AGAIN ON HOLLING'S PUZZLE

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Again on Holling's Puzzle

In [5] C. Holling introduces a new concept of Resilience as an important characteristic of the behavior of complex ecological systems. He writes

1) In mathematical analyses, stability has tended to assume definitions that relate to conditions very near equilibrium points.
2) Resilience determines the persistence of relationships within a system and is a measure of the ability of these systems to absorb changes of state variables, driving variables, and parameters, and still persist. In this definition resilience is the property of the system and persistence of probability of extinction is the result.
3) Stability on the other hand, is the ability of a system to return to an equilibrium state after a temporary disturbance.
4) The more rapidly it returns, and with the least fluctuation, the more stable it is. In this definition stability is the stability is the property of the system and the degree of fluctuation around specific states the result.

With these definitions in mind a system can be very resilient and still fluctuate greatly, i.e. have low stability.

These forms of definitions are rather vague and underestimate the achievements of modern stability theory. The subsequent examples do little to clarify the definitions. Meanwhile, defining stability as behavior not only near equilibrium but also in the large and allowing for existing oscillations even in stable systems, the concept of stability may be extended to a broader class of problems and in particular to Holling's concept of resilience. These broad definitions are in current use in stability theory [1,2,3,4].

The vague nature of Holling's approach resulted in the appearance of several mathematical definitions of resilience when this topic was discussed among the IIASA methodology staff in February, 1975.

This note is another attempt to solve a loosely specified problem and it is certainly open for any criticism and comments. As the concepts of stability and resilience appear very often together in Holling's presentation we shall try to relate them directly through rigorous concepts of stability theory.

Resilience versus Stability

Let us try to give a mathematical definition of resilience which may approximate Holling's description as given above.

To make this definition more illustrative we confine ourselves to considering the systems which are governed by a system of ordinary differential equations.

Assume we have an ecological system represented in the following way:

$$\frac{dz}{dt} = f(z,t,u) \quad , \qquad z(t_o) = z_o \quad , \tag{1}$$

$z(t)$ = an n-dimentional state vector at time t.

t = time (independent variable).

u = a vector of disturbances applied to the system and given as a parameter.

$u \varepsilon U$, U = a set of feasible disturbances

z_o = given initial conditions.

Introduce the notation:

$\rho(z,S)$ = a distance between the point z and a set S which is determined as

$$\rho(z,S) = \min_{\tilde{z} \varepsilon S} ||z - \tilde{z}|| \quad . \tag{2}$$

Ω_u = a set of equilibrium points for a given u, i.e.

$$\Omega_u = \{z\colon f(z,u,t) \equiv 0\} \quad . \tag{3}$$

Γ = a union of Ω_u for all $u \varepsilon U$, i.e.

$$\Gamma = \bigcup_{u \varepsilon U} \Omega_u \qquad (4)$$

If Γ is a bounded set then the following definitions may be introduced.

Definition 1: The solution $z(z_0, t)$ of the system (1) is said to be uniformaly stable with respect to $u \varepsilon U$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that for any z_0, satisfying the condition

$$\rho(z_0, \Gamma) < \delta(\varepsilon) \quad .$$

the inequality

$$\rho\{z(z_0, t), \ \Gamma\} < \varepsilon$$

will hold for all $t > t_0$.

Definition 2: The solution $z(z_0 \ t)$ of the system (1) is said to be uniformaly asymptotically stable in the large with respect to $u \varepsilon U$, if for any z_0 the following condition holds

$$\lim_{t \to \infty} \rho\{z(z_0, t), \quad \Gamma\} = 0 \qquad (5)$$

Example 1: Let the system (1) be

$$\frac{dz}{dt} = u - Az$$

where A = is a positive definite matrix, i.e. $z^T Az > 0$ for $\forall ||z|| \neq 0$,

$u \varepsilon U$, which is bounded. In this particular case

$$\Omega_u = \{z: z = A^{-1}u\} \quad ,$$

where A^{-1} is the inverse matrix.

Let us show that the solution of (6) satisfies the definition (1). Introduce a new variable y:

$$z = -A^{-1}u+y \quad . \tag{7}$$

Then y should satisfy

$$\frac{dy}{dt} = -Ay \quad , \qquad y_0 = +A^{-1}u + z_0 \tag{8}$$

Since A is positive definite the solution $y(y_0,t)$ is asymptotically stable in the whole i.e.

$$\lim_{t\to\infty} y(y_0,t) = 0 \quad ,$$

for any y_0.

From this follows

$$\lim_{t\to\infty} z(z_0,t) = A^{-1}u \;\varepsilon\; \Omega_u \;\varepsilon\; \Gamma \quad .$$

The above definitions of stability allow us to specify the whole set of stable points in the system state space. In practical systems, however, singular points may exist in this set. For example, in ecology a very important point is z = 0, which corresponds to extinction. Introduce the concept of resilience as some characteristic which represents a possibility to escape singular stable points.

Definition 3: The system (1) is said to be globally and ideally resilient with respect to the set U, if it satisfies definition 2 and $z = 0$ does not belong to the set Γ.

Example 2: If in Example 1 $A^{-1}u \neq 0$, for all $u \varepsilon U$, then the system is globally and ideally resilient as stated by definition 3.

Definition 4: The system (1) is said to be locally and ideally resilient with respect to set V, if it satisfies definition (1) and $z = 0$ does not belong to set Γ.

Definition 5: If $z = 0 \varepsilon \Gamma$, then there exists a point $u^* \varepsilon U^*$, $u^* \varepsilon U$ which generates $z = 0$ and the system is not ideally resilient.

In this case $U \setminus U^*$ makes the system (1) ideally resilient To deal with non-ideally resilient systems the domain of attraction of the simpler point $z = 0$ should be specified.

Definition 6: The domain of attraction, S of the point $z = 0$ is a set of initial points z_0 such that the solution of the system (1) $z(z_0,t)$ tends to zero as t tends to infinity, i.e.

$$S_{(u)} = \{z_0 \colon z(z_0,t) \to 0 \quad , \quad \text{as } t \to \infty\} \qquad (10)$$

To characterize resilience properties of non-ideally resilient systems let us introduce the concept of the area of the domain of attraction as

$$P_{(u)} = \int_S dz \quad , \qquad u \varepsilon U^* \qquad (11)$$

If the point z = 0 is not stable then S consists of only one point z = 0 and its area P = 0.

Definition 7: The measure of resilience for non-ideally resilient systems is

$$R = \frac{1}{P} \tag{12}$$

Example: Assume

$$\frac{dz}{dt} = zu \quad \text{and} \quad 0 < u < \infty \tag{13}$$

In this case Ω_u consists of a single point z = 0, hence, the system is not ideally resilient. The system (13) has the following solution: $z = z_0 e^{ut}$. Thus domain of attraction S consists of a single point $z_0 = 0$; and consequently

$$P = 0 \quad .$$

The system (13) which is not ideally resilient has an infinite measure of resilience R according to (12).

This represents the fact that the system (13) has an infinite number of alternative ways to persist. Any initial point $z_0 \neq 0$ and any feasible $u(-\infty < u < \infty)$ provide for an infinite life-time of the system and only z = 0 corresponds to extinction where $z(z_0,t) \equiv 0$.

Example: Assume

$$\frac{dz}{dt} = -\sin z\, u, \quad u > 0 \tag{14}$$

The set Ω_u consists of the points $\frac{2\pi}{u} \cdot k$, where $k = 0, \pm 1, \pm 2, \ldots,$. The system (14) is not an ideally resilient one.

Let us show that the domain of attraction of the point $z = 0$ consists of the points z which satisfy (15)

$$- \frac{\pi}{u} < z < \frac{\pi}{u} \qquad (15)$$

To do this we may either integrate the system (14) or use the Lynpunov functions. Assume as a Lynpunov function

$$v = 1 - \cos zu \qquad (16)$$

This function is positive over the entire interval $-\left(\frac{\pi}{u}, \frac{\pi}{u}\right)$ and is zero only if $z = 0$.

Its total time derivative along the integral curves of system (14) is

$$\frac{dv}{dt} = - \sin^2 zu \leq 0 \qquad (17)$$

Thus all the solutions of system (14) converge to zero if initial point z_0 satisfies (15). One can easily show with the same method that if initial point satisfies

$$+ \frac{\pi}{u} k < z < + \frac{\pi}{u} (k+2) \quad , \quad k = \pm 1, \pm 2. \cdots \qquad (18)$$

then solution of system (14) converges to

$$\frac{\pi}{u} (k+1) \neq 0 \quad , \qquad \text{if} \qquad k \neq -1 \quad . \qquad (19)$$

Thus the area of the domain of attraction of point $z = 0$ is $2\pi/u$, and the measure of resilience for our system is

$$R = \frac{u}{2\pi} \tag{20}$$

The bigger u is the smaller the area of attraction and the higher resilience.

All reasoning given heretofore assume the constant value of u over the analyses time. The results may be generalized for the case when $u = u(t)$ is a given function of time. System (1) can be rewritten then as

$$\frac{dz}{dt} = f(z,u(t),t) = \psi(z,t) \quad .$$

A further analysis may be performed on the basis of the Lynpunov method and all the concepts introduced above are still valid.

References

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[5] Holling, C.S., Resilience and Stability of Ecological Systems, IIASA Research Report, September, 1973.