Working Paper

A Differential Game with one Target and Two Players

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Abstract : We study a two players-differential game in which one player wants the state of the system to reach an open target and the other player wants the state of the system to avoid the target. We characterize the victory domains of each player as the largest set satisfying some geometric conditions and we show a "barrier phenomenon" on the boundary of the victory domains.

Résumé : Nous étudions un jeu différentiel à deux joueurs dans lequel un des joueurs cherche à faire entrer l'état du système dans une cible donnée tandis que l'autre joueur veut que l'état du système évite la cible. Nous caractérisons le domaine de victoire de chacun des joueurs comme le plus grand ensemble vérifiant certaines conditions géométriques, puis nous mettons en évidence un phénomène de barrière sur le bord des domaines de victoire.

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We study here the target problem: Two players, Ursula and Victor, control the dynamical system:

$$\text{(1)} \qquad \begin{cases} x'(t) = f(x(t), u(t), v(t)) \text{ for almost every } t \geq 0 \\ u(t) \in U, \ \ v(t) \in V \\ x(0) = x_0 \end{cases}$$

Let Ω be an open target of $\mathbb{R}^N$. Ursula, acting on u, wants the state of the system $x(\cdot)$ to reach the target Ω, while Victor, acting on v, wants the state of the system $x(\cdot)$ to avoid Ω. We want to determinate and characterize the victory domains of each player, i.e., the set of initial positions x_0 from where this player may win whatever does his adversary.

We study this game in the framework of nonanticipative strategies: See ([6],

[7], [8]). If we denote by

$$(2) \qquad \begin{cases} \mathcal{U} = \{u(\cdot) : [0,+\infty[\to U, \text{ measurable application } \} \\ \mathcal{V} = \{v(\cdot) : [0,+\infty[\to V, \text{ measurable application } \} \end{cases}$$

the sets of time-measurable controls, nonanticipative strategies are defined in the following way:
A map $\alpha : \mathcal{V} \to \mathcal{U}$ is a nonanticipative strategy (for Ursula) if it satisfies the following condition : For any $s \geq 0$, for any $v_1(\cdot)$ and $v_2(\cdot)$ of $\mathcal{V}$, such that $v_1(\cdot)$ and $v_2(\cdot)$ coincide almost everywhere on $[0, s]$, the image $\alpha(v_1(\cdot))$ and $\alpha(v_2(\cdot))$ coincide almost everywhere on $[0, s]$.
Nonanticipative strategies $\beta : \mathcal{U} \to \mathcal{V}$ (for Victor) are defined in the symetric way.

• THE VICTORY DOMAINS

Let us denote by $x[x_0, u(\cdot), v(\cdot)]$ the solution of (1) starting from $x_0 \in \mathbb{R}^N$, with $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$.
We are now ready to define the victory domains of each player.
- *Victor's victory domain* is the set of initial positions $x_0 \notin \Omega$ for which Victor can find a nonanticipative strategy $\beta : \mathcal{U} \to \mathcal{V}$ such that for any time-measurable control $u(\cdot) \in \mathcal{V}$ plaid by Ursula, the solution $x[x_0, u(\cdot), \beta(u(\cdot))]$ avoids Ω for any $t \geq 0$.
- *Ursula's victory domain* is the set of initial positions $x_0 \notin \Omega$ for which Ursula can find a nonanticipative strategy $\alpha : \mathcal{V} \to \mathcal{U}$, positive ϵ and T such that, for any $v(\cdot) \in \mathcal{U}$ plaid by Victor, the solution $x[x_0, \alpha(v(\cdot)), v(\cdot)]$ reaches the set $\Omega_\epsilon := \{x \mid d_{\Omega^c}(x) \geq \epsilon\}$ before T.
(We denote by $d_K(x)$ the distance from a point x to a closed set K.)

• ASSUMPTIONS AND NOTATIONS

Throughout this paper, we assume that f satisfies:

$$(3) \qquad \begin{cases} i) & \text{U and V are metric compact spaces. } V \in \mathbb{R}^d \text{ is convex.} \\ ii) & f : \mathbb{R}^N \times U \times V \to \mathbb{R}^N \text{ is continuous.} \\ iii) & f(\cdot, u, v) \text{ is a } \ell\text{-Lipschitz map for any } u \text{ and } v. \\ iv) & f \text{ is affine in } v. \end{cases}$$

We also always assume that Isaacs' condition is fulfilled:

$$\forall (x,p) \in \mathbb{R}^{2N}, \ \sup_u \inf_v < f(x,u,v), p >= \inf_v \sup_u < f(x,u,v), p >$$

We shall denote by B the closed unit ball of the state space $\mathbb{R}^N$.

1 - Discriminating domains and kernels

We first define, by the mean of geometric conditions, some sets - the discriminating domains and kernels - that shall play a great role in the sequel. For that purpose, let us first define generalized outward normals for closed sets which are not regular.

Definition 1 (Proximal normal) *A vector* $\nu \in I\!R^N$ *is a proximal normal to a closed set* K *at a point* $x \in K$ *if and only if :* $d_K(x+\nu) = \|\nu\|$.
We denote by $NP_K(x)$ *the set of proximal normals to* K *at* x.

If K is a $\mathcal{C}^2$ manifolds and x belongs to ∂K, any outward normal is a proximal normal up to a positive multiplicative coefficient.

Definition 2 (Discriminating domain) *A closed set* D *is a discriminating domain for* f *if*

$$\forall x \in D,\ \forall \nu \in NP_D(x),\ \sup_u \inf_v < f(x,u,v), \nu > \leq 0 \tag{4}$$

If a closed set K is not a discriminating domain, it contains a largest discriminating domain:

Theorem 1 (Discriminating kernel) *Let* K *be a closed subset of* $I\!R^N$. *There is a (may be empty) closed discriminating domain for* f *contained in* K *which contains any other discriminating domain for* f *contained in* K.
This set is called the discriminating kernel of K *for* f *and is denoted by* $Disc_f(K)$.

• Discriminating domains and Viability Theory
Let us now characterize the discriminating domains for dynamics f in terms of Viability Theory. The definitions and results of Viability Theorem can be found in the monograph ([1]).

Proposition 1 *A closed set* D *is a discriminating domain for* f *if and only if* D *is a viability domain for the set-valued maps* $x \rightsquigarrow f(x,u,V)$ *for any* $u \in U$.

Note that the discriminating domains for f are actually the discriminating domains defined in ([1]). The discriminating kernel of a closed set K can be computed as intersections of viability domains:

Proposition 2 (Algorithm for the discriminating kernel) *Let* K *be a closed subset of* $I\!R^N$. *Define the following decreasing sequence of closed sets:*

$$\begin{cases} K_1 := K \\ K_{i+1} := \bigcap_{u\in U} Viab_{f(\cdot,u,V)}(K_i) \end{cases} \tag{5}$$

Then the intersection of the K_i *is equal to* $Disc_f(K)$.

This results is used to prove Theorem 4 bellow.

2 - Interpretation of the discriminating domains

• When "Victor has a spy"
The following Theorem means that the discriminating domains are the sets in which Victor can ensure the state of the system to remain as soon as he knows which control Ursula plays:

Theorem 2 *A closed subset $D \subset \mathbb{R}^N$ is a discriminating domain for f if and only if, for any $x_0 \in D$, there exists a nonanticipative strategy $\beta : \mathcal{U} \to \mathcal{V}$ (for Victor), such that, for any $u(\cdot) \in \mathcal{U}$, the solution $x[x_0, u(\cdot), \beta(u(\cdot))]$ remains in D on $[0, +\infty)$.*

• WHEN "URSULA HAS A SPY"

If Ursula plays nonanticipative strategies, then discriminating domains are the sets in which Victor can "almost" ensure the state of the system to remain:

Theorem 3 *A closed subset $D \subset \mathbb{R}^N$ is a discriminating domain for f if and only if, for any x_0 of D, for any nonanticipative strategy $\alpha : \mathcal{V} \to \mathcal{U}$, for any positive ϵ and for any time $T \geq 0$, there exists a control $v(\cdot) \in \mathcal{V}$ such that the solution $x[x_0, \alpha(v(\cdot)), v(\cdot)]$ remains[1] in $D + \epsilon B$ on $[0, T]$.*

• APPLICATIONS TO THE TARGET PROBLEM

Theorem 4 (Alternative) *Set $K := \mathbb{R}^N \backslash \Omega$. Then:*
- Victor's victory domain is equal to $Disc_f(K)$.
- Ursula's victory domain is equal to $K \backslash Disc_f(K)$.

Note that the victory domains of the two players form a partition of the closed set K. A similar Alternative Theorem have been obtained by Krasovskii & Subbotin in the framework of the positional strategies (See [10]). Our results are thinest because there is no need to introduce the notion of "constructive motions" as in ([10]).
Moreover, we characterize the victory domains by means of geometrical conditions (it is the discriminating kernel of a closed set). This characterization is used in the joint work with M. Quincampoix & P. Saint-Pierre ([4]) to compute numerically the victory domains (see also [12] in the framework of control theory).

3 - Semi-permeable surfaces

Theorem 5 *Let K be a closed subset of $\mathbb{R}^N$ and x belong to the boundary of $Disc_f(K)$ and to the interior of K. Then, for any proximal normal ν to $\overline{\mathbb{R}^N \backslash Disc_f(K)}$ at x, one has:*

$$\sup_u \inf_v < f(x, u, v), -\nu > \geq 0, \tag{6}$$

Since $Disc_f(K)$ is a domain for f, it also satisfies (4) with $D := Disc_f(K)$. In particular, if $Disc_f(K)$ is a smooth (say $\mathcal{C}^2$) manifold in the neighbourhood of some point x of $\partial Disc_f(K) \backslash \partial K$, the outward normal ν of the boundary of $Disc_f(K)$ satisfies the so-called Isaacs' equation:

$$\sup_u \inf_v < f(x, u, v), \nu > = 0 \tag{7}$$

[1] We denote by $D + \epsilon B$ the set of points x such that $d_D(x) \leq \epsilon$.

thanks to (4) and (6). So, combining (4) and (6) gives a generalizalized solution of equation (7).
Smooth surfaces satisfying (7) are called semi-permeable surfaces by Isaacs. The reason is that each player can avoid the state of the system to cross this surface in one sense. In general, the boundary of the victory domains are not smooth and Isaacs' methods do not work any more. But we prove bellow that, in any case, the boundary of the victory domains is "almost semi-permeable". Similar results have been obtained by M. Quincampoix in the framework of Control Theory (See [11]).

Theorem 6 *Let K be a closed subset of $\mathbb{R}^N$, x_0 belong to $\partial Disc_f(K)$ but not to ∂K.*
If $\beta : \mathcal{U} \longrightarrow \mathcal{V}$ is a winning nonanticipative strategy[2] *for Victor, there is a time $T > 0$ such that, for any positive ϵ, Ursula can find a control $u(\cdot) \in \mathcal{V}$ such that the solution $x[x_0, u(\cdot), \beta(u(\cdot))]$ remains in $\partial Disc_f(K) + \epsilon B$ on $[0, T]$.*

This result means that Ursula can "almost" ensure the state of the system to remain in $\partial Disc_f(K)$. A similar result holds true if "Ursula has a spy".

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