POLYHEDRAL DYNAMICS - I: THE RELEVANCE OF ALGEBRAIC TOPOLOGY TO HUMAN AFFAIRS

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1. Introduction

The most serious single methodological obstacle in the analysis of large-scale systems has been the lack of a suitable mathematical apparatus capable of describing the global features of a system, given information about local (subsystem) behavior. It is perhaps not surprising that the heavy emphasis placed upon the use of tools of analysis has yielded very meager fruits in this regard, since the methods of classical analysis are inherently local, being based upon such concepts as derivatives, infinitesimals, power series expansions, and so forth which are all concerned with behavior in the neighborhood of a point. What is surprising, however, is that, with few exceptions, the other main roots of mathematics - algebra and geometry - have not been tapped to provide a new set of tools for the system theorist to probe the murky depths of large, complex systems. This oversight shows a singular lack of foresight since traditionally the problems in these fields have been of a global nature and centuries of work on the part of a veritable army of mathematicians has resulted in a very refined and sophisticated machinery suitable for answering global questions.

Fortunately, in the past few years several efforts have been made to rectify the foregoing deplorable state of affairs. Feverish activity by Kalman [1], Brockett [2], and others has injected a strong algebraic flavor into contemporary system theory which has already shown signs of providing a framework for further conceptual clarifications and advances. On the geometric front, work begun by Thom [3] and now being continued by Zeeman [4-5] and many others has given us a new mathematical apparatus, catastrophe theory, suitable for analyzing a large class of natural and social phenomena in which discontinuities in the system output play an important role.

The purpose of the current note is to explore another recent algebro-geometric approach to the structural analysis of large-scale systems. This approach, based upon ideas of algebraic topology, was introduced by Atkin [6-7] in a recent series of works which, unfortunately, have not yet received the circulation they deserve. By a very ingenious coupling of classical ideas in combinational topology and new notions of connectivity, patterns, and obstructions, this work presents a mathematical framework within which an extremely broad class of global systems questions can be precisely analyzed.

The objective of this work is two-fold: to present the basic theory of what we have chosen to call "polyhedral dynamics" as quickly as possible. This presentation includes the basic ideas of Atkin, plus extensions of our own which extend and broaden the original work. The second goal is to illustrate the concepts involved on a variety of problems relevant to ongoing IIASA activities.

2. Sets and Relations

Since the theory we present is based upon very basic notions of sets and relations, let us recall a few fundamental facts and definitions.

A set (finite or infinite) S is a collection of elements. The Cartesian product of the two sets A and B is a new set $A \times B$ which consists of all elements of the form (a,b), where $a \varepsilon A$, $b \varepsilon B$.

A relation λ from the set A to the set B is a rule which associates some of the elements of B with some of the elements of A. For example, if $A = \{1,2,3\}$, $B = \{0,4,8,10\}$ and λ is the relation "less than," then λ is the subset in $A \times B$ of those ordered pairs $\{(1,4),(1,8),(1,10),(2,4),(2,8),(2,10),(3,4),(3,8),(3,10)\}$. This is a relation from A to B; the associated relation from B to A, denoted by λ^{-1}, is written as $\lambda^{-1} \subset B \times A$.

When we represent the relation λ between two sets A and B as that subset of $A \times B$ such that the pair (a,b) is contained in the relation if and only if a is λ-related to b, then we naturally obtain a simple mathematical array which contains the relation. This array is called the incidence matrix of the relation and is an array of numbers λ_{ij}, with each λ_{ij} being either 0 or 1. The number λ_{ij} equals 1 if a_i is λ-related to b_j and is 0 otherwise. For the above example, the incidence matrix is

λ (A ↓) \ B →	0	4	8	10
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

3. Complexes and Relations

Our next task is to give a geometrical representation of a relation. It turns out that the appropriate vehicle for this is the simplicial complex.

We consider a finite set

$$V = \{v^i, i = 1,2,...,k\}$$

and a collection K of its subsets. Denote any one of these subsets consisting of p+1 distinct elements by σ_p. Such a subset is called a p-simplex. If σ_q is a q-simplex defined by a (p+1) subset of the (p+1) elements defining σ_p, then we say that σ_q is a face of σ_p and we write

$$\sigma_q < \sigma_p \quad .$$

The relation < defines a partial ordering on K.

The collection K is called a simplicial complex if and only if

i) each single element set $\{v^i\}$ is a member of K,

ii) whenever $\sigma_p \varepsilon K$ and $\sigma_q < \sigma_p$, then $\sigma_q \varepsilon K$.

The set V is called the vertex set of the complex K. Each p-simplex is said to be of dimension p; the largest integer n for which $\sigma_n \varepsilon K$ is called the dimension of K.

We can obtain a geometrical representation of a complex K in terms of connected convex polyhedra in the following manner. In the case p = 1, if v^1 and v^2 are the defining vertices of σ_1, then we associate points P_1 and P_2 with them and there is then a natural association of the 1-simplex $\langle v^1 v^2 \rangle$ with the convex set containing P_1 and P_2, i.e. with the line segment joining P_1 and P_2.

More generally, we can represent a p-simplex σ_p by a convex polyhedron with (p+1) vertices in some Euclidean space E^h of suitable dimension h. The fact that many of the simplices of the complex K share a common face suggests that a value of h smaller than the sum of all simplex dimensions will suffice. It turns out that h = 2n+1 is sufficient, when dim K = n.

Our next question is how to associate a simplicial complex $K_Y(X;\lambda)$, when we are given the finite sets X and Y, and a relation λ between them. This complex is constructed in the following manner. Let us assume that we have the incidence matrix

λ	X
Y	(λ_{ij})

where, for the sake of definiteness, we assume card X = n, card Y = m (card Z $\triangleq$ number of elements in the set Z). The set X is taken to be the vertex set for our complex $K_Y(X;\lambda)$ and a subset of (p+1) elements of X forms a p-simplex if there exists at least one element of Y which is λ-related to each of them. In terms of the incidence matrix, the columns corresponding to the (p+1) elements $X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_{p+1}}$ are all non-zero.

In the same way, by regarding Y as the vertex set we obtain the complex $K_X(Y;\lambda^{-1})$.

Let us take a simple example to fix the above ideas. We let X be a collection of social roles and Y a set of people. Thus,

X : X_1 = teacher, X_2 = parent, X_3 = town-councillor
X_4 = student, X_5 = householder, X_6 = motorist

Y : Y_1 = Smith, Y_2 = Jones, Y_3 = Anderson,
Y_4 = Williams, Y_5 = Carson.

Let the incidence matrix be

	X_1	X_2	X_3	X_4	X_5	X_6
Y_1	1	1	0	0	1	1
Y_2	0	1	1	0	0	0
Y_3	0	0	0	1	0	1
Y_4	0	0	1	0	1	0
Y_5	0	1	0	0	0	1

In $K_Y(X;\lambda)$ we have dim K = 3 and

$Y_1 = \langle X_1, X_2, X_5, X_6 \rangle$ = 3-simplex

$Y_2 = \langle X_2, X_3 \rangle$ = 1-simplex

$Y_3 = \langle X_4, X_6 \rangle$ = 1-simplex

$Y_4 = \langle X_3, X_5 \rangle$ = 1-simplex

$Y_5 = \langle X_2, X_6 \rangle$ = 1-simplex

The geometric representation is

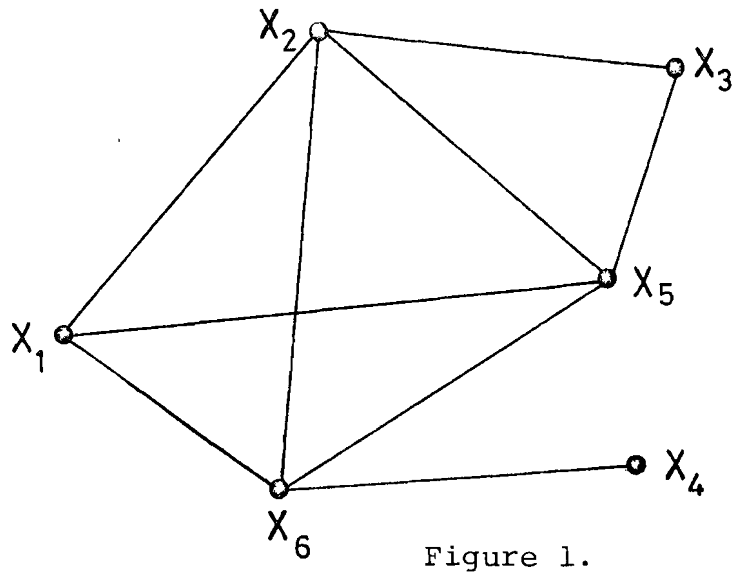


Figure 1.

We notice that Smith is a 3-simplex since he combines the roles of teacher, parent, householder, and motorist. The others are separate 1-simplices with Carson being a face of Smith via the edge X_2, X_6 of the tetrahedron.

<u>Exercise</u>: Construct the <u>conjugate</u> complex $K_X(Y;\lambda^{-1})$.

4. Connections, Patterns, and Obstructions

We now delve more deeply into the structure of a simplicial complex in order to express, in a precise way, the manner in which its simplices are connected to each other.

Given two simplices σ_p, σ_r in a complex K, we say they are joined by a chain of connection if there exists a finite sequence of simplices

$$\sigma_{\alpha_1}, \sigma_{\alpha_2}, \ldots, \sigma_{\alpha_n}$$

such that

(1) σ_{α_1} is a face of σ_p,

(2) σ_{α_n} is a face of σ_r,

(3) σ_{α_i} and $\sigma_{\alpha_{i+1}}$ have a common face say σ_{β_i}, $i = 1, \ldots, (h-1)$.

We shall say such a chain is of length (h-1) and that the chain is of q-connectivity if q is the smallest integer of the set

$$\alpha_1, \beta_1, \beta_2, \ldots, \beta_{h-1}, \alpha_n \quad .$$

(Remark: As a special case, a σ_p must be p-connected to itself by a chain of length zero.)

Referring to the example above and Fig. 1, we note that

Smith is 1-connected to Carson via $<X_2, X_6>$;

Smith is 0-connected to Jones via $<X_2>$, each chain being of length zero. On the other hand, Anderson is 0-connected to Williams via the chain $\langle X_6, X_5 \rangle$, a connection of length 1.

We can set up a relation γ_q between simplices of K by saying that two simplices σ_p, σ_r are in the relation γ_q if and only if they are q-connected. It is easy to see that γ_q is an equivalence relation on the complex K with the equivalence classes being the elements of the quotient set K/γ_q. We let Q_q denote the cardinality of K/γ_q, so that Q_q is the <u>number of distinct q-connected components</u> of K, a component being all members of an equivalence class under γ_q.

If we let q take on all integer values between 0 and dim K and find K/γ_q in each case, we will have performed a <u>Q-analysis</u> on K.

In the foregoing example, we obtain the following Q-analysis:

$q = 3$ (= dim K) , $Q_3 = 1$, Smith
$q = 2$, $Q_2 = 1$, Smith
$q = 1$, $Q_1 = 4$, <Smith,Carson>, <Anderson>, <Williams>, <Jones>
$q = 0$, $Q_0 = 1$, <Smith,Carson,Anderson, Williams,Jones>

Note that in performing the Q-analysis, the idea of the lengths of the chains of connection is not involved.

In the special case of a complex K in which $Q_0 = 1$

(i.e. the complex is in one piece), we introduce a vector which we call the obstruction vector. If

$$Q = (Q_n, Q_{n-1}, \ldots, Q_1, Q_0) \text{ with } Q_0 = 1 \quad ,$$

then the <u>obstruction vector</u>, denoted by $\hat{Q}$, is defined as

$$\hat{Q} = Q - U \quad ,$$

where $U = (1,1,\ldots,1)$ is the unit point in E^n.

For a particular simplex σ_r in K, it is possible to identify two special values called $\check{q}$ and $\hat{q}$. The integer $\check{q}$ is the smallest value of q for which σ_r is q-connected to another district simplex. The second value, $\hat{q}$, is the dimension of σ_r (in this case $\hat{q} = r$). Closely associated with $(\check{q},\hat{q})$ is a quantity called the <u>eccentricity</u> of σ. This is a rational number given by

$$\text{ecc}(\sigma) = \frac{\hat{q} - \check{q}}{\check{q} + 1} \quad .$$

The eccentricity is defined for all $\check{q}$ except $\check{q} = -1$, when we say $\text{ecc}(\sigma) = \infty$. This "infinite" eccentricity occurs when σ is totally disconnected from the rest of the complex. In general, $\text{ecc}(\sigma)$ is a measure of how well integrated σ is into the rest of the complex. A large value of $\text{ecc}(\sigma)$ signifies that σ is, in some sense, "aloof" or weakly connected at the remainder of K, while a small value indicates a high degree of integration with the complex.

By a <u>pattern</u> on a complex K, we shall mean a mapping

$$\pi: \left\{\sigma_p^i \quad , \quad 0 \le p \le N \quad , \quad \text{all } i\right\} \to J \quad ,$$

where J is (usually) the integers. Thus, π is defined on every simplex of K and, because these are graded by their q-values, it is natural to grade the pattern itself. Thus, we can write

$$\pi = \pi^0 \oplus \pi^1 \oplus \ldots \oplus \pi^N \quad ,$$

where N = dim K and where $\pi^t = \pi \,|\, \left\{\sigma_t^i; \quad \text{fixed } t\right\}$.

Each π^t is therefore a set function, defined on specified (t+1)-subsets of the vertex set X of K.

The complex K itself may be regarded as justifying the existence of a particularly simple pattern, namely the one which places a '1' on every simplex in K. Such a pattern is implied whenever we are given the existence of K. Changes from this basic pattern can then be interpreted either in terms of changes in the complex K (by addition or deletion of simplices) or by introducing the concept of a <u>force</u> on the complex. In the latter case, the complex is regarded as rigid and is not involved in the changing patterns; it acts as a <u>framework under stress</u> but its basic static geometry remains unchanged. A formal way of describing these complex forces is to measure the numerical changes in the pattern π. Indicating any such changes by $\Delta\pi$, we can identify the graded change via

$$\Delta\pi = \Delta\pi^{0} \oplus \Delta\pi^{1} \oplus \ldots \oplus \Delta\pi^{N} \quad .$$

When $\Delta\pi^{t} \neq 0$, we speak of a <u>t-force</u> acting in the static complex K.

An alternative approach is to regard the change in pattern as defining a new complex (often by replacing the original K by a number of new complexes.)

These two approaches mirror exactly the historical differences between the classical physical theories of Newton and the relativistic approach of Einstein. The <u>static backcloth</u> of the complex K is the geometrical structure attributed to space (or space-time). With a rigid view of the geometry, the gravitational theory of Newton was expressed in terms of classical forces (forces at a distance) existing in the complex; the relativistic approach was to demand that the phenomenon of gravitation should be interpreted as a modification of the space-time structure itself.

Of course, when we use the <u>t-force</u> definition of the change in a pattern we are adopting what might be loosely called the Newtonian view of the dynamics of the backcloth. In the Einsteinian view, we shall consider changes in the geometry which allow <u>free</u> changes in the patterns, where by free we shall mean that the changes are <u>compatible with the</u>

geometric backcloth. This is the significance of the obstruction vector $\hat{Q}$. It isolates those q-connected components of K in which a free change of pattern is prevented by the geometry of the situation. Moreover, it provides a quantitative measurement of the freedom for pattern changes in any part of the complex.

5. Connective Stability

It is possible to make use of the structural concepts discussed above to introduce a measure of how "stable" the complex is to perturbations. Intuitively speaking, one would be led to consider a given system "stable" if some qualitative property of the system remains invariant under perturbations. Specification of particular properties and the types of allowable perturbations lead to the various stability notions which fill the literature.

Roughly speaking, our term connective stability refers to the ability of a given complex K to retain its ability to sustain a flow of patterns in the face of structural perturbations to K. Thus, we are taking an Einsteinian point of view in that we regard the perturbations of interest as being external forces which change the structure of K itself, rather than being forces which induce stresses in a rigid complex. A precise definition of connective stability is that a complex K is connectively stable to degree r under a perturbation P if the r^{th} component, $\hat{Q}_r$, of the obstruction vector $\hat{Q}$ remains unchanged or decreases in the complex generated by P. Here, of course, P generates the new complex K_P by the mechanism of addition or deletion of vertices and/or edges from K. Note also that the definition makes sense only for those $r \leq \dim K_P$, which is not necessarily equal to dim K.

Thus, we see that connective stability is not a binary concept, but rather it is a multidimensional notion in which each level must be examined. Clearly, if a given complex is not connectively stable of degree r relative to a perturbation P, then the perturbation has changed the geometry of the system to the extent that the flow of patterns through r-dimensional faces has been impeded. This implies a restriction in the capability of the system to act as a channel of information flow.

Another way to look at the situation is to interpret connective stability of degree r as saying that the structure of the geometrical complex imposes no restrictions on the free flow of patterns through r-dimensional faces. From a managerial standpoint, this would imply that the managerial "decision" P has not restricted the future dynamics of the process at the r-level. In a decision-making environment, where one of the main objectives is to retain a measure of flexibility for future planners, the concept of connective stability provides a quantitative, multidimensional measure of the amount of future freedom lost (or gained) by current actions.

6. Mainly Examples

In this section we illustrate some possible uses of the methodology sketched above by applying it to some idealized examples appropriate to various IIASA projects. It will be clear that these examples are purely for illustrative purposes, any similarity between them and the real problems being fortuitous, but accidental. However, it will be seen that the gap between the real problems and the examples is not so large that a modest effort by a handful of people couldn't bridge it.

Example 1: A Predator-Prey Ecosystem

A favorite problem in the bio-world seems to be the study of interactions and interrelations between a collection of predators and their prey in a localized spatial environment. Let us approach the study of such a structure from the algebraic topological point of view.

For the sake of definiteness, we consider a single trophic level ecosystem in which the predator and prey have been divided into mutually disjoint sets. Let the predator set be given by

$$Y = \{\text{Man, Lion, Elephants, Birds, Fish, Horses}\}$$
$$= \{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6\} ,$$

while the set of prey are given by

$$X = \left\{ \begin{array}{l} \text{Antelope, Grains, Pigs, Cattle, Grass,} \\ \text{Leaves, Insects, Reptiles} \end{array} \right\}$$
$$= \left\{ X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8 \right\} \; .$$

We define a relation λ on $Y \times X$ by saying that Y_i is related to X_j if predator Y_i feeds on prey X_j. A plausible incidence matrix for this relation is

$\Lambda =$

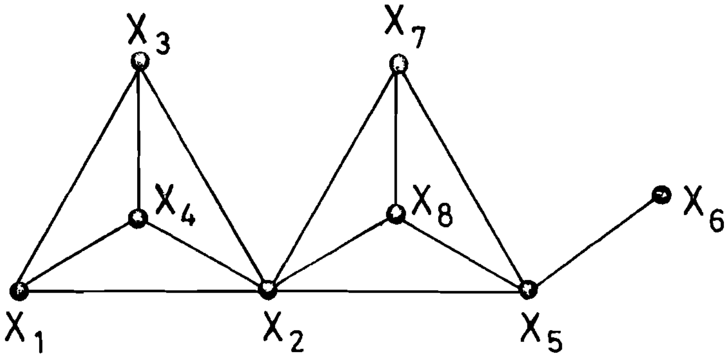
λ	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
Y_1	1	1	1	1	0	0	0	0
Y_2	1	0	1	0	0	0	0	0
Y_3	0	0	0	0	1	1	0	0
Y_4	0	1	0	0	1	0	1	1
Y_5	0	0	0	0	0	0	1	0
Y_6	0	1	0	0	1	0	0	0

Thus, if we consider the complex $K_Y(X;\lambda)$, we have

$\langle X_1 X_2 X_3 X_4 \rangle$ is a σ_3 whose name is Y_1

$\langle X_1 X_3 \rangle$ is a σ_1 whose name is Y_2,

and so forth. The geometrical representation of $K_Y(X;\lambda)$ is



We see that $K_Y(X;\lambda)$ consists (geometrically) of two 3-dimensional complexes $Y_1 = <X_1X_2X_3X_4>$ and $Y_4 = <X_2X_5X_7X_8>$ joined by the 0-dimensional simplex $Y_5 = <X_2>$, plus the 1-dimensional simplex $Y_3 = <X_5X_6>$. Already, the geometry suggests that the 0-simplex $Y_5 = <X_2>$, consisting of Grains, is going to be critical in the analysis of this ecostructure.

Referring to the algorithm given in the Appendix, the relevant connectivity matrix for this problem is

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	3	1	-	0	-	0
Y_2		1	-	-	-	-
Y_3			1	0	-	0
Y_4				3	0	1
Y_5					0	-
Y_6						1

Thus, the connectivity pattern is

at $q = 3$ we have $Q_3 = 2$, $\{Y_1\}, \{Y_4\}$

$q = 2$, $Q_2 = 2$, $\{Y_1\}, \{Y_4\}$,

$q = 1$, $Q_1 = 3$, $\{Y_1Y_2\}, \{Y_3\}, \{Y_4Y_6\}$,

$q = 0$, $Q_0 = 1$, $\{\text{all}\}$.

The structure vector for this complex is

$$Q = \begin{pmatrix} 3 & & & 0 \\ 2 & 2 & 3 & 1 \end{pmatrix} ,$$

with the obstruction vector

$$\hat{Q} = \begin{pmatrix} 3 & & & 0 \\ 1 & 1 & 2 & 0 \end{pmatrix} .$$

From the vectors Q and $\hat{Q}$, we see that our ecological complex K allows a free flow of pattern only at the 0-connectivity level, with the greatest level of obstruction being at the q-level 1. This is intuitively clear since K consists of 3 separate "pieces" at q-level 1 no two of which share a connecting link at this q-level. As a result, there is no "bridge" by which a pattern can cross from one of these components to another at this level of connectivity.

The eccentricities of the simplices Y_1 - Y_6 are

$$\text{ecc } Y_1 = 1 \;, \quad \text{ecc } Y_2 = 0 \;, \quad \text{ecc } Y_3 = 1$$
$$\text{ecc } Y_4 = 1 \;, \quad \text{ecc } Y_5 = 0 \;, \quad \text{ecc } Y_6 = 0 \;.$$

From these figures we are led to conclude that there is a great deal of homogeneity in the complex K, no one simplex exhibiting a significant degree of eccentricity. In other words, all of the predators are well-integrated into the ecosystem.

What happens when the "prey," X_2 (Grain) is removed from the system? It is clear from the geometry of the complex K that such an excision will result in a disconnected complex. What is not so apparent is what effect such a change will bring to other aspects of the system.

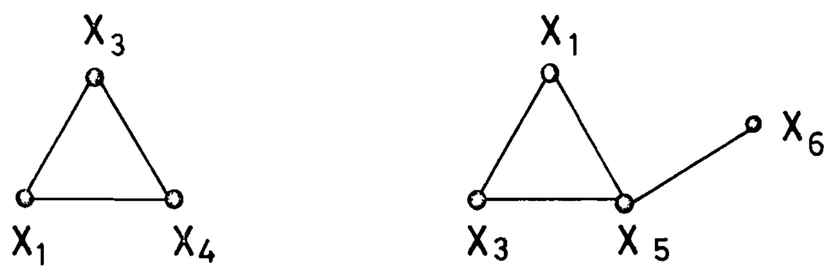
In order to satisfy our curiosity on this matter, we calculate the connectivity matrix using our previous incidence matrix Λ with the column X_2 removed. This results in the connectivity matrix

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	2	1	-	-	-	-
Y_2		1	-	-	-	-
Y_3			1	0	-	0
Y_4				2	0	0
Y_5					0	-
Y_6						0

Performing a Q-analysis, we find that

$$q = 2 \quad , \quad Q_2 = 2 \quad , \quad \{Y_1\}, \{Y_4\},$$
$$q = 1 \quad , \quad Q_1 = 3 \quad , \quad \{Y_1Y_2\}, \{Y_3\}, \{Y_4\},$$
$$q = 0 \quad , \quad Q_0 = 2 \quad , \quad \{Y_1Y_2\}, \{Y_3Y_4Y_5Y_6\} \quad .$$

Since $Q_0 > 1$, we see that the new complex is in two disjoint pieces consisting of the simplices $\{Y_1Y_2\}$ in one complex, $\{Y_3,Y_4,Y_5,Y_6\}$ in the other. The geometrical representation is



In performing further analysis, such as eccentricity calculations, obstruction analysis, etc., we must regard these pieces as being "decoupled" subsystems of the original ecosystem and analyze each separately. For example, the

Q-analysis for the complex K_1 consisting of the simplices Y_1 and Y_2 yields the Q-vector

$$Q^1 = \begin{pmatrix} 2 & & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

with the obstruction vector

$$\hat{Q}^1 = (0\ 0\ 0) \quad ,$$

showing that there is no geometrical obstacle to a free flow of patterns in K_1. In the complex $K_2 = \{Y_3, Y_4, Y_5, Y_6\}$, a similar analysis yields

$$Q^2 = \begin{pmatrix} 2 & & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad ,$$

$$\hat{Q}^2 = (0\ 1\ 0) \quad .$$

Thus, in this subsystem a free flow of patterns is restricted by the geometry at the level $q = 1$. The reason for this, of course, is that the simplex Y_3 shares only a 0-simplex with the remainder of the subsystem.

<u>Example 2</u>: (Economic Planning)

To illustrate the notion of a pattern on a complex and to further elucidate the role of the obstruction vector $\hat{Q}$, we consider a fictitious economic complex consisting of two sets X = {set of goods (resources)}, Y = {collection of economic sectors}.

Our relation λ will be defined as: good X_i is λ-related to sector Y_j if and only if X_i is utilized in sector Y_j.

For the sake of definiteness, suppose there are 11 goods, 6 sectors and the incidence matrix Λ for the complex $K_Y(X;\lambda)$ (regarding X as the vertex set) is

λ	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
Y_1	0	0	1	0	1	0	1	0	0	0	1
Y_2	0	0	1	0	1	1	1	1	0	0	0
Y_3	0	0	0	0	0	1	1	1	0	0	0
Y_4	0	1	0	0	1	0	1	1	0	0	0
Y_5	0	0	0	0	0	0	1	0	0	0	0
Y_6	1	0	0	0	0	0	0	0	1	0	1

The connectivity pattern then becomes

Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	$K_X(Y;\ )$
3	2	0	1	0	0	Y_1
	4	2	2	0	-	Y_2
		2	1	0	-	Y_3
			3	0	-	Y_4
				0	-	Y_5
					2	Y_6

with a structure vector

$$Q = \begin{Bmatrix} 4 & & & & 0 \\ 1 & 3 & 2 & 2 & 1 \end{Bmatrix}$$

with components

$$q = 4 \ : \ \{Y_2\} \ ,$$
$$q = 3 \ : \ \{Y_1\},\ \{Y_2\},\ \{Y_4\} \ ,$$
$$q = 2 \ : \ \{Y_1Y_2Y_3Y_4\},\ \{Y_6\} \ ,$$
$$q = 1 \ : \ \{Y_1Y_2Y_3Y_4\},\ \{Y_6\} \ ,$$
$$q = 0 \ : \ \{\text{all}\} \ .$$

The obstruction vector is

$$\hat{Q} = \begin{Bmatrix} 4 & & & & 0 \\ 0 & 2 & 1 & 1 & 0 \end{Bmatrix} \ .$$

Now let π be a pattern defined on $K_Y(X;\lambda)$. For example, π might be the total volume of all goods which flow through the sectors via the simplices Y. More specifically, we might have

$$\pi_0 : \{Y_1Y_2Y_3Y_4Y_5Y_6\} \longrightarrow 200$$

$$\pi_1 : \{Y_1Y_2Y_3Y_4\} \longrightarrow 150$$
$$\{Y_6\} \longrightarrow 75$$

$$\pi_2 : \{Y_1Y_2Y_3Y_4\} \longrightarrow 150$$
$$\{Y_6\} \longrightarrow 75$$

$$\pi_3 : \{Y_1\} \longrightarrow 50$$
$$\{Y_2\} \longrightarrow 60$$
$$\{Y_4\} \longrightarrow 20$$

$$\pi_4 : \{Y_2\} \longrightarrow 60$$

Note that the 'face' ordering must be obeyed in the definition of π, i.e. if σ_q is a face of σ_p, then $\pi(\sigma_q) = \pi|\sigma_q$, where "|" denotes the restriction map.

Any change via the values of π (change which is part of a free, uninhibited, unbiased redistribution of the values of π) effectively means a free flow of numbers throughout the complex $K_Y(X;\lambda)$ from one simplex to another. Hence, the dimensions of the common faces of two simplices is very important. If the pattern π^q is to change freely, then it needs a (q+1)-chain of connection to do so; a q-connectivity will not do. Hence, the number of separate q-components is an indication of the impossibility of free flow of any π^q. These numbers are directly displayed in the obstruction vector $\hat{Q}$. This discussion indicates that an increase in $\hat{Q}$ signifies an increase in the rigidity and this can happen at one q-level but not at another. This is why the vector components of $\hat{Q}$ need to be studied separately; it is <u>not</u> helpful to produce a single number, like the norm $||\hat{Q}||$, from $\hat{Q}$.

In our example, we see that the geometry of the complex imposes no restrictions on the flow of goods only at the q-levels 4 and 0, while the most serious impediment to free flow is at the level 3.

What about the change in a pattern from π to $\pi + \Delta\pi$? The problem of forming $\Delta\pi$ may be represented as an operator in the scheme

$$\Delta\pi : \pi^0 \overset{\Delta}{\rightarrow} \oplus \pi^1 \overset{\Delta}{\rightarrow} \oplus \pi^2 \overset{\Delta}{\rightarrow} \oplus \pi^3 \overset{\Delta}{\rightarrow} \oplus \pi^4 \overset{\Delta}{\rightarrow} \oplus 0 \quad ,$$

by which we mean, e.g. $\Delta\pi^1$ is free in the domain of π^2, etc. and $\Delta\pi^4$ is not free. The reason that $\Delta\pi^4$ is not free is that there is no 5-simplex in our economic complex. This means that changes in π^4 must be induced by "forces" of some kind which are of an external nature. Such external forces, of course, will produce a new complex with new connectivity patterns, thereby affecting all π^t. For example, if π^1 becomes a π^0 then the possibility of a free change $\Delta\pi$ has increased. In this way we can begin to describe the effects of the pressures in terms of the changes in patterns.

As an example of what we mean, suppose that over some interval of time, the pattern changes as follows

$$\Delta\pi^0 = -30 \quad , \quad \Delta\pi^1 = 0 \quad , \quad \Delta\pi^2 = 0 \quad , \quad \Delta\pi^3 = 0 \quad , \quad \Delta\pi^4 = +10 \quad .$$

The fact that $\Delta\pi^4 \neq 0$ can be interpreted by saying that there is an effective extra vertex (sector) which, if it were actually present, would allow a free change $\Delta\pi^4$ of the value +10. Thus, this change +10 is a measure of the lack of freedom to change, of the extraneous pressure or force which results in the change. Since the component π^0 is defined on a simplex which is a face of the one 4-simplex, this change can be viewed as a free change which can take place independently of the external pressures or forces. Consequently, we shall describe the situation

$$\pi \rightarrow \pi + \Delta\pi$$

as one which exhibits an <u>attractive force</u> at the 4-level, described by the value $\Delta\pi^4 = +10$. We call it attractive since it results in an increase in π^4.

The notion of a force suggests that we need to appeal to the idea of an external force only when the π^t pattern changes on (t+1)-disconnected components of the complex. Since this can happen at more than one value of t, we need to describe a force as a <u>t-force</u>.

When there is zero t-force for all values of t, then all changes in π which take place in the complex are free changes. Since, under these conditions, $\Delta\pi^t$ is a π^{t+1}, so every π^{t+1} can be regarded as a possible (source of) $\Delta\pi^t$. Thus, this kind of force-free pattern change is characterized by a flow of pattern values down the sequence of q-values (from a σ_2 to a σ_1, etc.), not up that sequence. Characteristically, a complete pattern change will be able to exhibit a flow of pattern values up the sequence of q-values, and this will include creation of an effective σ_{t+1} where one did not exist before.

<u>Example 3</u>: (Management Decision-making)

The last concept we wish to illustrate in this preliminary note is the treatment of weighted relations. In other words, a relation μ which takes account, not only of the connectivity of various subsystems, but also the strength of those connections.

Consider a manager who has several actions $A_1, \ldots, A_m$ at his disposal. Each of these actions produces some subset of the reactions $R_1, \ldots, R_n$ with a certain level of impact, i.e. we assume that action A_i has a certain impact level α_{ij} (measured on some subjective scale) on reaction (or effect) R_j. Thus, we can summarize this situation by the tableau

ω	R_1	R_2	---	R_n
A_1	α_{11}	α_{12}	---	α_{1n}
A_2	α_{21}	α_{22}	---	α_{2n}
.				
.				
.				
A_m	α_{m1}	α_{m2}	---	α_{mn}

Our problem is how to relate this tableau, associated with the weighted relation ω, to a meaningful incidence matrix, which will then allow us to construct an appropriate simplicial complex describing the situation.

We accomplish this task by introducing "slicing parameters," θ_{ij}, associated with each element α_{ij} of the tableau. These parameters represent certain impact levels, below which we consider the impact to be negligible. For example, suppose we slice by rows and consider only those impacts above level C_i in row i. Then we construct the appropriate incidence matrix Λ according to the rule

$$\lambda_{ij} = \begin{cases} 1 , & \alpha_{ij} \geq C_i \\ 0 , & \alpha_{ij} < C_i \end{cases} .$$

By varying the parameters C_i, we may enhance or diminish the effect of action A_i in the resultant complex.

We consider a numerical example. Assume that six actions are available, while the number of behavior modes is eleven. Furthermore, assume that the impacts are measured on a dimensionless scale of 0-200. Suppose that the weighted relation ω is given by the tableau

ω	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}
A_1	48	30	72	12	80	146	184	200	21	42	53
A_2	11	4	28	5	22	24	38	32	9	8	16
A_3	0	0	4	0	6	12	14	19	3	2	0
A_4	4	6	8	2	10	14	9	13	4	2	4
A_5	0	1	4	5	7	12	16	14	2	0	2
A_6	4	2	6	3	2	0	0	0	4	1	5

Slicing at an impact level of $\theta_{ij} = 20$ for all i,j, we obtain the incidence matrix

$\theta \geq 20$

λ	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}
A_1	1	1	1	0	1	1	1	1	1	1	1
A_2	0	0	1	0	1	1	1	1	0	0	0
A_3	0	0	0	0	0	0	0	0	0	0	0
A_4	0	0	0	0	0	0	0	0	0	0	0
A_5	0	0	0	0	0	0	0	0	0	0	0
A_6	0	0	0	0	0	0	0	0	0	0	0

Thus, at this impact level, only decisions A_1 and A_2 are operable.

Now suppose that we slice the tableau by columns and let $\theta_{ij} = 1$ if $j \neq 8$, $\theta_{ij} = 250$ if $j = 8$. Then in the resulting relation, there must be a column of zeros under R_8 and this means that effectively the response R_8 is absent from the system. Thus, we have "sliced out" the behavior under R_8, and this might correspond to the reality of closing out that particular line of behavior due to unacceptable social, political, or economic pressures. In a similar way, we could slice out various decision options by imposing a sufficiently large threshold value on the appropriate rows of the tableau.

7. Discussion and Conclusions

In this note we have demonstrated the potential applicability of algebraic-topological tools to the structural analysis of large-scale systems. Only a small part of the actual mathematical machinery available has been utilized in this presentation, but it seems clear that even the few basic ideas given here enable us to gain significant new insights into the connective patterns of many significant processes. However, there remain many important questions for future analysis, among them being

i) where do the other theoretical tools of algebraic topology such as homology, exact sequences, Betti numbers, etc. make their appearance felt in the context of large-systems. In other words, what are their system-theoretic implications and interpretations;

ii) how do the foregoing ideas interact with other techniques of systems analysis? In particular, how might the multistage decision-making apparatus of dynamic programming be linked with the somewhat static character of the simplicial complex analysis in order to inject a more "dynamic" flavor into the procedures given here;

iii) how can one introduce hierarchical concepts into the polyhedral framework?

Actually, all these questions are currently under consideration and potentially successful approaches to each of them have been made. These issues, plus others involving computational questions and more complicated (and realistic) examples will be discussed in future papers in this series.

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APPENDIX

Algorithm for Q-Analysis

If the cardinalities of the sets Y and X are m and n, respectively, the incidence matrix Λ is an (mxn) matrix with entries 0 or 1. In the product $\Lambda\Lambda'$, the number in position (i,j) is the result of the inner product of row i with row j of Λ. This number equals the number of 1's common to rows i and j in Λ. Therefore, it is equal to the value (q+1), where q is the dimension of the shared face of the simplices σ_p, σ_r represented by rows i and j. Thus, the algorithm is

(1) form $\Lambda\Lambda'$ (an mxm matrix),

(2) evaluate $\Lambda\Lambda'-\Omega$, where Ω is an mxm matrix all of whose entries are 1,

(3) retain only the upper triangular part (including the diagonal) of the symmetric matrix $\Lambda\Lambda'-\Omega$. The integers on the diagonal are the dimensions of the Y_i as simplices. The Q-analysis then follows by inspection.