Working Paper

Anti-Pollution Tax Policy: a Viability Approach

Jean Pierre Aubin

WP-93-52
September 1993

IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

Anti-Pollution Tax Policy: a Viability Approach

Jean Pierre Aubin

WP-93-52
September 1993

IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

FOREWORD

Why anti-pollution tax increases do happen in such a discontinuous way, excluding more progressive continuous evolution which looks more rational ?

Why does it happen in the last moment, long before it could be implemented ?

Why do some tax rates, once chosen, look to be locked-in forever ?

These are some questions — high in IIASA's agenda — that viability theory attempts to answer.

We shall present in the first section the main feature and objectives of viability theory and a nontechnical presentation of its main concepts. We then devote the second section to a very simple model describing possible antipollution tax policies to regulate polluting production *in face of demand, illustrating the general concepts introduced in the first section.*

Several lessons can be drawn from this example. For instance, a same recipe (keep the tax constant) can have opposite consequences on the growth of production. A second one is that the intuition elaborated in a static framework may sometimes mislead us when evolution is present, by suggesting that such or such result, which makes sense in a static world, may have perverse effects in a dynamic one.

But it is forbidden to draw more conclusions than these modest models conceal. However, if a given reasoning is not validated in the case of such a simple model, it will not be valid in a more complex one.

Anti-Pollution Tax Policy: a Viability Approach

Jean-Pierre Aubin

1 Contingency and Viability of Systems

Viability theory is a mathematical theory[1] that offers *mathematical metaphors of evolution of macrosystems under contingent uncertainty* arising in biology, economics, cognitive sciences, games, and similar areas. It concerns specifically macrosystems, which are systems the variable of which are divided into two categories:

1. the **state variables**, which constitute the components of the state of the system, on which act the **agents, actors, decision-makers**, etc. of the system,

2. the **regulee, or regulation controls**

The difference between states and regulees is that if we do know the agents (biochemical mechanisms in biology, economic agents in economics, individuals or actors in sociology and cognitive individuals in cognitive sciences) acting on the states (phenotypes in biology, commodities in economics, behaviors in sociology and sensori-motor states in cognitive sciences), we do not know — or we assume that we do not know — the agents governing the evolution of the regulee, which may be

[1] Viability Theory is the theme of the essay *La mort du devin, l'émergence du démiurge*, the purpose of which is to divulgate this mathematical theory motivated by the common features shared by many biological, economic, social and cognitive systems. The book *Contingence et viabilité des systèmes*, with a more academic flavor, provides a more rigorous account of the concepts and the results of this theory and its biological, economic and cognitive motivations. The general mathematical theory is presented in *Viability Theory*, (1991) Birkhäuser. This theory is based on set-valued analysis — which was for a large part motivated by viability theory — presented in *Set-Valued Analysis* by Hélène Frankowska and the author, (1990) Birkhäuser. The book *Neural Networks and Qualitative Physics: A Viability Approach*, Cambridge University Press, (1994) applies these mathematical tools to two important domains of Artificial Intelligence and the monograph *Dynamic Economic Theory: A Viability Approach* is devoted to the "economic" version of this theory.

1. *genotypes* or *fitness matrices* in genetics and population genetics (when the evolution of *phenotypes* of a population is regulated by sexual reproduction and mutations),

2. *prices or other fiduciary goods* in economics (when the evolution of commodities and services is regulated by Adam Smith's invisible hand or the market, the planning bureau, ...),

3. *cultural codes* in sociology (when the evolution of societies is regulated by every individual believing and obeying such codes),

4. *conceptual regulees* or *synaptic matrices* in pattern recognition mechanisms and neural networks (when the sensory-motor state is regulated by learning processes), etc..

In the example below, the state of the system is a commodity produced by a polluting process, and the refulee would be taxes to stimulate or slow-down production of this commodity.

If it is safe to posint in firts approximation that entrepreneurs govern the evolution of production, it is more delicate to assume than a given decision maker could pilot the evolution of taxes: agencies are mad of so many constituants that it is no longer possible to identify such a political agency with an actual decision maker. We propose instead to assume that it behaves as a regularoty mechanism, and not as a planner.

But then, if there are no longer actors who steer the regulee, it becomes impossible to forecast the future of the system, leaving room to "chance" that the vocation of science is to extrude, to a freedom of evolution that viability must master.

We shall specifically be concerned with three main common features:

- A nondeterministic (or contingent) engine of evolution, providing several (and even many) opportunities to explore the environment,

- Viability constraints that the state of the system must obey at each instant under "death penalty",

- An inertia principle stating that the "regulees" of the system are changed only when viability is at stake.

The first two features are best summarized by the deeply intuitive statement attributed to Democritus by Jacques Monod: *"Everything that exists*

Table 1: States and Regulees in some Systems

Field	State	Regulee	Viability	Actor
Economics	Commodities	physical commodities	fiduciary	consumers and producers
biological evolution	phenotypes	genotypes	viability	biochemical mechanisms
sociology	behaviors of individuals	cultural codes	sociability	individuals
cogni-sciences	sensori-motor states	conceptual codes	adaptabi-lity	orga-nisms

in the Universe is due to Chance and Necessity". The inertia principle is a mathematical formulation of the concept of *punctuated equilibrium* introduced recently in paleontology by Elredge and Gould. It runs against the teleological trend assigning aims to be achieved (in even an optimal way) by the state of the system and the belief that actors regulee the system for such purposes.

— **Nondeterminism:** We shall mean by this term that there is a large variety of possible evolutions.

In our simple example below, there as many feasible evolutions as tax amounts.

In other words, *les jeux ne sont jamais faits*, in the sense that at each instant, there are several available, or feasible, evolutions which depend upon the state, or even the history of the evolution of the state of the system up to this time. Therefore, the concept of evolution borrowed from Newtonian mechanics is no longer adequate for such systems. It has led to the misleading identification of mathematics with a *deterministic* paradigm, which implies that *the evolution of macrosystems can be predicted*. Even if we were to accept the existence of deterministic mechanisms[2] underlying the evolution of biological, economic and social macrosystems, we know that such systems often can be inherently unstable - and this places the actual computation of their solutions beyond the capabilities of even the

[2]And now we discover that some of our "perfectly deterministic" models can exhibit all sorts of different trajectories. These are *chaotic* systems, making prediction virtually impossible.

most sophisticated of present-day computers! To "run" models which have some inbuilt structural instability can serve no useful purpose.

Thus, we suppose here that the dynamics responsible for the evolution are not deterministic. This lack of determinism has many different features: it may be due to nonstochastic "uncertainty"[3], to "disturbances"and "perturbations" of various kinds, or to errors in the replicating systems in the course of evolution.

In many instances, the dynamics of the system are related to certain "regulees", which, in turn, are restricted by state-dependent constraints (closed systems.)

The systems made of living being have often a propensity to create, maintain or enlarge their own variability, multiplying possibilities, combining them, crossing them. This si the polymorphism in population genetics, the investment in flexibility in economics, where the number of fiduciary commodities increases every day, testing and exploring behaviors, multiplying the assumptions in cognitive processes and matching them, etc.

— **Viability:** For a variety of reasons, not all evolutions are possible. This amounts to saying that the state of the system must obey constraints, called *viability constraints.*

In our example below, the commodity to be produced must meet a demand, production constraints and limitations of the pollution caused by the production process.

In other examples, these constraints include homeostatic constraints in biological regulation, scarcity constraints in economics, state constraints in regulee, power constraints in game theory, ecological constraints in genetics, sociability constraints in sociology, etc. Therefore, the goal is to select solutions which are *viable in the sense that they satisfy, at each instant, these constraints.*

Viability theorems thus yield selection procedures of viable evolutions, i.e., characterize the connections between the dynamics and the constraints for guaranteeing the existence of at least one viable solution starting from any initial state. These theorems also provide the *regulation processes*

[3]No a priori knowledge of an underlying probability law on the state of events is made. *Fuzzy viability* provides models where the available velocities can be ranked through a membership cost function to take into account that some velocities are more likely to be chosen than others.

(feedbacks[4]*)* that maintain viability, or, even as time goes by, *improve* the state according to some *preference relation.*

Nonetheless, selection through viability constraints may not be discriminating enough. Starting from any state at any instant, several viable solutions may be implemented by the system, including equilibria, which are stationary evolutions[5].

Thus further selection mechanisms need to be devised or discovered. We advocate here a third feature to which a selection procedure must comply, the *Inertia Principle.*

— **Inertia Principle:** Since we assumed that no actors (or too many of them) govern the evolution of the regulees, the Inertia Principle states that *"the regulees are kept constant as long as viability of the system is not at stake".*

As long as the state of the system lies in the interior of the viability set (the set of states satisfying viability constraints), any regularity regulee will work. Therefore, the system can maintain the regulee inherited from the past. This happens if the system obeys the inertia principle. Since the state of the system may evolve while the regulee remains constant, it may reach the viability boundary with an "outward" velocity. This event corresponds to a period of *crisis*: To survive, the system must find another regulatory regulee such that the new associated velocity forces the solution back inside the viability set. Alternatively, if the viability constraints can evolve, another way to resolve the crisis is to relax the constraints so that the state of the system lies in the interior of the new viability set. When this is not possible, *strategies for structural change fail*: by design, this means the solution leaves the viability set and "dies".

Naturally, there are several procedures for selecting a viable regulee when viability is at stake. For instance, the selection at each instant of the regulees providing viable evolutions with *minimal velocity* is an example that obeys

[4] thus providing the central concept of cybernetics as a *solution* to the regulation problem.

[5] This touches on another aspect of viability theory - that concerned with complexity and robustness: It may be observed that the state of the system becomes increasingly robust the further it is from the boundary of the viability set. Therefore, after some time has elapsed, only the parts of the trajectories furthest away from the viability boundary will remain. This fact may explain the apparent discontinuities ("missing links") and hierarchical organization arising from evolution in certain systems.

this inertia principle. They are called *"heavy" viable evolutions*[6] in the sense of heavy trends in economics.

Heavy viable evolutions can be viewed as providing mathematical metaphors for the concept of *punctuated equilibrium*[7] introduced recently in paleontology by Elredge and Gould.

When the viability constraints are not consistent with the uncertain dynamics of the system, viability theory establishes the existence of the **viability kernel**, which is the subset of the states from which starts at least one viable evolution. This concept happens to be fundamental, and its properties are actively being investigated and algorithms are devised.

In particular, the viability kernel of the macrosystems under constant regulee (or zero inflation) is called the **viability niche** of the regulee. A regulee the viability niche of which is not empty is called a **punctuated equilibrium**. It is an equilibrium if the viability niche is reduced to an element, which is then the equilibrium.

Viability niches are **locked-in** by heavy evolutions: if an evolution enters the viability niche of a punctuated equilibrium, it remains in it forever.

In a nutshell, *the main purpose of viability theory is to explain the evolution of a system, determined by given nondeterministic dynamics and viability constraints, to reveal the concealed feedbacks which allow the system to be regulated and provide selection mechanisms for implementing them.*

It assumes implicitly an "opportunistic" and "conservative" behavior of the system: a behavior which enables the system to keep viable solutions as long as its potential for exploration (or its lack of determinism) — described by the availability of several evolutions — makes possible its regulation.

[6]When the regulees are the velocities, heavy solutions are the ones with minimal acceleration, i.e., maximal inertia.

[7]Excavations at Kenya's Lake Turkana have provided clear evidence of evolution from one species to another. The rock strata there contain a series of fossils that show every small step of an evolution journey that seems to have proceeded in fits and starts. Examination of more than 3,000 fossils by P. Williamson showed how 13 species evolved. The record indicated that the animals stayed much the same for immensely long stretches of time. But twice, about two million years ago and then, 700,000 years ago, the pool of life seemed to explode — set off, apparently, by a drop in the lake's water level. Intermediate forms appeared very quickly, new species evolving in 5,000 to 50,000 years, after millions of years of constancy, leading paleontologists to challenge the accepted idea of continuous evolution.

Contrary to *optimal control theory*, viability theory does not require any single decision-maker (or actor, or player) to "guide" the system by optimizing an *intertemporal* optimality criterion[8].

Furthermore, the choice (even conditional) of the controls is not made *once and for all* at some initial time, but *they can be changed at each instant so as to take into account possible modifications of the environment of the system*, allowing therefore for *adaptation* to viability constraints.

Finally, by not appealing to intertemporal criteria, *viability theory does not require any knowledge of the future*[9] (even of a stochastic nature.) This is of particular importance when experimentation[10] is not possible or when the phenomenon under study is not periodic. For example, in biological evolution as well as in economics and in the other systems we shall investigate, *the dynamics of the system disappear and cannot be recreated.*

Hence, *forecasting or prediction of the future are not the issues which are not addressed by viability theory.*

However, the conclusions of the theorems allow us to reduce the choice of possible evolutions, or to single out impossible future events, or to provide explanation of some behaviors which do not fit any reasonable optimality criterion.

Therefore, instead of using intertemporal optimization[11] that involves the future, viability theory provides selection procedures of *viable evolutions* obeying, at each instant, state constraints which depend upon the *present or the past.* (This does not exclude *anticipations*, which are extrapolations of past evolutions, constraining in the last analysis the evolution of the system to be a function of its history.)

---

[8] the choice of which is open to question even in static models, even when multicriteria or several decision makers are involved in the model.

[9] Most systems we investigate do involve myopic behavior; while they cannot take into account the future, they are certainly constrained by the past.

[10] Experimentation, by assuming that the evolution of the state of the system starting from a given initial state for a same period of time will be the same whatever the initial time, allows one to translate the time interval back and forth, and, thus, to "know" the future evolution of the system.

[11] which can be traced back to Sumerian mythology which is at the origin of Genesis: one Decision-Maker, deciding what is good and bad and choosing the best (fortunately, on an intertemporal basis, thus wisely postponing to eternity the verification of optimality), knowing the future, and having taken the optimal decisions, well, during one week...

2 Example: Controlling production through anti-pollution taxes

We illustrate the concept of viable and heavy solution in the case of a simple dynamical model (production regulated by anti-pollution taxes). The only advantage of these crude models is to provide a graphical description of some results and to compare, *mutatis mutandis*, the evolutions of the production and of the tax.

Let $K := [a, b]$ be the subset of commodities x produced by a polluting process. The lower bound a represents the minimum production needed to satisfy the basic needs of consumers. The production process causing pollution, the upper bound b denotes the amount of production generating the maximal production tolerated. Hence the viability constraint impose the amount of production to evolve between these two bounds.

The velocity $g(x(t))$ with which production evolves is assumed to be positive. In this case, the production increases according to the differential equation $x'(t) = g(x(t))$ and will exceed the upper bound set by anti-pollution policy.

A mechanism is needed to slow down the production. The one we retain here penalizes the production velocity by a nonnegative tax $p(t)$: the actual production velocity is the difference between the above velocity and a function proportional to the tax. In other words, the dynamical system corrected by taxes is described by

$$x'(t) = g(x(t)) - h(x(t))p(t), \text{ where } p(t) \geq 0$$

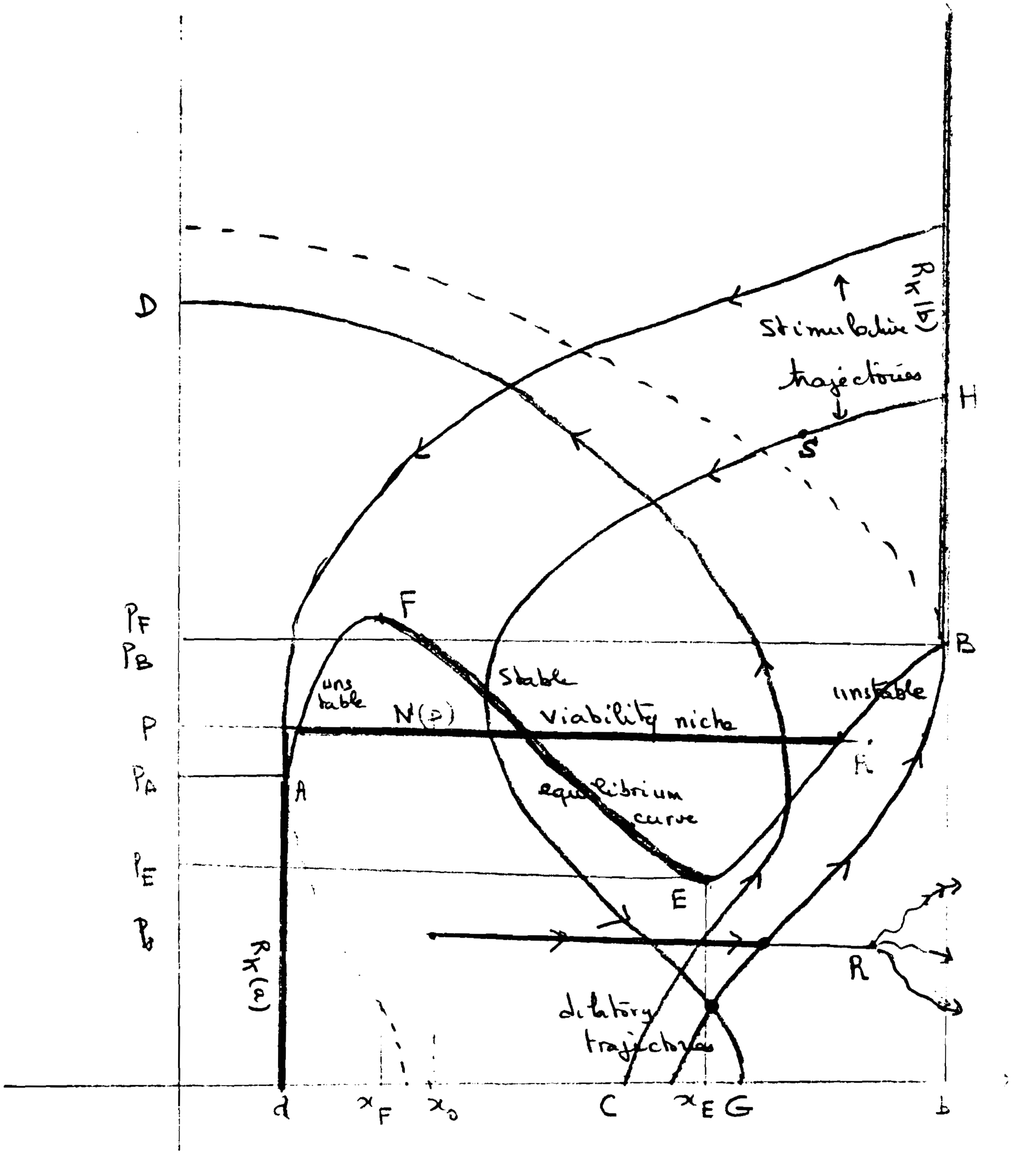
and where $h(x) \geq 0$.

2.1 Equilibria and Bifurcations

First, we can single out the taxes associated with equilibria: they are the ones for which the velocity vanishes (since an equilibrium is a stationary state). If we suppose that $\frac{f(x)}{g(x)}$ is defined for all $x \in K$, then $f : x \mapsto p = f(x)$ is the map associating with any production the tax for which it is an equilibrium. Its graph is called the **equilibrium curve**.

We are more interested by the inverse of this map, which associates with any tax the (possibly empty) set of equilibria for this tax.

In the case of figure 1, this set is empty if the tax is either smaller than p_E or larger than p_F. At p_E appears a **bifurcation**: there exists an equilibrium



D
H
S
Stimulative
trajectories
$R_K(b)$
F
P_F
P_B
B
unstable
Stable
viability niche
N(P)
P
P_A
A
equilibrium
curve
P_E
E
P_b
R
$R_K(a)$
dilatory
trajectories
a
x_F
x_0
C
x_E
G
b

x_E which gives birth to two equilibria when the tax lies between p_E and p_A. At p_A, a third equilibrium appears, and there are three equilibria in this example when p ranges over the interval $]p_A, p_B[$. We find again two equilibria when $p \in]p_B, p_F[$ and only one at p_F. In this example, there are four bifurcation taxes, p_E, p_A, p_B and p_F.

Below the equilibrium curve, one notices that the production velocities are always positive, and negative above. Hence the two increasing branches AF and EB of the equilibrium curve are made of **unstable equilibria**. Indeed, for a given tax, whether the initial production is slightly at the right or the left of the equilibrium, the production goes away from the equilibrium to the right and to the left respectively. Such equilibria are **repellers**.

Symmetrically, the decreasing branch FE of the equilibrium curve is made of **stable equilibria**. Indeed, for a given tax, whether the initial production is slightly at the right or the left of the equilibrium, the production goes towards to the equilibrium from the left and from the right respectively. Such equilibria are actually **attractors**, since the productions starting nearby converge monotonically to such equilibria.

Equilibria E and F are called **saddle points**: attractors in one direction, repellers in the other.

2.2 The Regulation Map

Actually, in an evolutionary perspective, we are not interested by equilibria and their stability, but by the evolution of both taxes and productions which comply the demand-versus-pollution constraint.

The (set-valued) **pricing map** Π_K which associates with production x the set of viable taxes: if the production is equal to a, a nonnegative velocity is required to reniew with growth, and this is possible with taxes ranging between 0 and p_A.

If the production belongs to the open interval $]a, b[$, then all taxes are possible because, for short periods, the production can increase or decrease without leaving the interval.

Finally, when the production is equal to b, nonpositive velocities are required to slow it down, and this happens with taxes larger than or equal to p_B.

In other words, the taxation map is defined by

$$\Pi_K(a) = [0, p_A], \ \Pi_K(x) = [0, \infty[\text{ when } \ a < x < b \ \& \ \Pi_K(b) = [p_B, +\infty[$$

This taxation map, being set-valued, is not deterministic: it leaves the possibility to errors. It imposes sufficiently low taxes when the lower bound is reached and sufficiently high when the upper bound is achieved.

The taxation map guides the evolution of taxes through the **regulation law**

$$\text{for all } t \geq 0, \quad p(t) \in \Pi_K(x(t))$$

It dictates what one must at least do to choose the taxes in order to comply the demand-versus-pollution constraints.

On can devise a multitude of "planning" mechanisms which are consistent with this regulation maps: **planning mechanisms** are selections $\pi(x) \in \Pi_K(x)$ of the taxation map used as **feedbacks** for which the differential equation

$$x'(t) = g(x(t)) - h(x(t))\pi(x(t))$$

have solutions (which are necessarily viable).

But using planning procedures is not the objective we pursue: we wish to show how the **inertia principle** works on this example.

2.3 Viability Niches

Consider an initial production-tax pair below the equilibrium curve. Then, keeping the tax constant, the velocity remains nonnegative and the production grows up.

At constant tax, the interval $[a, b]$ is not necessarily viable. The **viability kernel** of our dynamical system regulated by taxes is, by definition, the largest viability domain contained in the interval.

Take for instance a tax smaller than p_E. The production grows to the upper bound b which it reaches with a positive velocity, so that it violates the demand-versus-pollution constraint in **finite time**. The viability niche of such tax is thus empty. When the tax is p_E, the viability niche is reduced to the equilibrium x_E. When the tax lies between p_E and p_A, its viability niche is the interval the lower bound of which is a and the upper bound is the unstable equilibrium associated with this tax. The associated stable equilibrium is then an attractor of this system under constant tax in the niche.

When the tax ranges over $]p_A, p_B[$, its viability niche is the interval lying between the smallest unstable equilibrium and the largest unstable equilibrium, the stable equilibrium being an attractor of this system under constant tax. Between p_B and p_F, the viability niche is the interval whose lower bound is the unstable interval and the upper bound is b. Finally, for p_F, the viability niche is made of the equilibrium x_F.

If the initial production is outside of the viability niche of a given tax (which is always the case when its viability niche is empty), then the production evolving under constant tax shall violate the demand-versus-pollution constraint in finite time, either by traversing the upper bound if the tax is below the equilibrium curve, or by going below the lower bound a if the tax is above.

When the production reaches the upper bound b, one must increase at once the tax above the tax p_B prescribed by the taxation map. Or, if the tax is above the equilibrium tax, it must decrease ruggedly the tax below p_A.

A first lesson can be drawn from this example: a same recipe — maintain the tax constant — can produce strictly antagonistic consequences — increase of decrease production.

A second lesson is the danger caused by the inertia principle: when the production reaches the boundary of the interval, only discontinuous changes (impulses) can maintain the viability of the system.

2.4 Dilatory and Stimulative Solutions

When the initial production-tax pair is below the equilibrium curve and the initial production outside the viability niche of the tax, on can slow down production by increasing taxes at a fixed velocity, say equal to some $c \geq 0$ (in other words, the tax increases proportionally to time). Such solutions are called dilatory.

In this case, the production-tax pair ranges over a trajectory such as the curve CD[12]. This curve is increasing above the equilibrium curve, cuts vertically this equilibrium curve and is decreasing above. Starting from the

[12] These curves are the trajectories of the solutions $\rho_c^!$ to the differential equation

$$\frac{d}{dp}\rho_c^!(p) = \frac{g(\rho_c^!(p)) - h(\rho_c^!(p))p}{c} \tag{2.1}$$

As functions $p \mapsto x = \rho_c^!(p)$, they are increasing when the right-hand side is positive, i.e., below the equilibrium curve, vanish when they cross the equilibrium curve and decreasing above.

Indeed, when $p(t)$ is a solution to the differential equation $p'(t) = \pi(x(t), p(t))$, then the solution $x(t) := \rho(p(t))$ to the differential equation

$$x'(t) = g(x(t)) - h(x(t))p(t)$$

point C, both production and tax increase until the productions reaches its maximum and decreases, while the tax continues to grow up.

All the trajectories of dilatory production-tax pairs share the same behavior and do not cross.

We obtain the symmetric situation when the initial production-tax pair is below the equilibrium curve and the initial production is outside its viability niche. The production is accelerated by diminishing taxes at a constant velocity, say $-d$. Such solutions are called **stimulative**[13]. In this case, the production-tax pair ranges over a trajectory such as the curve GH. This curve is decreasing below the equilibrium curve, crosses it vertically and increasing above. Starting from H, both production and tax decrease until the production achieves its minimum and increases again, while the tax continuous to go down.

2.5 Bounded Tax Changes

One may reject constant tax scenarii as exceptional and thus, the rigid form of the inertia principle which may lead to brutal discontinuities (impulses) of taxes to maintain the viability of the system when the boundary of the interval is reached. But, when one thinks about it, it is not the inertia principle which has to be disregarded, but the primitive aspect of it, which does not forbid impulses, i.e., discontinuous mutations of taxes.

So, we add another assumption to this model: **taxes must respect speed limits**. In other words, one imposes **bounds to tax changes**: the tax velocity must remain between bounds $-d$ and $+c$. Therefore, the evolution of

---

satisfies

$$x'(t) = \left(\frac{d\rho(p)}{dp}\right) p'(t) \;=\; \left(\frac{d\rho(p)}{dp}\right) \pi(x(t), p(t)) = g(\rho(p(t))) - h(\rho(p(t)))p(t)$$

In the case of dilatory solutions $\rho_c^\sharp$, one takes $\pi(x,p) = c$.

[13] In the case of stimulative solutions $\rho_d^\flat$, one takes $\pi(x,p) = -d$. These curves are then the trajectories of the solutions $\rho_d^\flat$ to the differential equation

$$\frac{d}{dp}\rho_d^\flat(p) \;=\; -\frac{g(\rho_d^\flat(p)) - h(\rho_d^\flat(p))p}{d} \tag{2.2}$$

As functions $p \mapsto x = \rho_d^\flat(p)$, they are decreasing above the equilibrium curve, vanish when they cross it and increasing below.

production-tax pairs is governed by the system of differential inclusions

$$\begin{cases} i) & x'(t) = g(x(t)) - h(x(t))p(t) \text{ where } p(t) \geq 0 \\ ii) & \text{and } \; -d \leq p'(t) \leq +c \end{cases} \tag{2.3}$$

The model becomes more complicated to solve, and one cannot guess what is the **new** taxation map. On can envision that if the production is near the boundary with a tax such that the production-tax is below the equilibrium curve, it would be impossible to keep the production inside the interval without taxes violating the speed limit. This is because the old taxation map, valid for systems without speed limits, is no longer a taxation map for the system (2.3) completed by speed limits on taxes, i.e., under bounded tax change.

However, in this example, one can describe the new taxation map taking into account the bounded tax change[14].

For each production, the taxes must range between the part of the dilatory trajectory below the equilibrium curve which goes through B and the part of the stimulative trajectory above the equilibrium curve which goes through A[15].

2.6 Heavy Evolutions

A viable evolution is said to be **heavy** if the absolute value of the velocity of the viable tax is the smallest one.

[14] Its graph is the viability kernel of the rectangle $[a,b] \times [0,+\infty[$ for the system of differential inclusions (2.3)

[15] They are defined through the functions $\rho_c^\sharp$ and $\rho_d^\flat$ which are the solutions to differential equations (2.1) and (2.2) going through B and A respectively.

The equations of the upper curve is equal to

$$r_d^\flat(x) = p \text{ if and only if } x = \rho_d^\flat(p)$$

and the lower one is equal to

$$\begin{cases} r_c^\sharp(x) = 0 \text{ if } 0 \leq x \leq \rho_c^\sharp(0) \\ r_c^\sharp(x) = p \text{ if and only if } x = \rho_c^\sharp(p) \text{ when } \rho_c^\sharp(0) < x \leq b \end{cases}$$

The new taxation map associates with any production x the taxes p such that

$$r_c^\sharp(x) \leq p \leq r_c^\flat(x)$$

The growth properties of the functions r and ρ are exchanged, since one is the inverse of the other.

Let us begin by analyzing the case when the initial production-tax pair (x_0, p_0) is below the equilibrium curve, but above the dilatory curve going through B (which is the lower boundary of the graph of the new pricing rule).

If the initial production x_0 belongs to the viability niche of the initial tax p_0, we already saw that it converges to the stable equilibrium associated with p_0, and thus, a heavy solution *par excellence*. If not, when the initial production is outside the viability niche of p_0, the production increases while the tax remains constant, so that the trajectory of the production-tax pair is horizontal. This works until it hits the dilatory curve going through B (for a production smaller than the upper bound b). Then, and only then, the tax starts to evolve with the smallest velocity. A simple computation shows that there is only one viable tax velocity, equal to $+c$. The production-tax pair ranges over now the dilatory curve until the moment when it achieves the equilibrium B. Being an heavy solution, it remains at this equilibrium.

We thus built the heavy solution to the nondeterministic system (2.1). Hence, the region comprised between the dilatory curve going though B and the equilibrium curve is contained in the viability kernel of the system (2.3) under bounded tax change.

We also remark that the upper bound B is then locked-in by every heavy solution starting from a production-tax pair below the equilibrium pair and above the dilatory curve going through B, when x_0 does not belong to the viability niche of p_0 (otherwise, the stable equilibrium associated with p_0 is locked-in).

If the tax is maintained constant after crossing the dilatory curve going through B, it is impossible to maintain a viable production with an tax increase below c. Indeed, let $R := (\bar{x}, p_0)$ such a point. By braking the production by taking the highest tax velocity, the production-tax pair ranges over an dilatory curve below the one which passes through B, and which does not cross it. Therefore, the production reaches b at finite time with a positive velocity, since the tax is smaller than p_B. The solution is not viable. The situation is even worse for any other velocity $-d \leq p' \leq c$.

In other words, the speed limit on taxes conceals a warning signal to the agency deciding taxes, which must starts to increse taxes with the highest velocity as soon as the dilatory curve going through B is reached. Otherwise, they need a higher tax increase bound c_1 to remain viable (the tax increase c_1 is the one such that the pair $R := (\bar{x}, p_0)$ lies in the trajectory associated with the dilatory curve associated with $c_1 \geq c$ and going through B).

The situation is completely symmetric if the initial production-tax pair is above the equilibrium tax and below the stimulative curve going through A. Production goes down under constant tax until the production-tax pair hits the stimulative curve going through A. Hence the tax starts to decrease with velocity $-d$ until the production-tax pair reaches the equilibrium A, at which it remains forever.

We observe a **hysteresis phenomenon**: if the tax is too low, every heavy evolution starting outside its viability niche increases and reaches to the equilibrium B, and if the tax is too high, the heavy solution starting outside its viability niche reaches A.

We therefore checked that "viability" and "heaviness" suffice to determine a solution to the **nondeterministic** system (2.1).

2.7 Barrier Property

A theorem due to Marc Quincampoix states that the dilatory curve going through B and the stimulative curve going through A enjoy the barrier property: **once one of these curves is hit by a production-tax pair, the evolution of the production-tax pair cannot leave it until it reaches one of the equilibrium** A or B. In other words, on these two curves, the tax is doomed to be equal either to c or $-d$. Only heavy solutions can travel along these paths in order to maintain the viability of the system.

2.8 If an Agency Could Decide

If an agency could "really" command the evolution of taxes, he could design other evolutions than the heavy ones which either converge to a stable equilibrium or locks-in one of the two equilibria A or B.

It could, for instance, want to reach the equilibrium B at any cost. Assume for instance that it starts from the production-tax pair S (taken on the stimulative curve GH) above the equilibrium curve, where the production decreases. If the initial production does not belong to its viability niche, we saw that the heavy solution would arrive at the unwanted equilibrium A. In order to reach B, he can travel the stimulative curve GH with the tax velocity $-d$ as long as the production-tax pair does not reach the dilatory curve going through B. Despite tax reduction, the production continues to decrease until it reaches its minimum when the stimulative curve GH crosses the equilibrium curve, contradicting the static intuition that a stimulative policy would increase production. The production starts to increase only when the equilibrium curve is traversed, and then our agency must maintain during some time his stimulative policy for increasing production, until the production-tax pair reaches the dilatory curve going through B. Then, he must ruggedly change its policy by adopting an dilatory policy so that the production-tax pair persists to grow while traveling this dilatory curve which leads the production-tax pair to the desired equilibrium B.