\omega > 0$ and all $t \in [0,T]$,

$$\forall\, u \in \mathbf{R}^m, \quad \langle R(t)u, u\rangle \geq \omega \|u\|^2$$

We assume that $\varphi \in C^2$, $\lim_{\|x\| \to \infty} \varphi(x) = +\infty$, that $A(\cdot)$, $R(\cdot)$, $l(\cdot,\cdot)$, $\frac{\partial^2 l}{\partial x^2}(\cdot,\cdot)$ and $B(\cdot)$ are continuous,

$$\exists\, k \in L^1(0,T), \quad \left\| \frac{\partial l}{\partial x}(t,x) \right\| \leq k(t)(1 + \|x\|)$$

and that $l(t,\cdot)$ and φ are convex. Then

$$\forall\,(x,p) \in \mathbf{R}^n \times \mathbf{R}^n, \ \frac{\partial^2 H}{\partial x \partial p}(t,x,p) \ = \ A(t), \quad \frac{\partial^2 H}{\partial x^2}(t,x,p) \ = \ -\frac{\partial^2 l}{\partial x^2}(t,x)$$

$$\frac{\partial^2 H}{\partial p^2}(t,x,p) \ = \ B(t)R(t)^{-1}B(t)^\star$$

Since

$$\forall\, x \in \mathbf{R}^n, \ \frac{\partial^2 l}{\partial x^2}(t,x) \geq 0 \quad \& \quad \varphi''(x) \geq 0$$

by Corollary 3.7, the solution $P(\cdot)$ to the corresponding matrix Riccati equation is defined on $[0,T]$ for every choice of continuous $(x(\cdot), p(\cdot))$. Hence the conclusions a) - c) of Theorem 4.1 are valid. Furthermore, by Corollary 3.7, $\frac{\partial^2 V}{\partial x^2}(t, x(t)) = -P(t) \geq 0$. Thus $V(t,\cdot)$ is convex.

4.3 Local Regularity of the Value Function

In the general case we do not have existence of solutions to the matrix Riccati equations for all the extremals (x,p). However from a priori bounds on the data, it is possible to estimate the interval of time $[t_0, T]$ during which there is no shocks and so the value function is continuously differentiable on $[t_0, T] \times \mathbf{R}^n$.

Consider the problem

$$(P) \qquad \text{minimize} \int_{t_0}^{T} \left(l(x(t)) + \frac{1}{2}\langle Ru, u\rangle \right) dt \ + \ \varphi(x(T))$$

over solution-control pairs (x, u) of the control system

$$x'(t) = f(x(t)) + g(x(t))u(t), \quad x(t_0) = x_0, \quad u(t) \in \mathbf{R}^m \tag{24}$$

where $t_0 \in [0, T]$, $x_0 \in \mathbf{R}^n$,

$$f : \mathbf{R}^n \mapsto \mathbf{R}^n,\ g : \mathbf{R}^n \mapsto L(\mathbf{R}^m, \mathbf{R}^n),\ l : \mathbf{R}^n \mapsto \mathbf{R},\ \varphi : \mathbf{R}^n \mapsto \mathbf{R}$$

are twice continuously differentiable and $R \in L(\mathbf{R}^m, \mathbf{R}^m)$ is a self-adjoint operator such that for some $\omega > 0$ and all $u \in \mathbf{R}^m$, $\langle Ru, u \rangle \geq \omega \|u\|^2$.

We assume that

$$f,\ g,\ f',\ g',\ l',\ \varphi',\ f'',\ g'',\ l'',\ \varphi'' \text{ are bounded} \tag{25}$$

The Hamiltonian H of this problem is given by

$$H(x,p) \ = \ \langle p, f(x) \rangle + \frac{1}{2} \left\langle R^{-1} g(x)^\star p, g(x)^\star p \right\rangle - l(x)$$

and for $C \in L(\mathbf{R}^m, \mathbf{R}^m)$ such that $CC^\star = R^{-1}$

$$\frac{\partial H}{\partial x}(x,p) = f'(x)^\star p + \frac{d}{dx}(C^\star g^\star(\cdot)p)(x) - l'(x) \ \ \& \ \ \frac{\partial H}{\partial p}(x,p) = f(x) + g(x)R^{-1}g(x)^\star p$$

So the Hamiltonian system is

$$\begin{cases} x'(t) & = & f(x(t)) + g(x(t))R^{-1}g(x(t))^\star p(t) \\ -p'(t) & = & f'(x(t))^\star p(t) + \frac{d}{dx}(C^\star g^\star(\cdot)p)(x(t)) - \nabla l(x(t)) \\ p(T) & = & -\nabla\varphi(x(T)) \end{cases} \tag{26}$$

By (25) the norms of the co-states $p(\cdot)$ are bounded by a constant independent of $x(T)$. Thus there exists $c > 0$ such that every solution (x, p) of (26) satisfies

$$\|x'(\cdot)\|_\infty \ + \ \|p(\cdot)\|_\infty \ + \ \|p'(\cdot)\|_\infty \ \leq \ c$$

Fix $\varepsilon > 0$ and set

$$D(t) \ = \ \frac{\partial^2 H}{\partial p^2}(t, x(t), p(t)) \ + \ \varepsilon\, I$$

Then $D(t) \geq \varepsilon I$. By Theorem 3.5 and our assumptions we may reduce the matrix Riccati equation

$$\begin{cases} P' + \frac{\partial^2 H}{\partial p \partial x}(t, x(t), p(t))P \; + \; P \, \frac{\partial^2 H}{\partial x \partial p}(t, x(t), p(t)) + \\ P\left(\frac{\partial^2 H}{\partial p^2}(t, x(t), p(t)) + \varepsilon I\right) P + \frac{\partial^2 H}{\partial x^2}(t, x(t), p(t)) = 0, P(T) = -\varphi''(x(T)) \end{cases} \tag{27}$$

to the new Riccati equation

$$S' + S^2 + Q_{(x(\cdot),p(\cdot))}(t) \; = \; 0, \quad S(T) = S_{(x(\cdot),p(\cdot))} \tag{28}$$

with $Q_{(x(\cdot),p(\cdot))}(t)$ and $S(T) = S_{(x(\cdot),p(\cdot))}$ self-adjoint and such that

$$\forall \, t \; \in \; [0, T], \;\; Q_{(x(\cdot),p(\cdot))}(t) \; \leq \; \lambda I, \;\; S_{(x(\cdot),p(\cdot))} \; \leq \; \lambda I$$

where λ is independent from the solution (x, p) of (26), because of the boundedness assumption (25). Setting

$$S(t) \; = \; \lambda I + (T - t)\gamma I$$

and choosing γ large enough we prove that for some $t_0 \in [0, T[$ and

$$\forall \, t \in [t_0, T], \quad S'(t) + S^2(t) + Q_{(x(\cdot),p(\cdot))}(t) \; \leq \; 0$$

for all (x, p) solving (26). By Theorem 3.6 the solution to (28) is defined at least on $[t_0, T]$. By the comparison Theorem 3.2, also the solution of (27) with $\varepsilon = 0$ is defined on $[t_0, T]$ for all (x, p) solving (26). Thus $V \in C^1$ on $[t_0, T] \times \mathbf{R}^n$.

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