Working Paper

Viability in a Keynesian Model: A Preliminary Approach

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September, 1994

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1 Presentation

The purpose of this paper is to study an elementary dynamic Keynesian model by means of the viability approach. The mathematical theory of viability[1] allows one to raise and answer questions which cannot be solved with usual dynamic tools. It deals with the question whether for a given dynamical system, there can be found solutions which satisfy some a priori given constraints, and with the analysis of subsets of the phase space where viable evolutions, i. e., evolutions satisfying the given constraints, are possible. Equilibria are examples of such viable sets, but the viability approach is much more general then the equilibrium approach. In contrast to traditional ecodynamics where local stability and asympototic properties of models are the heart of the matter, the viability approach is concerned with contingent evolutions over time of dynamic systems.

In section 2 we describe an elementary dynamic Keynesian model. Viability questions are evoked in section 3 along with some intuitive answers, and rigorous results and developments will be found in section 4.

2 An elementary Keynesian Model

2.1 The economy

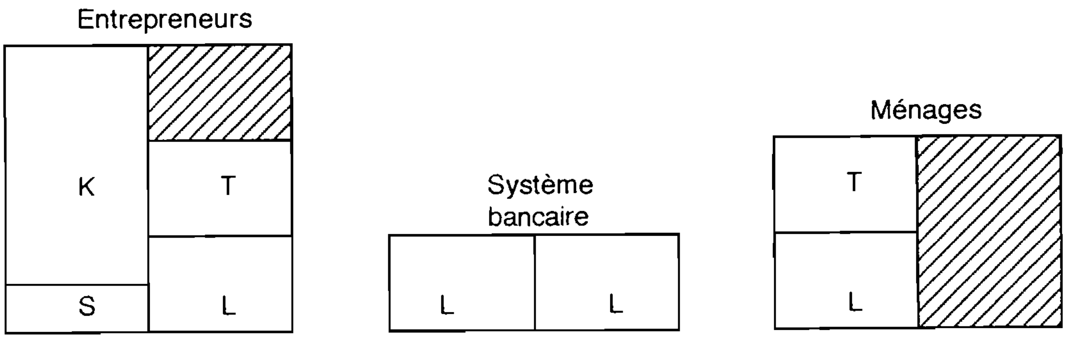
Let be a closed one-commodity economy with entrepreneurs , households (both reduced to representative units) and a banking system (including a centralbank). Entrepreneurs own the capital K, which is a quantity of the unique commodity held for the sake of production, and a quantity S of the same commodity which is nothing but the excess of production over sales, cumulates over time. S differs from K in that S reveals an involuntary stockpiling. A negative S is a cumulate

[1] Viability theory is due to J.P. Aubin, H. Frankowska and other mathematicians. For a rigorous presentation, see: J.P. Aubin, *Viability Theory*, Birkhaüser, 1991. Non mathematical readers will content themselves with: J. P. Aubin, *La mort du devin, l'émergence du démiurge.*

excess of orders over production. Entrepreneurs are indebted. Total debt D is twofold: perpetual bonds B and short-term debt to the banking system L.

Households "own" their personal abilities, called labour. At t, a quantity $N(t)$ is used by entrepreneurs in combination with $K(t)$. As a consequence, households receive wages. They hold financial assets which are the counterparts of entrepreneurs's indebtedness. These counterparts are either direct (bonds) or indirect, the banking system acting as an intermediary. In the balance–sheet of households, L means liquidity (demand – or term – deposits).
Banking system's unique resources are households's deposits L. It lends all the deposits as short-term credits L. Intermediation is supposed to be neutral. In addition, the banking system determines the rate of interest i. At t, our very simple economy is described by the balance–sheets in figure 1:

Figure 1: The economy



Production takes place according to a continuous and twice- differentiable production function defined by

$$Q(t) = F[K(t), N(t)] \tag{1}$$

with $F'_{K,N} > 0$ and $F''_{K,N} < 0$. For the sake of illustration, we can assume a Cobb–Douglas function given by

$$Q(t) = K(t)^{(1-w)}N(t)^{w}. \tag{2}$$

The real wage is equal to the marginal productivity of labour given by the term $wK(t)^{(1-w)}N(t)^{-(1-w)}$. Keynes called this proposition: the first classical postulate and took it for granted. Total wage is equal to $N(t)$ times the real wage, which is equal to $wQ(t)$. Households consume a constant fraction c of their wages. We neglect payments of interest. Consumption is thus determined by

$$C(t) = aQ(t), \tag{3}$$

where $a = cw$. Households divide their financial assets between liquidity and bonds according to Keynes's theory of liquidity preference:

$$L(t) = [1 - bi(t)]D(t). \tag{4}$$

Entrepreneurs's investment is ruled by the state of long-term expectations A, the amount of involuntary stockpiling S and by the rate of interest i:

$$K'(t) = [A - ei(t) - mS(t)/K(t)]K(t). \tag{5}$$

2.2 The model

Three state variables make up the desription of the dynamics of our economy. The first one is the capital K. Equation (5) shows the evolution of K over time. The second one is the involuntary stockpiling S which, incidently, is equivalent to the gap between expected and realized profit. Its evolution is given by the differential equation

$$S'(t) = (1 - a)Q(t) - K'(t). \tag{6}$$

The third state variable is entrepreneurs's debt to the banks, L. Total debt's evolution is defined by

$$D'(t) = (w - a)Q(t), \tag{7}$$

whereas variation of L is the time derivative of equation (4):

$$L'(t) = [1 - bi(t)]D'(t) - bi'(t)D(t). \tag{8}$$

In order to reduce the number of the state variables from three to two, it is convenient to redefine our variables as ratios to K:

$$s = S/K, \quad l = L/K, \quad q = Q/K. \tag{9}$$

We introduce two affine functions G and T of i by

$$G(i) = A - ei \quad \text{and} \quad T(i) = (1 - bi)(w - a). \tag{10}$$

$G(i)$ denotes the "animal spirits" of the Keynesian tradition (the inducement to invest) and $T(i)$ formalizes the incentives to liquidity. We obtain finally the following dynamical system determinating the temporal path of the economy:

$$(11)\begin{cases} i. & s'(t) = ms(t)^2 + (m - G(i(t)))s(t) - G(i(t)) + (1 - a)q(t) \\ ii. & l'(t) = ml(t)s(t) - (G(i(t)) + bi'(t)/(1 - bi(t)))l(t) + T(i(t))q(t) \end{cases}$$

with the initial conditions $s(0) = s_0$ and $l(0) = l_0$. Here, the functions $q(\cdot)$ and $i(\cdot)$ are considered as a priori given so that the solutions of system (11) depend not only on the initial condition (s_0, l_0), but also on $q(\cdot)$ and $i(\cdot)$. In fact, they are actually the two *regulees* of our economy in a viability framework (see next section). Note that the system above is decomposable because the differential equation for the first state s is independent from the second state l.

2.3 A first qualitative analysis of the model

In a first approach we assume the two functions $q(\cdot)$ and $i(\cdot)$ to be constant, and look for the stationary solutions and the qualitative behaviour of the dynamic system (11), which appears now in the form

$$(12)\begin{cases} i. & s'(t) = ms(t)^2 + (m - G(i))s(t) - G(i) + (1-a)q(t), \quad s(0) = s_0 \\ ii. & l'(t) = ml(t)s(t) - G(i)l(t) + T(i)q(t), \quad l(0) = l_0, \end{cases}$$

where $q > 0$ and $i \geq 0$.

We start with the analysis of the first equation. The stationary points are given by

$$(13) \quad s = \left(-(m - G(i)) \pm \sqrt{(m - G(i))^2 + 4m(G(i) - (1-a)q)}\right) / 2m.$$

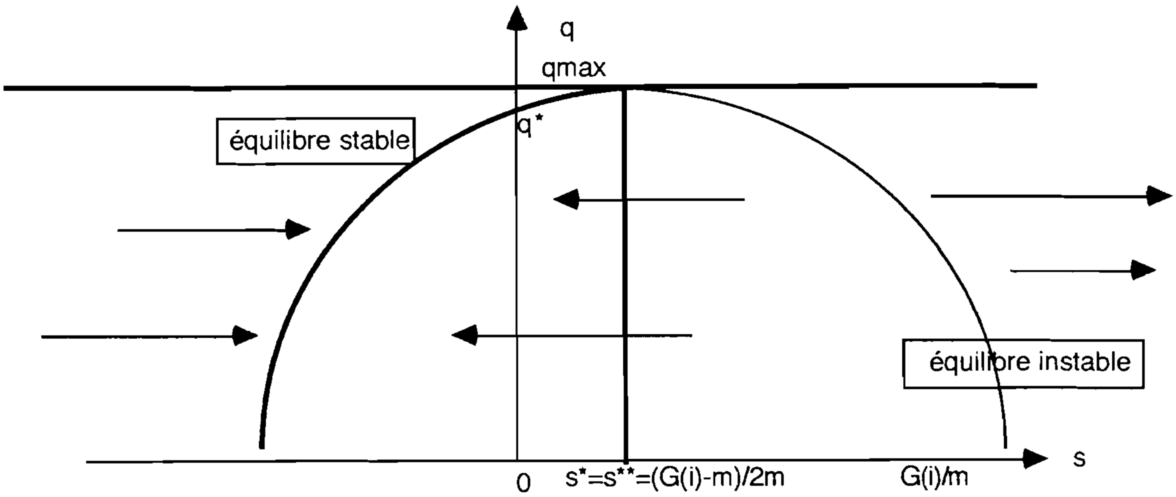
Real roots exist if $((m - G(i))^2 + 4m(G(i) - (1-a)q) \geq 0$, which implies a maximum value for q equal to $q_{max} = (G(i) + m)^2/4m(1-a)$. For $q = q_{max}$, roots are equal: $s^* = s^{**} = (G(i) - m)/2m$. In order to get $s^* = s^{**} = 0$, i. e., a full equilibrium, we must have $G(i) = m$. No demand pressure, coming either from investment or consumption exists when the economy evolves along the full equilibrium path. In this case $q_{max} = q^* = G(i)/(1-a)$, which is the usual Keynesian multiplier formula. Condition $G(i) = m$ is never fulfilled but through the banking system settling i at $i^* = (A - m)/e$. Nothing guarantees such a behaviour of the banking system.

We study now the influence of the choice of the parameters q and i on the position of the equilibria and on the monotonicity of the state s. For fixed s, the parameter providing the equilibrium solution is given by

$$(14) \qquad q(s) = -(ms^2 + (m - G(i))s - G(i))/(1-a).$$

This function defines a parabola dividing the s-q-plane in two monotonic zones: below the parabola, $s'(t)$ is negative and above, $s'(t)$ is positive (see figure 2). The relative position of the curve $q(s)$ depends uniquely on $G(i)$ and m. When animal spirits are strong enough, as compared with m, which is the intensity of reaction to involuntary stockpiling, the parabola's maximum is at the right of the vertical axis $s = 0$. It stands to the left in the opposite case. In both cases,

Figure 2: The sign of $s'(t)$



full equilibrium does not correspond to q_{max}; in other words, full equilibrium and full employment do not coincide except for $G(i) = m$.

Still fixing q and i, let us now look for the complete system (12). In the s–l–plane, the isoclines where $s'(t) = 0$ and $l'(t) = 0$ respectively are given by the two vertical lines

$$(15) \quad s = \left(-(m - G(i)) \pm \sqrt{(m - G(i))^2 + 4m(G(i) - (1 - a)q)}\right) /2m,$$

and by the hyperbola

$$(16) \qquad l = T(i)q/(G(i) - ms).$$

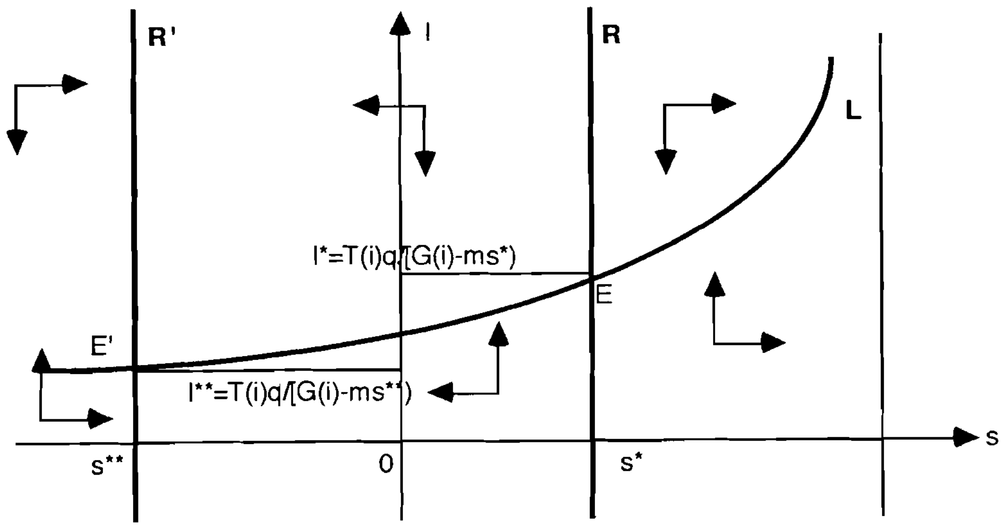
Both isoclines are drawn on the phase diagram in figure 3. The equilibria E and E' given by the intersection of the two isoclines are a saddle point and a sink respectively. But the sink E' has to be discarded as economically meaningless. These conclusions are not what we are looking for. The description above has only paved the way for a study along the viability approach.

3 Viability for an economy: a brief view

3.1 The problem

Dynamic properties of stationary equilibria, although important, are not the end of the story. Several reasons may be invoked in favour of a further analysis.

Figure 3: Phase diagram



The first one is that local stability analysis gives poor intuitions about the effective temporal path out of equilibrium. That E is a saddle-point suggests that the economy is pushed farther and farther away from equilibrium (except for rational expectations hypothesis), but nothing is known about the velocity of this process and the span of time during which economy is not too far from equilibrium. "In the long term, we are all dead',' Keynes reminded us. Asymptotic properties are not sufficient for all the economists interested by the evolution *hic et nunc* of the economy.

A second reason for being not really satisfied by traditional economic dynamics is that economies in the real world are normally out of equilibrium. This fact does not seem to put at stakes the survival of our society. Rather, disequilibrium situations appear to induce more or less efficient reactions of economic agents and to be the condition *sine qua non* of the evolution (Schumpeter has put particular emphasis on the role of entrepreneurs in this process).
However, common sense suggests that economy must not be too far from equilibrium. Limits exist beyond which adaptative or innovative reactions cease to be possible. The crucial point is not whether economy is or not in equilibrium but whether evolution away of equilibrium is or not compatible with the fundamental data and ordinary rules of the economy. It becomes then natural to specify the minimal conditions under which an economy can work and to check if the dynamic evolution of the economy does not violate these conditions. Is it possible to constrain effective trajectories to evolve in a given region of the phase space?

In our model, economy is described by two ratios, s and l. It seems sensible to admit that limited deviations from s^* and l^* have not dramatic consequences but also that some critic values must not be passed over. Beyond these limits, normal working of the economy is no longer possible. Let us call *constraint set* the set defined by such limits. The viability problem consists in checking if there exists or not for a given dynamical system with a given constraint set for every point in the constraint set at least one trajectory starting from there and remaining in the constraint set over time. If for all values in the constraint set there exists such a trajectory, it is called a *viability domain*. A constraint set is in general not a viability domain, but there may exist a nonempty subset of the constraint set which satisfies the viability condition in every point, and we call the largest subset exhibiting this property the *viability kernel* of the constraint set.

A third reason to adopt a viability point of view is to be found in the special character of economic regulation. In the great tradition of political economy, the adjustment of some variables to disequilibria plays a prominent role. The so-called "law of supply and demand" states that prices react in a determined direction in response to a difference beetween supply and demand in the market: the price of a particular commodity is assumed to vary according to the sign of the excess demand of this commodity. Another famous law of adjustment concerns the variation of the quantity in reaction to the profitability of its production. For instance, in Walras theory of production, the quantity varies according to the sign of the profit (sales minus costs). In our model, two variables are candidates to play this role: i and q. System (11) generates trajectories depending on i and q. It would seem natural to assume that q changes according to the sign of $s'(t)$ or $s(t)$ and, conversely, that i varies with the sign of $l'(t)$. Despite the respectability of the tradition, we follow another track. Instead of reasoning with a law of adjustment *a priori* given, we fix limits to the speed of variation of the regulees but the sign of variation is not preassigned. These limits are more or less narrow, depending on the flexibility of the system. Intuitively, to more flexible economies correspond wider viability domains or viability kernels. Thus, i and q are supposed to vary within certain limits but without any *a priori* law of variation.
Does it mean that we must renounce to the very notion of a law of adjustment? Absolutely not. On the contrary, we may hope to derive one from the study of the viability, which is far better than to postulate one beforehand. We shall give below a more explicit formulation for the viability problem. Before that, it is necessary to give further indications about the economy we have modelled.

3.2 Viability constraints

A market economy keeps working only if entrepreneurs do not incur bankruptcy. This implies in turn that a state of confidence rules over the entire economy.

Such a state of confidence may persist only if certain limits are not reached. For instance, the indebtedness of entrepreneurs must be moderate enough to guarantee debtors against insolvency. In the same way, the profitability of entrepreneurship must be high enough to convince people to commit themselves in such an activity. In our model, two thresholds are significant. The first one is $\bar{s}$, the maximum value for s, which is a rentability indicator. Beyond $\bar{s}$, it is no longer possible for entrepreneurs to remain active, because the difference between expected and realized profit is too big. The ratio s must not be too large if negative, because it would mean that entrepreneurs are not able to supply a sufficient fraction of the demand. We have hence the following constraint:

$$\underline{s}K(t) \leq S(t) \leq \bar{s}K(t) \qquad \forall t \geq 0. \tag{17}$$

The second one is a solvency indicator. The indebtedness of entrepreneurs to banks cannot exceed the value $\bar{l}$. If it were the case, creditors would loose confidence and entrepreneurs would go into bankruptcy. For evident reasons, l must be positive: otherwise, entrepreneurs would be creditors of banks and not the reverse! We have then a second constraint:

$$0 \leq L(t) \leq \bar{l}K(t) \qquad \forall t \geq 0. \tag{18}$$

The economy is described by two state variables, $s(t) = S(t)/K(t)$ and $l(t) = L(t)/K(t)$, which have to remain in the constraint set defined by the two thresholds (17) and (18).
Two regulees intervene into the working of the economy:

- The first one is the production per unit of capital $q(t)$ which depends within certain limits on entrepreneurs. The regulee $q(t)$ may be interpreted also as a rate of utilization of capital (or of labour if it is recalled that $q = f(n)$). Due to the inertia of entrepreneurs or to a lack of flexibility of institutions, we admit that limits exist to the adjustment of q:

  $$-u \leq q'(t) \leq u \qquad \forall t \geq 0. \tag{19}$$

  Furthermore, q is also submitted to the constraints

  $$\underline{q} \leq q(t) \leq \bar{q} \qquad \forall t \geq 0. \tag{20}$$

- The second is the level of the rate of interest which is governed by the banking system. Some limits exist also to the action of the banks, so that

  $$-v \leq i'(t) \leq v \qquad \forall t \geq 0. \tag{21}$$

  Naturally, the interest rate has to be positive:

  $$i(t) \geq 0 \qquad \forall t \geq 0. \tag{22}$$

We obtain hence the following control system

$$(23)\begin{cases} i. & s'(t) = ms(t)^2 + (m - G(i(t)))s(t) - G(i(t)) + (1-a)q(t) \\ ii. & l'(t) = ml(t)s(t) - (G(i(t)) + bi'(t)/(1-bi(t)))l(t) + T(i(t))q(t) \\ iii. & q'(t) \in [-u, u] \\ iv. & i'(t) \in [-v, v], \end{cases}$$

under the constraints

$$(24)\qquad \begin{cases} i. & \underline{s} \le s(t) \le \overline{s} \\ ii. & 0 \le l(t) \le \overline{l} \\ iii. & \underline{q} \le q(t) \le \overline{q} \\ iv. & 0 \le i(t) \end{cases}$$

for all $t \ge 0$. Naturally, there exists not for all initial condition (s_0, l_0, q_0, i_0) in the constraint set $K = [\underline{s}, \overline{s}] \times [\underline{l}, \overline{l}] \times [\underline{q}, \overline{q}] \times [0, \infty[$ a viable solution, i. e., a state–control solution $(s(\cdot), l(\cdot), q(\cdot), i(\cdot))$ of the control system (23) satisfying the constraints (24) at each instant $t \ge 0$.
The problem viability theory deals with is to find viable subsets $\mathcal{D}$ of the constraint set K, where a viable evolution is always possible, i. e., subsets $\mathcal{D}$ satisfying the following *viability property*:

> For all $(s_0, l_0, q_0, i_0) \in \mathcal{D}$ there exists a state–control solution $(s(\cdot), l(\cdot), q(\cdot), i(\cdot))$ of control system (23) starting at (s_0, l_0, q_0, i_0), and satisfying the constraints
>
> $$\begin{array}{ccccc} \underline{s} & \le & s(t) & \le & \overline{s} \\ 0 & \le & l(t) & \le & \overline{l} \\ \underline{q} & \le & q(t) & \le & \overline{q} \\ 0 & \le & i(t) & & \end{array}$$
>
> for all $t \ge 0$.

In particular we will look for the largest viable subset of the constraint set K, which is the viability kernel $Viab(K)$ of K for the control system (23). The viability kernel can be seen as the graph of a *regulation map* governing the evolution of those controls providing viable solutions. Whenever a solution $(s(\cdot), l(\cdot), q(\cdot), i(\cdot))$ satisfies $(q(t), i(t)) \in R(s(t), l(t))$ for all $t \ge 0$, this solution is viable.

The existence of viable solutions does not mean that the economy described by system (23) will never collapse. For a trajectory starting in the viability kernel, it is not sure that it will remain in the viability kernel and hence remain viable over time, but one can guarantee the possibility to avoid the breakdown by choosing proper values at proper times for the regulees. In particular, we can consider *heavy solutions*, i. e., solutions regulated by controls being constant as long as viability is not at stake. Emptyness of the viability kernel would mean that an incompatibility exists between the dynamics derived from the behaviour of agents (system (11)) and the general flexibility of the economy considered which is given by (19) and (21).

4 Numerical results

4.1 The problem under the mathematical point of view

The analytical characterization of viability kernels is very difficult. However, they can be determined by the viability kernel algorithm [5] [8] [9], which has been implemented for two dimensional systems and which is in preparation for three dimensional ones. Our control system (23) is in fact four dimensional (two states – two controls), and has to be simplified in order to obtain numerical results.

When looking for heavy solutions, i. e., solutions regulated by controls being constant as long as viability of the solution is not at stake, we admit a hierarchy between the two regulees: The evolution of the interest rate i has more inertia than the evolution of the relative production rate q. To simplify, we shall therefore assume that $i'(t) = 0$ or $i = constant$. We shall hence consider the following three dimensional system

$$(25) \quad \begin{cases} i. & s'(t) = ms(t)^2 + (m - G(i))s(t) - G(i) + (1-a)q(t) \\ ii. & l'(t) = ml(t)s(t) - G(i)l(t) + T(i)q(t) \\ iii. & q'(t) \in [-u, u], \end{cases}$$

under the constraints

$$(26) \quad \begin{cases} i. & \underline{s} \leq s(t) \leq \overline{s} \\ ii. & 0 \leq l(t) \leq \overline{l} \\ iii. & \underline{q} \leq q(t) \leq \overline{q} \end{cases}$$

for all $t \geq 0$, and compare the results for different – constant – interest rate i. For the reduced control system

$$(27) \quad \begin{cases} i. & s'(t) = ms(t)^2 + (m - G(i))s(t) - G(i) + (1-a)q(t) \\ ii. & q'(t) \in [-u, u], \end{cases}$$

under the constraints

$$(28) \quad \begin{cases} i. & \underline{s} \leq s(t) \leq \overline{s} \\ ii. & \underline{q} \leq q(t) \leq \overline{q} \end{cases}$$

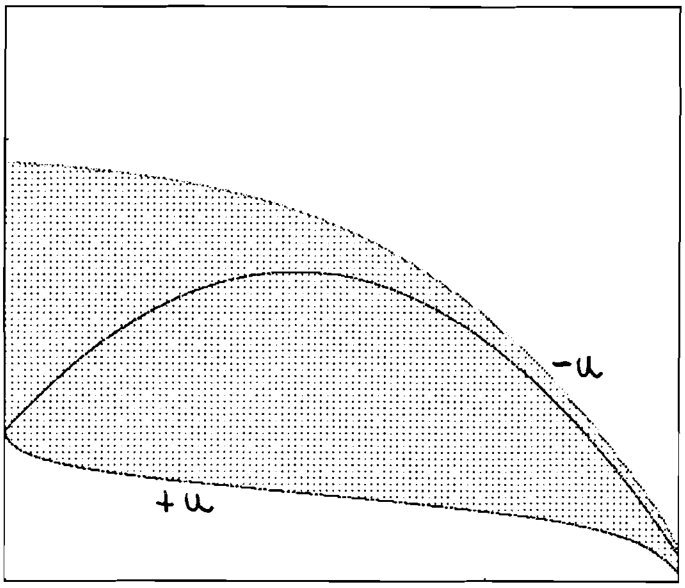
for all $t \geq 0$, consisting only on the equation for s and the inclusion for q, we can even characterize the viability kernels analytically and compute them.

4.2 Numerical results for the two dimensional subsystem

We consider still the two dimensional subsystem

$$(29) \quad \begin{cases} i. & s'(t) = ms(t)^2 + (m - G(i))s(t) - G(i) + (1-a)q(t) \\ ii. & q'(t) \in [-u, u], \end{cases}$$

Figure 4: Viability kernel together with the equilibrium parabola



under the constraints

$$(30) \qquad \begin{cases} i. & \underline{s} \leq s(t) \leq \overline{s} \\ ii. & \underline{q} \leq q(t) \leq \overline{q} \end{cases}$$

for all $t \geq 0$. For fixed interest rate i, we want to find the viability kernel $Viab(K)$ of the constraint set $K = [\underline{s}, \overline{s}] \times [\underline{q}, \overline{q}]$ for the control system (29), i. e., the set

$$(31) \qquad Viab(K) = \{(s_0, q_0) \in K; \ \exists \text{ solution } (s(\cdot), q(\cdot)) \text{ of (29) starting in } (s_0, q_0) \text{ and satisfying } \underline{s} \leq s(t) \leq \overline{s}, \ \underline{q} \leq q(t) \leq \overline{q} \ \ \forall t \geq 0\}.$$

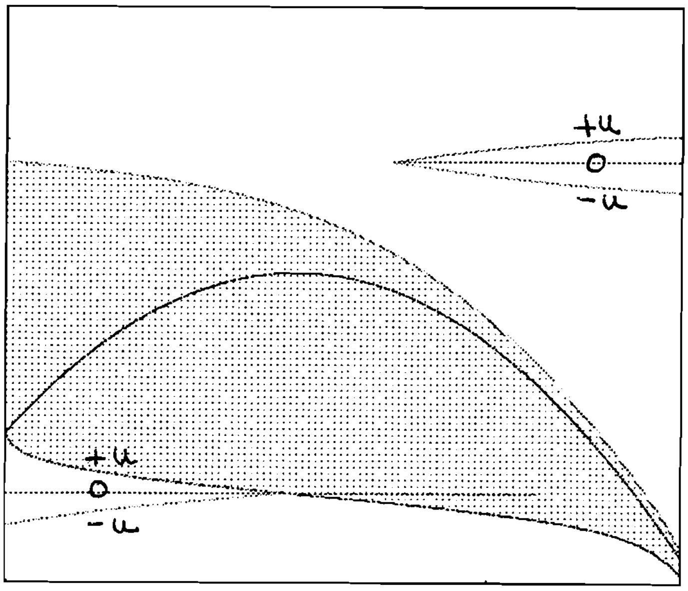
One can show [7] that for this type of control system, the viability kernel is limited by two trajectories starting on the respective intersection points of the equilibrium parabola (14) and the constraint set K such that q increases or decreases maximally, i. e.,

$$(32) \qquad q'(t) = +u \quad \text{repectively} \quad q'(t) = -u.$$

(See figure 4. [2])

[2]For the computation we took the values $m = 0.2, A = 0.25, e = 1, a = 0.8, w = 1, b = 4, u = 0.01, \underline{s} = -0.6, \overline{s} = 0.8, \underline{q} = 0.3, \overline{q} = 1.6, \overline{l} = 0.36$.

Figure 5: Heavy solution and solution starting outside of the viability kernel



The set $Viab(K)$ can be seen as the graph of a *regulation map* $R : [\underline{s}, \overline{s}] \longrightarrow [\underline{q}, \overline{q}]$,

(33) $$q \in R(s) \iff (s, q) \in Viab(K).$$

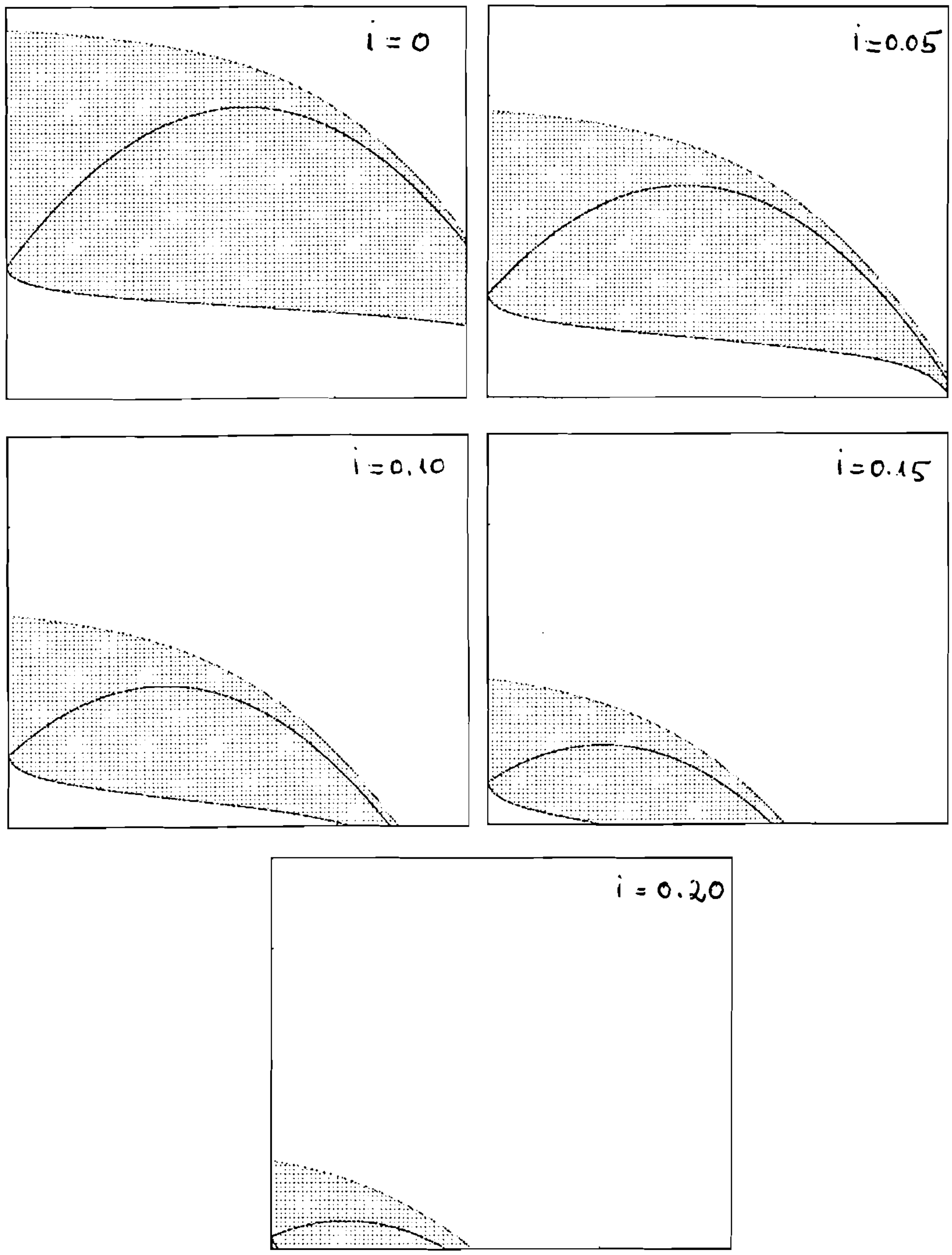
This defines a (feedback) *regulation law* for the viable evolutions: a state–control solution $(s(\cdot), q(\cdot))$ of control system (29) is viable if and only if it satisfies the regulation law

(34) $$q(t) \in R(s(t)) \qquad \forall t \geq 0.$$

Heavy evolutions, where the control is constant as long as the viability of the solution is not at stake are of particular interest. Figure 5 shows an example for a heavy solution. It starts in the interior of the viability kernel regulated by a constant control until arriving on the boundary of the viability kernel. When the control q does not move, the solution leaves the viability kernel and hence finally the constraint set, because all solution starting from outside the viability kernel has to leave the constraint set in finite time per definition (31). (See figure 5 as well.) In this example, the control has to be increased with maximal velocity $+u$, so that the solution can remain on the boundary of the viability kernel and converge to a viable equilibrium.

We end this section by showing a sequence of viability kernels varying with different interest rate i. For this computation we took the values $i = 0, i = 0.05, i = 0.10, i = 0.15, i = 0.20$.

Figure 6: Different viability kernels for i varying



4.3 Analysis of the s–l–subsystem

Since at the moment, we are not able to compute three dimensional viability kernels, we shall derive some informations about the control system (25) of the generalized control system

$$(35) \quad \begin{cases} i. & s'(t) = ms(t)^2 + (m - G(i))s(t) - G(i) + (1-a)q(t) \\ ii. & l'(t) = ml(t)s(t) - G(i)l(t) + T(i)q(t), \\ & \text{where } q(t) \in [\underline{q}, \overline{q}] \text{ for all } t \geq 0, \end{cases}$$

under the constraints

$$(36) \quad \begin{cases} i. & \underline{s} \leq s(t) \leq \overline{s} \\ ii. & 0 \leq l(t) \leq \overline{l} \end{cases}$$

for all $t \geq 0$. The regulee $q(\cdot)$ is not only not fixed, but no limit on the variation of $q(\cdot)$ is imposed as in system (25). The projection of the viability kernel of the complete system (25) is contained in the viability kernel of the constraint set $K = [\underline{s}, \overline{s}] \times [0, \overline{l}]$ for system (35), because for all viable state–control solution $(s(\cdot), l(\cdot), q(\cdot))$ of (25), $(s(\cdot), l(\cdot))$ is a viable solution of (35).

We first investigate the monotonic behaviour of the solutions of (35).
Figure 7 shows a partition of the s–l–plane in *monotonic cells* for the system (35), i. e., in subsets of the plane where all solutions of (35) have the same monotonic behaviour. The two vertical lines limit the set of isoclines for the first equation. Easily, one derives that in the sets left of the left line and right of the the right line, s' is positive for all solutions of system (35), whereas it can have all sign in the zone between the two lines. The two hyperbolas — the upper one corresponding to the parameter $\overline{q}$, the lower one corresponding to $\underline{q}$ — limit the set of isoclines for the second equation. Below the lower hyperbola, l' has to be positive, above the upper hyperbola negative, and between the two hyperbolas, l' may have all sign. Le line contained in the set K_{ii}, where the sign of both s' and l' is indifferent, is the set of all equilibria of the system (35). Only a small part of it is contained in the constrained set (the cube). On the boundary of K, we marked the set $K_{\Longrightarrow} =]s_H, \overline{s}] \times \{\overline{l}\} \cup \{\overline{s}\} \times [0, \overline{l}]$, where $s_H = (g(i) - T(i)\underline{q}/\overline{l})/m$ is the point where the lower hyperbola meets the line $l = \overline{l}$. The set $K_{\Longrightarrow}$ is the subset of K, where all solution of (35) has to leave the constraint set K immediately.

Figure 8 shows the viability kernel for the generalized system. It is limited by the solution of (35) passing through the point $(s_H, \overline{l})$ and regulated by the constant (minimal) control $q(t) = \underline{q}$.

What happens when the interest rate i is varying?
Figure 9 shows that, for our choice of the parameters in the computation, the viability kernel effectively increases in size when the interest rate i decreases. But, at the same time, the set of viable equilibria *becomes smaller and smaller*

Figure 7: Monotonic cells of the generalized s–l–system

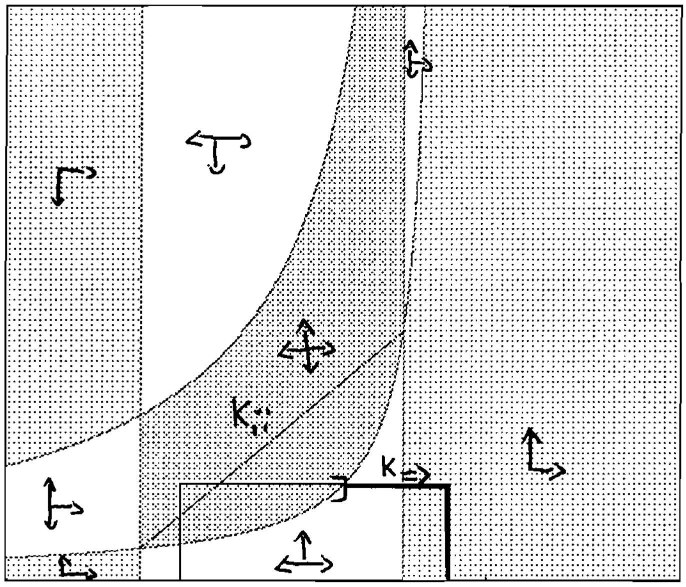
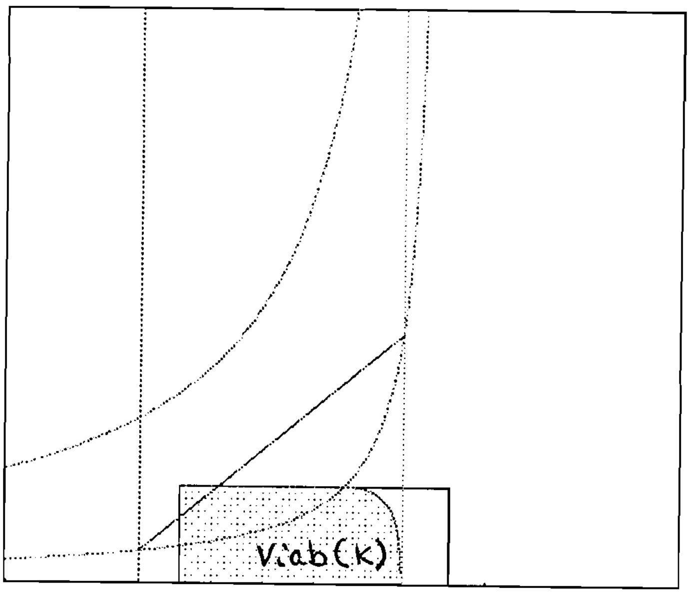
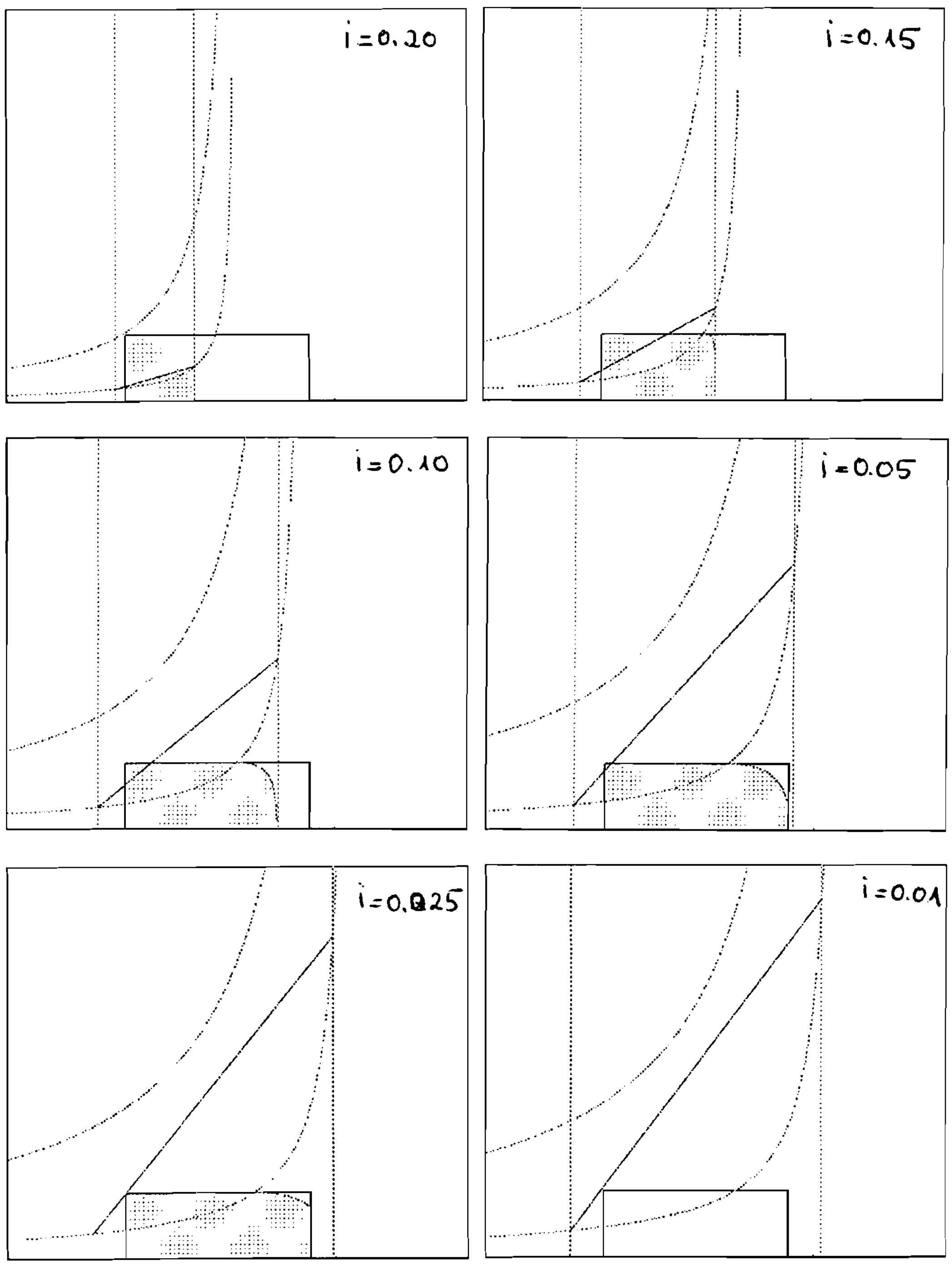


Figure 8: Viability kernel for the generalized system



until $i = 0.025$, where the only viable equilibrium (producing the maximal viability kernel!) is the edge point $(\underline{s}, \bar{l})$. For i smaller then $i_0 = 0.025$, the constraint set contains no equilibria anymore, and the viability kernel is empty.

Figure 9: Sequence of viability kernels for $i = 0.20, 0.15, 0.10, 0.05, 0.025, 0.01$



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