Working Paper

Decentralized Ecodynamics: an Alternative View

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FOREWORD

It is known that the classical Walrasian tâtonnement is neither viable nor necessarily asymptotically stable. This failure can be overcome, one can find a way to get out of this impasse of the orthodox economic theory, if we agree to perform some modifications of the mathematical translation of the basic ideas and to change the questions, as we shall explain in this paper.

Instead of starting with a supply and demand law such as the Walrasian tâtonnement, even though it aggregates sensible individual behaviors, even though it is so seducing — prices goes up when demand exceeds supply and go down when supply exceeds demand — to be "believed" in, we propose to start with the actual behavior of consumers among which scarce resources have to be allocated and derive *what are the supply and demand laws which allow an allocation decentralized by the prices. The supply and demand law is no longer a primitive of the model, but a conclusion, in a sense made precise later: the supply and demand law shall emerge from the confrontation of the diverse wishes of the consumers and the scarcity of available commodities.*

The next problem we examine is then to select among such supply and demand laws compatible with the scarcity of constraints the ones which satisfy supplementary requirements. We shall suggest to chose the ones which satisfy the inertia principle, *in the sense that prices evolve only when scarcity is at stakes.*

Decentralized Ecodynamics: An Alternative View

Jean-Pierre Aubin & Jean Cartelier

1 The Issues

We shall devote this proposal to the simplest economic problem we can think of:

how to allocate scarce resources among consumers

by complying to the basic economic constraint

It is impossible to consume more physical goods than available

In other words, let us introduce the set of **allocations** of these scarce resources among the consumers. If M denotes the set of scarce resources, then the set of allocations of scarce commodities to n consumers[1]. is the set K of $x = (x_1, \ldots, x_n)$ satisfying $y = \sum_{i=1}^{n} x_i \in M$

This means that each consumer receives a commodity the sum of which is **viable** in the sense that the total consumptions is an available resource.

This problem looks at first glance somewhat silly and simple minded, since it amounts to pick up an element in this allocation set (i.e., to choose an allocation) in the case of static models, or to evolve in this set, regarded as a **viability set**, in the case dynamical systems. However, it elucidates the basic difficulties characteristic of economic theory, which has to explain the viability of allocation mechanisms, from "dictatorial" mechanisms where a planning bureau computes and imposes an allocation to more sophisticated

[1] "Exchange economies" are the particular case when $M := \{0\}$ is reduced to 0 or $M := -\mathbf{R}^l_+$ is the negative orthant. In this case, x denotes the net exchange. For such exchange economies, viability theorems are trivial.

mechanisms ("social rules" imposed by institutions) involving **decentralization** of decisions through prices, shortages, taxes, etc. which allows consumers to choose freely and independently their commodities in such a way that the scarcity of constraints is satisfied.

In the framework of a an allocation model decentralized by prices (and only by prices), our ambition is to present a mathematical metaphor of a mechanism for which one can characterize what are all the "supply and demand laws" regulating the evolution of prices which allow continuous transactions of consumers in a decentralized way. The fact that in a stationary environment such mechanisms converge to an equilibrium (stationary allocation) is an independent issue which is not the one which is primarily addressed, even though it became the main criterion of economic relevance.

Indeed, static models assign one or several elements in the allocation set. But it may be time to answer the wish J. von Neumann and O. Morgenstern expressed in 1944 at the end of the first chapter of their monograph "Theory of Games and Economic Behavior":

"Our theory is thoroughly static. A dynamic theory would unquestionably be more complete and therefore, preferable..."
"Our static theory specifies equilibria ... A dynamic theory, when one is found — will probably describe the changes in terms of simpler concepts."

We study here some mechanisms which govern the **evolution** of allocations of scarce resources[2]. In these dynamical models, the laws which govern the evolution of allocations are most often represented by differential equations (or differential inclusions) with or without memory (functional differential inclusions).

Static models are particular cases of (time-independent) dynamical models yielding "constant evolutions", which are also called "equilibria". By the way, the concept of equilibrium often covers two different meanings in economics:

1. The first one, the meaning we use in these lectures, is derived from mechanics, where an equilibrium is a constant evolution, or a "rest point" of an underlying dynamical system. In this case, equilibrium means **stationarity**.

[2]By the way, in dynamical models, we can assume that the subset of allocations evolves with time, and may also depend upon the history of the evolution. In this paper, the set M of allocations is assumed to remain constant, i.e., resources are not depleted by consumption.

When a solution of a dynamical system converges asymptotically to a limit, this limit is an equilibrium. The question arised conversely to obtain an equilibrium which is **asymptotically stable**, i.e., an equilibrium which is obtained as a limit of the solutions to the dynamical process starting nearby.

2. The second meaning is covered here by what we call the **viability constraints**, such as the total consumption must be less than or equal to the total supply, etc.

Stationarity — the first meaning of equilibrium — cannot be an issue whenever the behavior of consumers evolve with time and whenever the set of scare resources can be depleted by consumption or enlarged by production, technological advances and so on. These are reasons which led some of us to abandon such stationarity requirements.

However, in a time independent environment, stationarity may describe an aspect of satisfaction: when satisfied, no one has any reason to change the situation. But even *homo oeconomicus* may be frustrated most of the time[3].

But viability constraints — even fiduciary ones — cannot be violated, and we have to devise evolutionary models which provide consumption paths which respect them at each instant.

This is then the first issue which is dealt with by viability theory: find evolutions which obey at each instant the scarcity constraints. A second issue is, in a time independent environment, to prove whether among these solution one can find equilibria. A third issue is then to know whether some of these equilibria are asymptotically stable.

It is known that the classical Walrasian tâtonnement is neither viable nor necessarily asymptotically stable: For some of them, there may exist an equilibrium, even unique, which is not asymptotically stable. This failure can be overcome, one can find a way to get out of this impasse, if we agree to perform some modifications of the mathematical translation of the basic ideas and to change the questions, as we shall explain now.

Instead of starting with a supply and demand law such as the Walrasian tâtonnement, even though it aggregates sensible individual behaviors, even though it is so seducing — prices goes up when demand exceeds supply and go down when supply exceeds demand — to be "believed" in, we propose to

[3]If *homo oeconomicus* is not born at equilibrium — and thus, eternally happy — he only reaches blissfulness when he dies at very old — infinite — age!

start with the actual behavior of consumers among which scarce resources have to be allocated and derive what are the supply and demand laws which allow an allocation decentralized by the prices. The supply and demand law is no longer a primitive of the model, but a conclusion, in a sense made precise later: the supply and demand law shall emerge from the confrontation of the diverse wishes of the consumers and the scarcity of available commodities.

The next problem is then to select among such supply and demand laws compatible with the scarcity of constraints the ones which satisfy supplementary requirements. We shall suggest to chose the ones which satisfy the inertia principle, in the sense that prices evolve only when scarcity is at stakes.

2 Decentralization

2.1 Centralized and decentralized mechanisms

We begin by distinguishing between centralized and decentralized models. In the first category of models, consumers delegate their decision power to another "agent" who, knowing the behaviors of the consumers and the set of scarce resources, solves the problem at the global level.

For instance, consumers must agree to describe their behavior by a collective utility function

$$x := (x_1, \dots, x_n) \mapsto U(x) = U(x_1, \dots, x_n) \in \mathbf{R}$$

Then, this agent (planning bureau, big computers or big brothers, ...) knowing U and the subset M, decides to maximize U over the allocation set K: Find an allocation $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ such that $\sum_{i=1}^{n} \bar{x}_i \in M$ and

$$U(\bar{x}_1, \dots, \bar{x}_n) = \max_{\sum_{i=1}^{n} x_i \in M} U(x_1, \dots, x_n)$$

The problem is then transferred to the question of choosing the collective utility function U.

Or, in the dynamical version, they agree to represent their behavior by, say, a system of differential equations

$$x_i'(t) = h_i(x_1(t), \dots, x_n(t)) \quad (i = 1, \dots, n)$$

where the variations of the consumption of each agent depend upon the knowledge of **both** the whole set of scarce resources and the choices of every other agents. Nagumo's Theorem — the first viability theorem for ordinary differential equations published in German in 1943, and thus, forgotten and rediscovered at least fourteen times — provides necessary and sufficient conditions for the dynamics h_i to provide solutions to this system of differential equations satisfying the viability condition

$$\forall\, t \geq 0,\ \sum_{i=1}^{n} x_i(t) \in M$$

Such necessary and sufficient conditions can be regarded as "social" rules imposed on the consumers for respecting the scarcity conditions. Actually, there are many dynamical systems satisfying such rules[4].

In both cases, we deal with **centralized** models.

In a **decentralized mechanism**, the information on the problem is split and mediated by, say, a "message" which summarizes part of the information. In our case, we use the "price" p as a main example of message. Knowing the price p, consumers are supposed to know how to choose their consumption bundle, **without**

- knowing the behavior of their fellow consumers
- knowing the set of scarce resources

Then the problem is to find what is the message which carries the relevant information.

Actually, we have to ask whether it is possible to find such a relevant message and then, **how to find it**.

If it is possible to answer the first type of question, it is much more difficult to investigate the second, leaving such problems to mythical players such as Adam Smith's "invisible hand", etc. We shall bethink that these players are not really operating on the price system, which we shall propose

[4] which, for exchange economies, take the form

$$\sum_{i=1}^{n} h_i(x_1, \ldots, x_n) = 0$$

so that there are as many mechanisms than ways to allocate 0 in the form of velocities of consumers.

to regard as a **regulatory control** (a "regulee") to help the consumers to respect the scarcity constraints by delivering them proper informations on the behavior of all consumers and the set of available resources.

There are many other decentralized models, such as "rationing" mechanisms which involve shortages (and lines), or "frustration" of consumers, or "monetary" mechanisms, or others.

Naturally, there is no "pure" decentralization, since the choice of the decentralization message is in some sense centralized. The prices help consumer to make their choice in a decentralized way, but the difficulty is postponed to explain the evolution of price.

Decentralization has a meaning in the context of a hierarchical organization in two levels (at least). Furthermore, one can conceive "cascades" of decentralization mechanisms using several kinds of messages (prices, quantities, lines or queues, advertisement, rumors, etc.) at each level of more complex hierarchical organizations in Russian dolls or Chinese boxes. At each level, an adequate class of messages allows us to decentralize the level below through a specific "institution". Hierarchical decentralization allows to hide the ultimate difficulties for the explanation of the evolution of the messages of the upper level.

We shall proceed from now on by considering the mechanism of decentralization by prices only. This is the context in which Adam Smith, Léon Walras, Kenneth Arrow, Gérard Debreu and many others designed their contribution.

2.2 Adam Smith's Invisible Hand

Indeed, there is no doubt that Adam Smith is at the origin of what we now call **decentralization**, i.e., the ability for a complex system moved by different actions in pursuit of different objectives to achieve an allocation of scarce resources[5]. The difficulty to grasp such a disordered way of regulation of economic processes, contrary to apparently more logical (or simple minded?) attractive organizational processes based on several varieties of planning procedures[6] led him to express it in a poetic manner. Let us quote the

[5]With his compatriot David Hume, one can say that he is also at the origin of cybernetics and that he deeply influenced Charles Darwin. This is this cybernetical tradition that we chose to pursue here. In chapter 7 of his most famous book, he proposes an explicit dynamical process which is viable in spirit because transactions among agents are carried out at every step, contrary to the Walrasian tâtonnement.

[6]in favor among military organizational schemes.

celebrated quotation of his book AN INQUIRY INTO THE NATURE AND THE CAUSES OF THE WEALTH OF NATIONS published in 1776, two centuries ago:

"Every individual endeavours to employ his capital so that its produce may be of greatest value. He generally neither intends to promote the public interest, nor knows how much he is promoting it. He intends only his own security, only his own gain. And he is in this led by an **invisible hand** *to promote an end which was no part of his intention. By pursuing his own interest, he frequently thus promotes that of society more effectually that when he really intends to promote it"*

We had to wait a century more for Léon Walras, a former engineer, to propose that this invisible hand "operates" on economic agents through prices, gaining enough information on the desires of the agents and the available commodities for guaranteeing their consistency, or the viability of the allocation system.

He presented in 1874 the general equilibrium concept in ÉLÉMENTS D'ÉCONOMIE POLITIQUE PURE as a solution to a system of nonlinear equations. At that time, when only linear systems were understood, the fact that the number of equations was equal to the number of unknowns led him and his immediate followers to make the optimistic assumption that a solution should necessary exist[7].

2.3 Walras' Choice

In modern terms, the behavior of each consumer is described by a demand function $d_i(p,r)$ associating to a price p and an income r the i^{th} consumption $x_i = d_i(p,r)$ of Mrs. i. A demand function subsumes a **passive way** to choose automatically a commodity knowing the price and the income, in the sense that a dynamical system describes rather an **active way**, in which the consumers **changes** his consumption and thus, acting on the system, becoming an economic **agent** in the real sense of the word.

This is a decentralized mechanism, since Mrs. i ignores the behavior of her fellow consumers and the size and the nature of the set M of scarce resources.

A **price** p associates linearly with any commodity y its value denoted

[7]But it took another century, until 1954, for Kenneth Arrow and Gérard Debreu to find a mathematical solution to this problem. This solution, however, could not have been obtained without the fundamental Brouwer Fixed Point Theorem in 1910, which in turn required much modification to tailor it to this specific problem — by proving theorems whose assumptions could bear the same degree of economic interpretation as the conclusion.

by $\langle p, y\rangle$ (expressed in numbers of monetary units). It thus associates with the set M of available commodities its **total income** $\sigma_M(p) := \sup_{y\in M}\langle p, y\rangle$, which is the largest value of the available commodities y of M at price p. In Walrasian economies, it is assumed that this total income is allocated among consumers. Each consumer receives a share $r_i(p)$ of the total income $\sigma_M(p)$:

$$\forall\, p,\ \sum_{i=1}^{n} r_i(p) \ \le\ \sigma_M(p)$$

Once such an income allocation among consumers is made, Mrs. i chooses the commodity $x_i = d_i(p, r_i(p))$ knowing only the price p. The problem is then to find a price $\bar{p}$ (the Walrasian equilibrium price) such that $(d_1(\bar{p}, r_1(\bar{p})), \ldots, d_n(\bar{p}, r_n(\bar{p})))$ forms an allocation.

This is a **decentralized model** because consumers do not need to know neither the choices of other consumers nor the set M of available commodities.

The basic Arrow-Debreu Theorem states in this case that such an equilibrium exists whenever a **budgetary rule known as Walras law** — *it is forbidden to spend more monetary units than earned* — is obeyed by consumer's demand functions:

$$\forall\, i = 1, \ldots n,\ \ \forall\, p,\ \ \langle p, d_i(x, r)\rangle \le r$$

In other words, if the demand functions obey the Walras law, there exists at least a solution $\bar{p}$ to the allocation problem

$$\sum_{i=1}^{n} d_i(\bar{p}, r_i(\bar{p})) \ \in\ M$$

Such a solution $\bar{p}$ is called a **Walrasian equilibrium**.

Example The classical example of a Walrasian demand function is the one which is derived from the maximization of a consumer's utility function under the budgetary constraint $\langle p, x\rangle \le r$, in such a way that the Walras law is automatically satisfied. When a unique solution to such an optimization problem exists, it provides an example of a demand function satisfying the Walras law. If several solutions do exist, one has to extend the problem to the case when demand functions are replaced by set-valued demand maps. □

But why did Walras call it an equilibrium ? He had in mind an underlying dynamical process regulating the evolution of prices for which the solution $\bar{p}$ of the above problem is an equilibrium in the mechanical sense, i.e., for which $\bar{p}$ is a constant evolution. Furthermore, if the solution $p(t)$ of this dynamical system converges to some value $\bar{p}$, then this asymptotic price is an equilibrium. Therefore, such dynamical process can be regarded as a (continuous) algorithm to obtain an equilibrium: *"Voyons à présent comment ce même problème de l'échange de plusieurs marchandises entre elles, dont nous venons de trouver la solution scientifique, est aussi celui qui se résout empiriquement sur le marché par le mécanisme de la concurrence... Cela se fait après réflexion, sans calcul, mais exactement comme cela se ferait par le calcul en vertu du système des équations d'équivalence des quantités demandées et offertes et de la satisfaction maxima complété par les restrictions convenues. ... Que faut-il donc prouver pour établir que la solution théorique et la solution du marché sont identiques ? Tout simplement que la hausse ou la baisse sont un mode de résolution par* tâtonnement *du système des équations d'égalité de l'offre et de la demande."*, he wrote in the twelfth lecture of his ELÉMENTS D'ÉCONOMIE THÉORIQUE.

In our framework, the *"système des équations d'égalité de l'offre et de la demande"* is described by the **excess demand map** E associating with any price p the difference

$$E(p) := \sum_{i=1}^{n} d_i(p, r_i(p)) - M$$

between the total demand $\sum_{i=1}^{n} d_i(p, r_i(p))$ and the total supply (the set M of scarce resources).

The **Walrasian tâtonnement**[8], in its continuous version, is defined by the **differential inclusion**

$$p'(t) \in E(p(t))$$

Hence, according to this law of supply and demand, the price increases whenever the excess demand is positive and decreases in the opposite case.

Furthermore, a Walrasian equilibrium price $\bar{p}$ is indeed an equilibrium of this underlying dynamical process because

$$0 \in E(\bar{p})$$

[8]Tâtonnement means "tentative process", "trial and error" — literally, cumbersomely walking in obscurity by touch (tâter). The translation of the verbal description by Walras into the framework of differential equations has been made by Kenneth Arrow and Léon Hurwicz.

We observe that if $p(t)$ is a price supplied by the Walras tâtonnement process and if it is not an equilibrium, **it cannot be implemented** because the associated total demand $\sum_{i=1}^{n} d_i(p, r_i(p))$ is not necessarily available.

Hence, this model **forbids consumers to transact as long as the prices are not equilibria.** It is as if there was a super auctioneer calling prices and receiving offers from consumers. If the offers do not match, he calls another price according to the above dynamical process, but **does not allow transactions to take place** as long as the offers are not consistent, and this happens only at equilibrium!.

Tâtonnement is therefore not viable.

But the question which was asked despite this drawback is the convergence of solutions $p(t)$ of the above tâtonnement process to a limit, which then is an equilibrium. This issue, as well as either the Lyapunov stability[9] or the asymptotic stability[10] of a Walrasian equilibrium, require the use of adequate Lyapunov functions (introduced by Lyapunov in 1892) which decrease along the price paths $p(t)$ in such a way that one can derive the convergence of $p(t)$ when t goes to infinity. For example, an excess demand satisfying the property

$$\forall\, p \neq \bar{p}, \ \left\langle p - \bar{p}, \sum_{i=1}^{p} d_i(p, r_i(p)) \right\rangle \ < \ \sigma_M(p - \bar{p})$$

is shown to converge to an equilibrium[11]. It had been difficult to provide

[9]This means that for any neighborhood of the equilibrium, there exists a smaller neighborhood from which the solutions remain in the given neighborhood.

[10]The means that there exists a subset, called the basin of attraction, from which the solutions converge to the equilibrium.

[11]Each differentiable Lyapunov function $p \mapsto V(p)$ entails a condition on the excess demand map because the property

$$V(p(t)) \ \leq \ V(p(0))e^{-\alpha t}$$

holds for any price path if and only if

$$\left\langle V'(p), \sum_{i=1}^{p} d_i(p, r_i(p)) \right\rangle + \alpha V(p) \ \leq \ \sigma_M(V'(p))$$

The function $V(p) := \frac{1}{2}\|p - \bar{p}\|^2$ provides the above condition. Another (nondifferentiable) Lyapunov function proposed in the economic literature has been $V(p) := \max_{i=1,\ldots,n} \frac{p_i}{\bar{p}_i}$. (Lyapunov method can be extended to any lower semicontinuous function!) Observe however a satisfying economic interpretation to such additional assumptions (to the Walras law). Furthermore, Scarf proposed in 1960 an example of excess demand function for which there exists a unique equilibrium which basin of attraction is empty. Hence, convergence of the Walras tâtonnement holds only for restrictive classes of tâtonnement processes.

On the other hand, Debreu, Mantel, Sonnenschein, among many others, showed that any continuous function can be regarded as an excess demand function for some underlying Walrasian economic model, destroying the hope that **every possible tâtonnement process** should converge to an equilibrium.

Finally, it may be too much to ask the entity which regulates the price (the market, the invisible hand, the Gosplan, ...) to behave as a real decision-maker whereas consumers act passively according to their demand functions.

For these reasons, and also because these issues are treated in depth in may other books and papers, we shall let aside the Walrasian tâtonnement, the existence of a Walrasian equilibrium and the asymptotic theorems.

However, the legitimate admiration that Léon Walras deserves should not imply a dogmatic respect of his contribution by his followers: the equilibrium concept was a simplifying step in the attempt to grasp some essential economic feature in an otherwise complex maze of concepts. This concept had its use, as a first approximation, despite the fact that it rarely happened in economic history. The Walrasian tâtonnement still keeps its attractiveness among some economists despite its failure to explain the achievement of a (viable) allocation, let it be an equilibrium.

So, its **dépassement**, as well as the observation that the Walrasian tâtonnement is not viable and should be replaced by a viable dynamical system, should not be regarded by the faithfuls as a crime of **lèse majesté**. On the other hand, these shortcomings should not be used to claim that any decentralized mechanism using prices is merely a fantasy dreamed by theoreticians from their ivory towers — an "empty box", as it has been written — and even, to reject the relevance of mathematical metaphors in economics[12].

that for any desired Lyapunov function V, a modified tâtonnement of the form

$$p'(t) \in E(V'(p(t)))$$

satisfies $t \mapsto V(p(t))$ does not increase under the usual Walras law.

[12]This is a typical instance of impatience and the totalitarian desire for monist explanations.

2.4 Nontâtonnement Models

Since the Walrasian tâtonnement is not viable, many authors proposed nontâtonnement dynamical processes (in particular, Arrow-Hahn, Negishi, Smale, Uzawa, among many others).

Here, both the consumers act on their consumptions and Adam Smith's invisible hand act on the prices, according to a system of differential equations of the form

$$\begin{cases} \forall\, i = 1,\ldots,n, \ \ x_i'(t) \ = \ f_i(x_1(t),\ldots,x_n(t),p(t)) \\ p'(t) \ = \ g(x_1(t),\ldots,x_n(t),p(t)) \end{cases}$$

Second, this dynamical system must yield viable solutions, in such a way that at each instant, consumers may transact and consume available resources.

In such models, the price dynamics described by g is regarded as a supply and demand law, assumed to describe how the invisible hand governs the evolution

But, in order to keep the consumptions viable, the Nagumo theorem requires to assume conditions on the rules f_i which:

1. guarantee the viability for all prices, so that such nontâtonnement models can no longer provide explanations of the role of prices in decentralization mechanisms,

2. and are not decentralized because the maps f_i have to depend upon the consumptions of the other consumers.

These requirements can be regarded as "rationing" conditions imposed on the consumers for respecting the scarcity conditions, providing "nonprice" decentralization mechanisms.

Actually, the issues studied by the students of nontâtonnement mechanisms is the asymptotic convergence of the solutions to Pareto optimal allocations (which are Walrasian equilibria for adequate allocations of the total income).

One naturally may add to the price mechanism any "rationing" mechanism, which is valid any time that decentralization by prices only is not possible. But before complying to that strategy, one may ask whether this is due to the fact that the supply and demand law described by the dynamics g is imposed *a priori* independently of the behavior of the consumers and the set M of scarce resources. We shall attempt to answer positively this question by characterizing the family of supply and demand laws which are consistent with the behavior of the consumers and the set of scarce resources. In other words, to answer the questions whether one can find price dynamics $g(x_1, \ldots, x_n, p)$ which can regulate the evolution of prices in such a way that visible consumers modify continuously their consumption in a decentralized way.

2.5 Visible Consumers and Viable Processes

We suggest to go a step further from the Walrasian tâtonnement by

1. conserving the dynamic (active) behavior of consumers as in nontâtonnement models, but in a decentralized way,
2. abandoning any *a priori* supply and demand rule regulating the evolution of prices.

In other words, instead of taking a supply and demand rule as a primitive of the model as it is done in both tâtonnement and nontâtonnement models, we shall derive them (and compute them) from the knowledge of the decentralized dynamics describing the behavior of the consumers and the knowledge of the set M of scarce resources.

It may be wise indeed to let the real decision-makers, the consumers in our case, to govern the evolution of their consumption through differential equations

$$x_i'(t) = c_i(x_i(t), p(t))$$

parametrized (or controlled) by the price $p(t)$, so that consumers change their consumptions knowing only the price $p(t)$ at each time t, without taking into account neither the behavior of the other consumers nor the knowledge of the set M of scarce resources.

Hence, a consumer is an economic agent, whose dynamical behavior is described by the function $(x, p) \mapsto c_i(x, p)$, called change function[13].

Hence it shares with the Walrasian static model its decentralization property.

The problem is then to find a time-dependent price $p(t)$ such that the associated solutions $x_i(t)$ of the above differential equations do form an allocation

[13]Once can associate with any change function c_i a demand (set-valued) map D_i defined by $x \in D_i(p)$ if and only if $c_i(x, p) = 0$

Table 1: Comparison between Walrasian tâtonnement and viable process

Process:	**Walrasian**	**Viable**
Description of the behavior of consumers	demand functions $d_i(x, r)$ $x_i = d_i(x_i, r_i(p))$	change functions $c(x_i, p)$ $x_i'(t) = c_i(x_i(t), p(t))$
Derivation from utility function	$d_i(p, r)$ maximizes U_i under $\langle p, x\rangle \leq r$	$c_i(x, p)$ $= U_i'(x) - p$ $= \partial \left(U_i\|_{\langle p,x\rangle \leq r}\right)(x)$
Equilibrium: stationarity and (static) viability	$\forall\, i,\ \bar{x}_i = d_i(x_i, r_i(\bar{p}))$ such that $\sum_{i=1}^{n} \bar{x}_i \in M$	$\forall\, i,\ c_i(\bar{x}_i, \bar{p}) = 0$ such that $\sum_{i=1}^{n} \bar{x}_i \in M$
Budget rule	$\langle p, d_i(x, p)\rangle \leq r$	$\langle p, c_i(x, p)\rangle \leq 0$
(dynamic) Viability		$\exists\ p(t)$ such that $\sum_{i=1}^{n} x_i(t) \in M$
Characterization of the viability		$\forall\ (x_1, \ldots, x_n)$, $\Pi_M(x_1, \ldots, x_n) \neq \emptyset$
Regulation law		$p(t) \in \Pi_M(x_1(t), \ldots, x_n(t))$
Supply and Demand	$p'(t) \in \sum_{i=1}^{n} d_i(p(t)) - M$	$p'(t) \in G_M(x_1(t), \ldots, x_n(t), p(t))$

at each time t:

$$\forall\, t \geq 0,\ \sum_{i=1}^{n} x_i(t) \in M$$

We prove that this viability property holds true under a budget rule which is a dynamical version of the static Walras law. We shall even prove under the same assumption the **existence of a viable equilibrium** $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ of this dynamical system, which is a solution to

$$\begin{cases} \forall\, i = 1, \ldots, n, & c_i(\bar{x}_i, \bar{p}) = 0 \\ \sum_{i=1}^{n} \bar{x}_i \in M \end{cases}$$

Example As in classical Walrasian economics, one can derive change functions from the consumer's wish to increase its utility under the budgetary constraint $\langle p, x\rangle \leq r$. The idea then is to use the steepest ascent method to the restriction of the utility function U to the budget set (which is no longer differentiable). Using nonsmooth analysis, i.e., replacing gradients by generalized gradients, we may take

$$c_i(x,p) := \partial\left(U_i|_{\langle p,x\rangle \leq r}\right)(x) = U_i'(x) - p$$

(when the utility function U_i is assumed to be differentiable. Otherwise, one can take the generalized gradient of U_i, and obtain a set-valued change map. We thus have to replace differential equations by differential inclusions to describe the behavior of consumers, and this is possible.)

Actually, under adequate convexity assumptions, the static maximization mechanism: Find an allocation $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ such that $\sum_{i=1}^{n} \bar{x}_i \in M$ and

$$\sum_{i=1}^{n} U_i(\bar{x}_i) = \max_{\sum_{i=1}^{n} x_i \in M} \sum_{i=1}^{n} U_i(x_i)$$

providing Pareto-optimal allocations conceals the three kind of algorithms: a Walrasian tâtonnement, a nontâtonnement process and a viable one. Indeed, convex analysis provides three characterizations of such an optimal solutions $(\bar{x}_1, \ldots, \bar{x}_n)$ and its Lagrange multiplier $\bar{p}$:

1. $(\bar{x}_1, \ldots, \bar{x}_n)$ maximizes the restriction of the function $\sum_{i=1}^{n} U_i$ to the set of allocations
2. $\bar{p}$ minimizes the dual problem
3. $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is a saddle point of the associated Lagrangian

The (generalized) gradient method of each of the above problems are

1. a viable process

$$\begin{cases} x_i'(t) = U_i'(x(t)) - p(t) \\ \text{where} \\ p(t) \in N_M(x_1(t), \ldots, x_n(t)) \end{cases}$$

2. a nontâtonnement process of the form

$$\begin{cases} x_i'(t) = U_i'(x(t)) - p(t) \\ \text{where} \\ p'(t) \in \sum_{i=1}^{n} x_i(t) - S_M(p(t)) \end{cases}$$

(where the supply map S_M associates with any price p the set of resources yielding the maximal income)

3. a Walrasian tâtonnement process

$$p'(t) \in E(p(t))$$

where $E(p) = \sum_{i=1}^{n} D_i(p) - S_M(p)$ is the excess demand when $D_i(p)$ denotes the demand map derived from the utility function.

The Crandall-Pazy theorem implies the existence of solutions to each one of these three mechanisms (since those three dynamics are maximal monotone) and the convergence of both the viable process and the Walrasian tâtonnement, since the dynamics are (generalized) gradient methods. This provides at least instances where the Walras tâtonnement converges, offering positive examples countering Scarf's counter-example. □

Actually, we shall prove that the budgetary rule is just a sufficient condition, easy to implement for the viability of such a decentralized dynamical system and the existence of an equilibrium.

The viability theorem allows us to characterize the above viability property in terms of a **pricing map** Π_M associating with any allocation $(x_1, \ldots, x_n)$ a subset $\Pi_M(x_1, \ldots, x_n)$ of prices. This pricing map is built from the knowledge of M and the change functions c_i of the consumers.

The basic viability theorem implies that *the above dynamical system is viable if and only if for any allocation* $(x_1, \ldots, x_n)$*, the subset* $\Pi_M(x_1, \ldots, x_n)$ *is not empty. In this case, the existence of an equilibrium is also guaranteed.*

Furthermore, the prices which govern the evolution of allocations evolve according the (set-valued) "regulation law"

$$\forall\, t, \;\; p(t) \;\in\; \Pi_M(x_1(t), \ldots, x_n(t))$$

The above law provides at each instant the set of "viable" prices which govern the evolution of allocations of scarce resources, stating *a posteriori* how the invisible hand should choose the prices (such prices can be regarded as "open loop" controls in the terminology of systems theory).

Once this has been established, one can raise other questions. For instance, whether the allocations $(x_1(t), \ldots, x_n(t))$ converges asymptotically to a limit $(\bar{x}_1, \ldots, \bar{x}_n)$, which is then an allocation. Whether this convergence is at each instant Pareto-improving. This can be solved, each further demand bridling the initial regulation map Π_M, replacing it by a smaller one which associates the set of prices $p(t)$ providing allocations satisfying these supplementary demands.

3 Deriving Decentralizing Supply and Demand Laws

The above regulation law is not yet a "supply and demand" law, because it does not tell how the velocities of prices vary. By supply and demand law, we mean here a dynamical process of the form

$$p'(t) = g(x_1(t), \ldots, x_n(t), p(t))$$

as in nontâtonnement models[14].

Naturally, if the pricing map $\Pi_M = \pi_M$ were actually single-valued and differentiable, the chain rule allows us to derive from the regulation law

$$\forall\, t,\; p(t) = \pi_M(x_1(t), \ldots, x_n(t))$$

the law

$$\forall\, t,\; p'(t) = \pi'_M(x_1(t), \ldots, x_n(t))(c_1(x_1(t), p(t)), \ldots, c_n(x_n(t), p(t)))$$

which can be regarded as a supply and demand law. For such a law, the nontâtonnement system

$$\begin{cases} \forall\, i = 1, \ldots n,\; x'_i(t) = c_i(x_i(t), p(t)) \\ p'(t) = \pi'_M(x_1(t), \ldots, x_n(t))(c_1(x_1(t), p(t)), \ldots, c_n(x_n(t), p(t))) \end{cases}$$

[14] for which we do not require *a priori* that price goes up when demand exceeds supply. One can not exclude that, for certain economies where the set M of resources is not a product, prices of certain commodities interfere with the modification of prices of other commodities in order to maintain continuous transactions (the viability condition).

yields at each instant allocations of scarce resources (in a decentralized way).

When the pricing map is set-valued, the differential calculus of set-valued maps[15] allows also to differentiate the regulation law. It is possible to define a concept of derivative of set-valued maps, called **contingent derivative**. Denote by $\Pi'_M(x_1, \ldots, x_n, p)$ the contingent derivative of the pricing map[16] Π_M at a point $(x_1, \ldots, x_n, p)$ of its graph, which is still a set-valued map associating with marginal commodities $(v_1, \ldots, v_n)$ a set

$$\Pi'_M(x_1, \ldots, x_n, p)(v_1, \ldots, v_n)$$

of marginal prices.

From now on, we shall set

$$G_M(x_1, \ldots, x_n, p) := \Pi'_M(x_1, \ldots, x_n, p)(c_1(x_1, p), \ldots, c_n(x_n, p))$$

One can prove a "set-valued chain rule" which allows to differentiate the regulation law and yields

$$\forall\, t,\ p'(t) \in G_M(x_1(t), \ldots, x_n(t), p(t))$$

We shall regard this differential inclusion as a "Supply and Demand Map" specifically designed to guarantee the viability of the decentralized nontâtonnement mechanism

$$\begin{cases} \forall\, i = 1, \ldots n, \ x'_i(t) = c_i(x_i(t), p(t)) \\ p'(t) \in G_M(x_1(t), \ldots, x_n(t), p(t)) \end{cases}$$

The set-valued map $G_M(x_1, \ldots, x_n, p)$ is a kind of envelope of all single-valued supply and demand maps $g(x_1, \ldots, x_n, p)$ for which the **decentralized nontâtonnement process**

$$\begin{cases} \forall\, i = 1, \ldots n, \ x'_i(t) = c_i(x_i(t), p(t)) \\ p'(t) = g(x_1(t), \ldots, x_n(t), p(t)) \end{cases}$$

provides at each instant viable allocations of scarce resources.

[15] which was motivated by this very problem in the first place.

[16] Actually, we shall differentiate a "smaller" regulation map which governs the evolution of almost everywhere differentiable prices. This a technical issue which is too long to explain here.

In the language of viability theory, such selections g of G_M are called "dynamical closed loops".

The question arises how to find such selection procedures, and among them, to find the ones which retain some economic meaning.

We shall now advocate the ones which satisfy the **inertia principle**.

4 The Inertia Principle and Heavy Evolutions

Actually, if the behavior of the consumers is well defined, what about either the market or the planning bureau, the task of which is now to find the prices $p(t)$ in $\Pi_M(x_1(t), \ldots, x_n(t))$? They do not behave as actual decision makers, knowing what is good or not (this is the case of even a planning bureau as soon as it involves more than three bureaucrats!). Hence, their role is only a **regulatory one**. If they are not able to optimize, we may assume that they only are able to correct the prices when the viability of the economic system is at stake, i.e., when the total consumption is no longer available.

Hence, we assume that the Adam Smith's "invisible hand" or the planning bureau are able to "pilot" or "act" on the system by choosing such controls according to the **inertia principle:**

> **Keep the price constant as long as the evolution provides allocations of available resources, and change them only when the viability is at stakes.**

Indeed, as long as the state of the system lies in the interior of the allocation set (the set of states satisfying scarcity constraints), any price will work. Therefore, the system can maintain the price inherited from the past. This happens if the system obeys the inertia principle. Since the allocations may evolve while the price remains constant, the total consumption may reach the boundary of the set of scarce resources with an "outward" velocity. This event corresponds to a period of *crisis*: To survive, the system must find another price such that the new associated velocity forces the solution back inside the allocation set. Alternatively, if the scarcity constraints can evolve, another way to resolve the crisis is to relax the constraints (by technological progress, for instance) so that the state of the system lies in the interior of the new allocation set. When this is not possible, **strategies for structural change fail**: by design, this means that the solution leaves the allocation set and "dies".

This management by crisis or **bankruptcy** has been observed in economic history, so that we suggest to take these phenomena into account in the framework of this Inertia Principle[17]. Crisis could be defined here as either discontinuity of the price evolution or "fast" evolutions of prices.

The extreme form of the inertia principle leads to the question whether one can obtain the evolution of allocations of scarce resources **under a constant price.** We shall say that a price $\widehat{p}$ is a **punctuated equilibrium** if there exists a nonempty subset $N(\widehat{p})$ of allocations, called its **viability niche,** which is viable for a price:

$$\begin{cases} \forall\,(x_1^0,\ldots,x_n^0) \in N(\widehat{p}), \\ \text{the solutions to the system} \\ \forall\, i = 1,\ldots n,\;\; x_i'(t) \;=\; c_i(x_i(t),\widehat{p}) \\ \text{remain in the viability niche } N(\widehat{p}) \end{cases}$$

One can expect that the viability niches of most of the prices are empty. Naturally, **when the viability niche of a punctuated equilibrium is reduced to an unique allocation, this allocation is an equilibrium.**

Therefore, we have to select "supply and demand" functions (dynamical closed loops) $g(x_1,\ldots,x_n,p)$ which obey the inertia principle, thus providing rules for choosing prices when viability

is at stakes in order to obtain allocations of scarce resources.

The simplest one (and most often, the most reasonable one) is to assume that at each instant, the prices are changed as slowly as possible. This is obtained by taking for map g the map ϖ_M defined by:

$$\begin{cases} \varpi_M(x_1,\ldots,x_n,p) \;\in \Pi'_M(\varpi_M(x_1,\ldots,x_n,p))(c_1(x_1,p),\ldots,c_n(x_n,p)) \\ \text{is the element of minimal norm in} \\ \Pi'_M(\varpi_M(x_1,\ldots,x_n,p))(c_1(x_1,p),\ldots,c_n(x_n,p)) \end{cases}$$

[17] This Inertia Principle provides an explanation of the concept punctuated equilibrium introduced in 1972 by Elredge and Gould in paleontology. Excavations at Kenya's Lake Turkana have provided clear evidence of evolution from one species to another. The rock strata there contain a series of fossils that show every small step of an evolution journey that seems to have proceeded in fits and starts. Examination of more than 3,000 fossils by P. Williamson showed how 13 species evolved. The record indicated that the animals stayed much the same for immensely long stretches of time. But twice, about two million years ago and then, 700,000 years ago, the pool of life seemed to explode — set off, apparently, by a drop in the lake's water level. Intermediate forms appeared very quickly, new species evolving in 5,000 to 50,000 years, after millions of years of constancy, leading paleontologists to challenge the accepted idea of continuous evolution.

It obviously satisfies the inertia principle because if the velocity 0 of the price is available, it is picked by such a selection procedure, so that the price remains constant as long as 0 belongs to $G_M(x_1(t),\ldots,x_n(t),p(t))$.

Evolutions obeying this specific choice are called "heavy[18] evolutions", in the sense of heavy trends. Hence heavy evolution is obtained by requiring at each instant the (norm of the) velocity of the price to be as small as possible.

Heavy solutions enjoy the property of "locking-in" punctuated equilibria.

Indeed, assume that for some instant $t_f > 0$, $p(t_f)$ is a punctuated equilibrium, then $p(t) = p_{t_f}$ for all $t \geq t_f$ and thus, $(x_1(t),\ldots,x_n(t))$ *remains forever in the viability niche of this punctuated equilibrium (and thus, an allocation of scarce resources).*

Therefore, for implementing this inertia principle, we have to provide conditions under which relevant prices $p(\cdot)$ are differentiable (almost everywhere), to built the differential inclusion which governs the evolution of differentiable relevant prices and then, select a differential equation in this differential inclusion (called a "dynamical closed loop") which will obey the inertia principle.

In summary, **given the decentralized behavior of the consumers described by the differential equations $x_i' = c_i(x_i,p)$ and the set of scarce resources, we can built the dynamics ϖ_M governing the behavior of the market, so that the** evolution of the economic system is described by the system of differential equations

$$\begin{cases} i) & x_i'(t) = c_i(x_i(t),p(t)) \quad (i=1,\ldots,n) \\ ii) & p'(t) = \varpi_M(x(t),p(t)) \end{cases}$$

Contrary to nontâtonnement models, this law governing the evolution of prices is not a modeling assumption, but a consequence of the modeling data of this elementary model satisfying the inertia principle.

In summary, we assume implicitly that the "invisible hand" follows an "opportunistic" and "conservative" behavior of the system: a behavior which enables the system to allocate scarce resources among consumers as long as any price makes possible its regulation and to keep this price as long as it is possible.

[18]This is justified by the fact that the velocity of the price is related to the acceleration of the consumptions, which, being minimal, has the maximal inertia.

We then attempt to explain the evolution of allocations and prices and to reveal the concealed feedbacks which allow the system to be regulated by prices.

5 Planning Procedures

Another way to find prices satisfying the regulation law is to obtain them by planning procedures.

This means that the planning bureau has to associate with any allocation $(x_1, \ldots, x_n)$ a price $\pi(x_1, \ldots, x_n)$ which it sends back to the consumer.

The viability theorem states that whenever the map

$$(x_1, \ldots, x_n) \mapsto \pi(x_1, \ldots, x_n)$$

is a selection of the pricing map Π_M in the sense that

$$\forall\,(x_1, \ldots, x_n),\ \ \pi(x_1, \ldots, x_n) \ \in\ \Pi_M(x_1, \ldots, x_n)$$

then the evolution of consumption according

$$\forall\, i = 1, \ldots, n,\ \ x_i'(t) \ =\ c_i(x_i(t), \pi(x_1(t), \ldots, x_n(t)))$$

yields allocations of scarce resources.

Planning procedures introduced by Drèze, de la Vallée-Poussin, Malinvaud, etc. fall in this category, and can be interpreted as closed loop controls.

One can obtain such selection by static optimization techniques (or, more generally, by game theoretical methods, or any other kind of technique). For instance, we can choose the element $\pi^0(x) \in \Pi(x)$ of minimal norm. Despite the lack of continuity of such a selection, we still can prove that the system of differential equations

$$x_i'(t) \ =\ c_i(x_i(t), \pi^0(x_1(t), \ldots, x_n(t)))$$

has viable solutions, which are called "slow allocations". However, this type of selection may not enjoy economic meaning, contrary to the heavy allocations we proposed earlier.

These selection procedures are myopic contrary to intertemporal optimization mechanisms.

Indeed, in the dynamical case, selection procedures split in two categories: we have to distinguish between "intertemporal optimization" problems and "myopic or instantaneous optimization" problems.

In intertemporal optimization, we maximize intertemporal utility functions of the form

$$U(x(\cdot),p(\cdot)) := \int_0^T u(t,x(t),p(t))dt + v(x(T),p(T))$$

under the constraint $(x(\cdot),p(\cdot)) \in \text{Graph}(\Pi)$.

These are questions with which Calculus of Variations and Optimal Control Theory deal with.

But Optimal Control Theory does require Adam Smith's invisible hand to "guide" the system by optimizing such an intertemporal optimality criterion, the choice of which is open to question even in static models, even when multicriteria or several decision makers are involved in the model.

Furthermore, the choice (even conditional) of the controls is made once and for all at some initial time, so that they cannot be modified at each instant so as to take into account possible modifications of the environment of the system, forbidding therefore adaptation to scarcity constraints.

Finally, intertemporal optimization viability theory does require knowledge of the future (even of a stochastic nature.) This requires the possibility of experimentation or the belief that the phenomenon under study is periodic. Experimentation, by assuming that the evolution of the state of the system starting from a given initial state for a same period of time will be the same whatever the initial time, allows one to translate the time interval back and forth, and, thus, to "know" the future evolution of the system.

But in economics, as well as in biological evolution, experimentation is not possible[19]. Furthermore, the dynamics of the system disappear and cannot be recreated. Most economic systems do involve myopic behavior; while they cannot take into account the future, they are certainly constrained by the past.

Hence, forecasting or prediction of the future are not the issues which we shall address here. *La prévision est un rêve duquel l'événement nous tire,* wrote Paul Valéry.

[19]The twentieth century Soviet type (or military type) economic experimentation showed experimentally the limits of centralized operation of complex systems.

Therefore, instead of using intertemporal optimization[20] that involves the future, we shall advocate here myopic selection procedures, either by dynamical closed loops (supply and demand maps) or by closed loop prices (planning procedures), for providing selection procedures of **viable evolutions** obeying, at each instant, scarcity or more generally, viability constraints which depend upon the **present or the past**. (This does not exclude **anticipations**, which are extrapolations of past evolutions, constraining in the last analysis the evolution of the system to be a function of its history.)

Viability theory deals with "dynamics under constraints", playing the role of "optimization under constraints" or "existence of equilibrium under constraints" in a dynamical manner. The viability theorem, which is an extension of Nagumo's theorem to differential inclusions, actually plays the role of Lagrange's or Kuhn-Tucker's theorem. Both use the concept of tangent cones to "differentiate" the constraints, so to speak, and by duality, involve the concepts of normal cones to the constraints, allowing the dual interpretation by prices and "budgetary" conditions to pop up to complement conditions bearing on physical commodities. In the same way than optimization under constraints is much more difficult to handle than optimization without constraints, viability theory raises more obstacles to overcome than the study of ordinary systems of differential equations. In some operational sense, the basic viability theorem can replace the Kakutani fixed point theorem each time it is used in a static model to make it "dynamical". This is the strategy which we have tried to explain in the case of the problem of decentralized dynamical allocation of resources.

[20]which can be traced back to Sumerian mythology which is at the origin of Genesis: one Decision-Maker, deciding what is good and bad and choosing the best (fortunately, on an intertemporal basis, thus wisely postponing to eternity the verification of optimality), knowing the future, and having taken the optimal decisions, well, during one week...

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