SIMPLIFIED VARIANTS OF PENALTY FUNCTION METHODS

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Introduction

The idea of utilization penalty functions for nonlinear programming was suggested first by R. Courant. Later this approach was developed, generalized, and extended significantly and a great number of important results have been obtained. Extensive literature exists on penalty function methods. For the basic result refer to Fiacco and McCormick [1]. The list of references at the end of this paper includes only articles closely relevant to the methods presented. Penalty function methods have several disadvantages. The main ones are as follows.

1) The methods are time-consuming. They do require multiple solving of unconstrained minimization sub-problems.

2) Solution of minimization sub-problems becomes exceedingly cumbersome when the penalty coefficient increases, since a minimization function becomes ill-conditioned.

3) The usual penalty methods cannot be used for determining a solution with great accuracy. These methods are subject to numerical instabilities because the derivatives of the penalty functions increase without bound near the solution as computation proceeds.

The methods suggested below essentially simplify standard penalty function procedure and remove, to some extent, the first two shortcomings.

Statement of Problem and Some Definitions

We consider the following primal nonlinear programming problem:

$$\text{minimize } F(x) \tag{1}$$

subject to constraint

$$x \in X = \{x \in E_n | g(x) = 0, \quad h(x) \leq 0\}$$

where F, g, h are given functions defined on E_n, Euclidean n-space, $x = (x^1, x^2, \ldots, x^n)$ is a point in E_n, functions F, g, h define the mappings $F: E_n \to E_1$, $g : E_n \to E_e$, $h : E_n \to E_c$, where E_i is i-dimensional Euclidean space.

Introduce the solution set X_* of problem (1) and the strictly interior points set of X:

$$X_* = \{x_* | \min_{x \in X} F(x) = F(x_*) \quad , \quad x_* \in X\} ,$$

$$X_0 = \{x \ | g(x) = 0 \quad , \quad h(x) < 0\} .$$

Problem (1) is called a convex programming problem if F(x), h(x) are convex functions and g(x) is affine.

In this paper a number of methods will be suggested for solving problem (1). They can be used when (1) is a general nonlinear programming problem, but here we shall prove the convergence of the methods only for the simplest case when (1) is a convex programming problem.

We shall consdier mainly continuous versions of numerical methods which would be governed by an ordinary differential equation

$$\dot{x} = f(x, t) \tag{2}$$

where a super dot denotes differentiation with respect to time independent variable t, and $f(x, t)$ is a continuous function of both arguments.

The solution of system (2) with a given initial condition $x = x_0$ at $t = 0$ is denoted by $x(x_0, t)$. We shall choose the function $f(x, t)$ in order to obtain such a system (2) that its solution $x(x_0, t)$ converges to a point which belongs to solution set X_*.

For simplicity, we assume the existence of a unique solution of (2) in some vicinity of the point $t = 0$ for any given x_0. Uniqueness is not an important restriction and can be easily omitted.

The distance $\rho(x, M)$ of the point x from the set M is defined as

$$\rho(x, M) = \inf ||x - y||, \quad y \in M$$

where $|| \cdot ||$ is the euclidean norm.

Definition: The system (2) is called Lagrange stable if 1) a solution (may be non-unique) $x(x_0, t)$ exists for any x_0 and for all $t \geq 0$; 2) a bound $B(x_0)$ exists such that $||x(x_0, t)|| < B(x_0)$ for all $t \geq 0$ (and all solutions).

Definition: The set M is called invariant with respect to the system (2) if, for any $x_0 \in M$, the solution $x(x_0, t)$ belongs to M for all $t \geq 0$.

Definition: The set $\omega(x_0)$ is called the positive limit set of a bounded motion $x(t, x_0)$ if, for any point p in $\omega(x_0)$ a sequence of times $\{t_i\}$ exists tending to infinity as $i \to \infty$

so that

$$\lim_{i\to\infty} ||x(t_i) - p|| = 0 \quad .$$

Definition: The method (2) converges globally (on set M) to set X_* if positive limit set $\omega(x_0) \subset X_*$ for any $x_0 \in E_n$ (for any $x_0 \in M$).

In other words method (2) converges to the set X_* if any limit point of the solution $x(x_0, t)$ of system (2) solves problem (1).

Methods similar to (2) are suitable for use on an analog computer. If we solve the problem on a digital computer, then instead of (2), the following simplest discrete version of (2) can be utilized

$$x_{s+1} = x_s + \alpha_s f(x_s, t_s), \qquad s = 0,1,\cdots \tag{3}$$

where x_0 is given, step length α_s is a monotonically decreasing sequence which satisfies the following conditions

$$0 < \alpha_s \quad , \quad \alpha_s \to 0 \quad , \quad \lim_{k\to\infty} \sum_{s=0}^{k} \alpha_s \to \infty \quad . \tag{4}$$

All definitions presented above can be reformulated for difference system (3). Convergence of (2) does not imply the convergence of its discrete version (3). Nevertheless, some results obtained for (2) are of importance for investigation of system (3). In [2] it was shown that if (2) is an autonomous system, then proof of convergence of a method (2) ensures convergence of the discrete version (3) under condition (4) and another rather simple assumption. Investigation of a continuous system is much simpler than investigation of a discrete one. Therefore, the result we obtained for (2) will be considered as the first step of the investigation of system (3).

Exterior Point Technique

To simplify and shorten the presentation here, we will henceforth assume that $F(x)$ and $h(x)$ are continuously differentiable functions. The auxilliary exterior penalty function for problem (1) is defined as

$$P(x, \tau) = F(x) + \tau\, S(x),$$

$$S(x) = \sum_{i=1}^{e} \Psi(|g^i(x)|) + \sum_{i=1}^{c} \Psi(h_+^i(x)) \quad .$$

Here $0 \leq \tau$ is a scalar, $h_+^i(x) = \max\,[0, h^i(x)]$, and $\Psi(y)$ is a scalar-valued function of the single variable y, defined for all positive y. Suppose that this function is twice differentiable and satisfies the following conditions:

$$\Psi(0) = 0, \quad \Psi'(0) = d\Psi(0)/dy = 0 \quad ,$$

$$d^2\Psi(y)/dy^2 \geq \mu > 0 \quad \text{for all } y \geq 0 \quad . \tag{5}$$

It is easy to verify that if $F(x)$, $h(x)$ are convex differentiable functions, $g(x)$ - affine then $P(x, \tau)$ is also convex and differentiable in x function.

If we use the routine penalty function technique, then we have to select a monotonically increasing sequence $\{\tau_i\}$ such that $\tau_i > 0$ and $\tau_i \to \infty$ as $i \to \infty$, and compute x_i which minimizes $P(x, \tau_i)$ on E_n for $i = 1, 2, \cdots$. The limit of the sequence $\{x_i\}$ will belong to solution set X_*. A minimizing point x_i can be found using the following differential equation

$$\dot{x} = - P_x(x, \tau_i) \quad , \quad x(0) = x_0 \tag{6}$$

where

$$P_x = F_x + \tau\left[\sum_{i=1}^{e} \Psi'(|g^i|)g_x^i + \sum_{i=1}^{c} \Psi'(h_+^i)h_x^i\right]$$

is the column vector of derivatives.

Method (6) is analogous to Cauchy's method of steepest descent. Via convexity the solution $x(x_0, t)$ of system (6) converges to x_i as $t \to \infty$ for any $x_0 \in E_n$.

The penalty function procedure can be simplified significantly if, instead of multiple solving of system (6), we solve only once a system similar (6) with a continuously variable parameter $\tau = \tau(t)$. For example the following system can be used

$$\dot{x} = - P_x(x, \tau(t)) \quad , \quad x(0) = x_0 \tag{7}$$

where $\tau(t)$ is a differentiable function which satisfies the inequalities

$$0 < \tau(t) \leq d\tau(t)/dt \quad \text{for any } t \geq 0 \quad . \tag{8}$$

We will now prove below that in the case of a convex programming problem under certain assumptions, every limit point of the solution of system (7) belongs to solution set X_*. If the Slater condition holds (X_0 is non empty), then for any

$x_* \in X_*$ vectors $p_* \in E_e$ and $w_* \in E_c$ exists such that

$$\begin{aligned} & w_* \geq 0, \quad w_*^i h^i(x_*) = 0, \quad 1 \leq i \leq c \quad , \\ & F_x(x_*) + \sum_{i=1}^{e} g_x^i(x_*) p_*^i + \sum_{i=1}^{c} h_x^i(x_*) w_*^i = 0 \quad . \end{aligned} \tag{9}$$

Denote $\gamma = \left[\sum_{i=1}^{e} (p_*^i)^2 + \sum_{i=1}^{c} (w_*^i)^2 \right] / 2\mu$.

We shall now establish a preliminary lemma which will be followed by the convergence theorem for method (7).

<u>Lemma 1</u> If (1) is a convex programming problem, the set X_* is compact, X_* and X_0 are non empty sets, then for any $x_* \in X_*$, $x \in E_n$, $0 < \tau \leq T$ the following inequalities hold

$$F(x_*) - \gamma/\tau \leq P(x, \tau) \leq P(x, T). \tag{10}$$

This lemma was proved in [3, 4]. Nevertheless, taking into account the importance of this result for further consideration we shall give brief proof of this lemma.

Applying the Taylor formula for second-order expansions and taking into account (5), we obtain

$$2\Psi(y) \geq \mu\, y^2 \geq 0 \tag{11}$$

By convexity and (9) we have for any $x_* \in X_*$, $x \in E_n$

$$\begin{aligned} F(x) - F(x_*) \geq (F_x(x_*), x - x_*) = & \sum_{i=1}^{e} p_*^i (g_x^i(x_*), x_* - x) \\ & + \sum_{i=1}^{c} w_*^i (h_x^i(x_*), x_* - x) \end{aligned} \tag{12}$$

where $(\cdot,\cdot)$ is euclidean scalar product.

Combining (9), (11), and (12) we find that left-hand side inequality (10) holds

$$P(x,\ \tau) - F(x_*) \geq - \sum_{i=1}^{e} p_*^i g^i(x) - \sum_{i=1}^{c} w_*^i h_+^i(x) +$$

$$+ \frac{\tau\mu}{2}\left[\sum_{i=1}^{e} [g^i]^2 + \sum_{i=1}^{c} [h_+^i]^2\right] \geq - \left[||p_*||^2 + ||w_*||^2\right]/2\tau\mu$$

Since $P(x,\ \tau)$ is increasing function of τ for all $x \in E_n$, we obtain that $P(x,\ \tau) \leq P(x,\ T)$ for any $0 \leq \tau \leq T$.

The following theorem guarantees the convergence of system (7) to the solution set X_*.

Theorem 1 If $F(x)$, $h(x)$ are continuously differentiable functions of x, $g(x)$ is affine, X_* and X_0 are non empty, X_* is compact, inequalities (5), (8) hold, then method (7) globally converges to the solution set X_*.

We first prove that system (7) is Lagrange stable. Introduce the real scalar function $v(x,\ \tau) = \frac{1}{2}\rho^2(x,\ X_*) + \gamma/\tau^{-1}$ which is analogous to the Liapunov function [5]. This function in contrast to the Liapunov function is not equal to zero for any $x \in E_n$ and any finite $\tau > 0$. Function v is differentiable in x and t for any x, $t > 0$. Let $\dot{v}(x,\tau)$ be the total derivative of $v(x,\ \tau)$ along the solutions of (7) passing through the state x at t. It is given by:

$$\dot{v} = (P_x(x,\tau),\ x_* - x) - \gamma\tau^{-2}d\tau/dt \tag{13}$$

where

$$x_* \in X_*,\ ||x_* - x(t)|| = \rho(x(t),\ X_*)\quad .$$

Making use of (8), (10) and taking into account the convexity of $P(x, \tau)$ in x, we conclude that

$$\dot{v} \leq F(x_*) - P(x, \tau) - \gamma/\tau \leq 0 \quad . \tag{14}$$

Hence along the motion $x(x_0, t)$

$$\rho^2(x(x_0, t), X_*) \leq \rho^2(x_0, X_*) + 2\gamma/\tau(0) \quad . \tag{15}$$

Let the bounded set R_1 be defined by

$$R_1 = \{x : \rho^2(x, X_*) \leq \rho^2(x_0, X_*) + 2\gamma/\tau(0)\} \ .$$

From (14) it follows that a trajectory $x(x_0, t)$ generated by (7) can never leave the set R_1 for $t \geq 0$. Hence trajectory $x(x_0, t)$ exists and is bounded for all $t \geq 0$, the positive limit set $\omega(x_0)$ is non empty and $\omega(x_0) \subset R_1$.

Now to prove the theorem it is enough to show that $\omega(x_0) \subset X_*$ for any $x_0 \in E_n$. Let $\bar{x}$ be an arbitary point belonging to the set $\omega(x_0)$ and $\{t_i\}$ is an increasing sequence of time t_i tending to infinity as $i \to \infty$, such that $x(x_0, t) \to \bar{x}$, $t \in \{t_i\}$. The sequence $v(x_0, t_i), \tau(t_i))$ is non increasing and is bounded below by zero. It therefore has a limit $v(x(x_0, t_i), \tau(t_i)) \to \frac{1}{2}\rho^2(\bar{x}, X_*)$. Here we take into account inequalities (8) which imply that $\tau(t)$ tends to infinity as $t \to \infty$. From boundness of $v(x, \tau(t))$ on R_1 it follows that a subsequence $\{t_s\} \subset \{ t_i\}$ exist such that $\dot{v}(x(x_0, t), \tau(t) \to 0$, $x(x_0, t) \to \bar{x}$ if $t \in \{t_s\}$. Otherwise there are some positive numbers δ and $T(\tau)$ such that $\dot{v}(x(x_0, t), \tau(t)) \leq - \delta$ for all $t \geq T(\delta)$. By integrating this inequality we would find $v(x(x_0, t), \tau(t) \leq - \delta(t - T)$ and it would follow that

$v(x(x_0, t), \tau(t)) \to -\infty$ as $t \to \infty$, contradicting the positiveness of $v(x, t)$ on R_1.

Using convexity and exploiting the continuity of $F(x)$ we obtain from (13) that

$$F(x_*) - F(\bar{x}) = \lim_{t \in \{t_s\}} \tau(t) \; S(x(x_0, t)) \tag{16}$$

$$\lim_{t \in \{t_s\}} (P_x(x(x_0, t), \tau(t)), \; x_* - x(x_0, t)) = 0 \; . \tag{17}$$

Since the left-hand side in (16) is restricted on the set R_1 and $\tau(t)$ tends to infinity as $t \to \infty$, each term in $S(x(x_0, t))$ must go to zero. Hence we arrive at an important conclusion $\bar{x} \in X$, i.e. $\bar{x}$ is a feasible point for problem (1). If the right-hand side in (16) has a limit equal to zero then $F(x_*) = F(\bar{x})$ and therefore $\bar{x} \in X_*$. Otherwise condition (17) implies

$$F_x(x(x_0, t_i)) + \tau(t_i) \; S_x(x(x_0, t_i)) \to 0, \quad t_i \in \{t_i\} \tag{18}$$

Define

$$p^i(x_0, t) = \tau(t) \; \psi'(|g^i(x(x_0, t))|) \quad ,$$

$$w^j(x_0, t) = \tau(t) \; \Psi'(h_+^i(x(x_0, t)) \text{ for } 1 \leq i \leq e, \; 1 \leq j \leq c \quad .$$

Introduce the following set of integers

$$B = \{i | h^i(\bar{x}) = 0 \quad , \quad i \leq i \leq c\} \quad .$$

The limits of $p^i(x_0, t)$, $w^i(x_0, t)$ as $t \in \{t_s\}$ exist and are equal to $\bar{p}^i$ and $\bar{w}^i$ respectively. To prove this, note that via condition (5) $w^i(x_0, t) \geq 0$ for any $1 \leq i \leq c$ and

$t \geq 0$. If $i \bar{\varepsilon} B$ then $h^i(x_0, t) < 0$ for sufficiently large t and therefore $\lim_{t\varepsilon\{t_s\}} w^i(x_0, t)$ exists and is equal to zero.

Let

$$C(x_0, t) = \sum_{i=1}^{e} |p^i(x_0, t)| + \sum_{i\varepsilon B} w^i(x_0, t) \quad ,$$

$$a^i = p^i/C \quad , \quad b^j = w^i/C \quad , \quad 1 \leq i \leq e \quad , \quad 1 \leq j \leq C \quad ,$$

$$\bar{C} = \lim_{t\varepsilon\{t_s\}} C(x_0, t), \ \bar{a}^i = \lim_{t\varepsilon\{t_i\}} a^i(x_0, t), \ \bar{b}^i = \lim_{t\varepsilon\{t_i\}} b^j(x_0, t)$$

If $\bar{C} = +\infty$ then dividing (18) by C and taking the limit as $t \varepsilon \{t_s\}$ yields

$$\sum_{i=1}^{c} \bar{a}^i g_x^i(\bar{x}) + \sum_{i\varepsilon B} \bar{b}^j h_x^j(\bar{x}) = 0 \quad ,$$

where all $\bar{b}^j \geq 0$. But this contradicts the Slater conditions. Thus $\bar{C} < \infty$ and from (18)

$$F_x(\bar{x}) + \sum_{i=1}^{e} \bar{p}^i g_x^i(\bar{x}) + \sum_{i\varepsilon B} \bar{w}^i h^i(\bar{x}) = 0 \quad .$$

Hence vectors $\bar{p}$ and $\bar{w}$ associated with the limit point $\bar{x}$ satisfy the Kuhn-Tucker necessary and sufficient conditions for $\bar{x}$ to be a solution of problem (1). Therefore $\bar{x} \varepsilon X_*$ $\lim_{t\varepsilon\{t_s\}} v(x(x_0, t), \tau(t)) = 0$. Sequence $v(x(x_0, t), \tau(t))$ monotonically decreases, and possesses subsequence $v(x(x_0, t), \tau(t))$, $t \varepsilon\{t_s\}$ which converges to zero. Therefore the entire sequence must converge to zero (see [2, 6]). For any convergent subsequence $x(x_0, t_j)$ the sequence $v(x(x_0, t_j), \tau(t_j))$

has the same limit equal to zero, consequently $\omega(x_0) \subset X_*$. The starting point x_0 is arbitary, hence method (7) converges globally, for any $x_0 \in E_n$. This completes the proof of the theorem.

We also obtain an important additional by-product result: every limit points of $p^i(x_0, t)$, $w^i(x_0, t)$ coincide with dual variables p_*^i, w_*^i respectively (see (9)).

As an illustration of this approach consider the simplest example. We seek a solution to the problem

$$\text{minimize } x \quad \text{subject to } x = 0.$$

The solution to this problem is trivial $x = 0$. Use a particular penalty function

$$P(x, \tau) = x + \gamma e^t x^4/4 \quad ,$$

where $0 < \gamma$ is arbitary scalar. Using method (7), we obtain the following differential equation

$$\dot{x} = -1 - \gamma e^t x^3 \quad , \quad x(0) = x_0 \quad .$$

Solution $x = 0 = X_*$ is not an equilibrium point for this system and is not stable in the sense of Liapunov. Meanwhile any solution $x(x_0, t)$ converges to X_* for any $\gamma > 0$ and any x_0.

Consider the following maximin problem associated with problem (1).

$$I = \max_{\tau \leq T} \min_{x \in E_n} P(x, \tau) \tag{19}$$

where $0 < T$ is some fixed number. Introduce two new sets:

$$Z = \{z \mid \min_{x \varepsilon E_n} P(x, T) = P(z, T)\} \quad ,$$

$$R_2 = \{x \mid \rho^2(x, Z) \leq 2T + \rho^2(x_0, Z)\} \quad .$$

A pair $(T, \tilde{x})$, where $\tilde{x} \varepsilon Z$, solves maximin problem (19). If $\tilde{x} \varepsilon Z$, $x_* \varepsilon X_*$ then $F(x_*) - \gamma/T \leq P(\tilde{x}, T) \leq F(x_*)$. If function $F(x)$ is bounded below ($F(x) \leq \delta$ for all $x \varepsilon R_2$) then

$$S(\tilde{x}) \leq [F(x_*) - \delta]/T.$$

By making T sufficiently large we can thereby find an appropriate solution to problem (1) with any required accuracy. For solving maximin problem (19) it is sufficient to solve the following problem: minimize $P(x, T)$ over all $x \varepsilon E_n$. Regretably this unconstained problem is extremely difficult to solve. Since for large T the function $P(x, T)$ is ill-conditioned. It is more convenient (see [7]) to let the parament τ vary continuously from zero to T and solve differential equation of the form

$$\dot{x} = - P_x(x, \tau), \ \dot{\tau} = S(x)(T - \tau), \ x(0) = x_0, \ \tau(0) = 0 \quad . \tag{20}$$

The simplest discrete version of this method is

$$x_{s+1} = x_s - \alpha_s P_x(x_s, \tau_s), \ \tau_{s+1} = \tau_s + \alpha_s S(x_s)(T - \tau_s),$$
$$s = 0, 1, 2, \cdots \tag{21}$$

We shall call the constraints essential in problem (1) if the unconstrained infinum of $F(x)$ differs from the solution to (1).

<u>Theorem 2</u> Let F, h be convex, continuously differentiable functions, g(x) be affine function, X_* and Z be non empty compact sets, the constraints be essential, and the inequalities (5) hold. Then method (20) converges globally to solution set Z for any $x_0 \in E_n$. Discrete method (21) globally converges to Z if α_s is a monotonically decreasing sequence satisfying (4) and if α_0 is sufficiently small.

To prove this theorem we shall use the following Liapunov function

$$v(x, \tau) = T - \tau + \rho^2(x, Z)/2 \quad .$$

Making use of convexity, we obtain that the total derivative of $v(x, \tau)$ along the solution of (20) satisfies inequality

$$\dot{v}(x, \tau) \leq P(\bar{x}, \tau) - P(\bar{x}, T) + P(\bar{x}, T) - P(x, T) \leq 0$$

where

$$x = x(t), \ \tau = \tau(t), \ \bar{x} = \bar{x}(t) \in Z, \ \rho(x(t), Z) = ||x(t) - \bar{x}(t)|| \quad .$$

Hence, along the motion $x(x_0, t)$

$$\rho^2(x(x_0, t), Z) \leq 2\tau + \rho^2(x_0, Z) \leq 2T + \rho^2(x_0, Z)$$

Therefore for any $t \geq 0$ all trajectory $x(x_0, t)$ belongs to the bounded set

$$R_2 = \{x | \rho^2(x, Z) \leq 2T + \rho^2(x_0, Z)\}$$

and $x(x_0, t)$ can never exit from R_2. It is obvious that $\tau(t) \leq T$ for all $0 \leq t$. Consequently, system (20) is Lagrange stable. Positive limit set $\omega(x_0)$ of a motion

$x(x_0, t)$ is non empty and is contained in R_2.

The functions $v(x, \tau)$, $-\dot{v}(x, \tau)$ are positive definite functions of vector x on $E_n \times Z$, i.e. $v(x, \tau) > 0$, $-\dot{v}(x, \tau) > 0$ for any $x \bar{\varepsilon} Z$, $0 \leq t < T$ and $v(x, \tau) = \dot{v}(x, \tau) = 0$ if $x \varepsilon Z$. Prove that $P(\bar{x}, \tau) < P(\bar{x}, T)$ for any $\bar{x} \varepsilon Z$, $0 \leq \tau < T$. We shall construct a contradiction. Suppose that a number $\tau = \tau_1 < T$ exists such that $(t - \tau_1)\, S(\bar{x}) = 0$. It is possible only if $S(\bar{x}) = 0$, i.e. $\bar{x} \varepsilon X$. Therefore $F(\bar{x}) = \min_{x \varepsilon E} P(x, \tau_1)$ Since $\bar{x}$ maximizes $P(x, \tau_1)$, it is necessary that $P_x(\bar{x}, \tau_1) = F_x(\bar{x}) = 0$. Hence $\bar{x}$ is a stationary point of convex function $F(x)$ and consequently is a global minimum of $F(x)$. This contradicts our assumption that constraints are essential in problem (1). The monotonically decreasing along the trajectories of (20) Liapunov function $v(x(x_0, t), \tau(t))$ is always positive and therefore a sequence $t_i \to \tau$ exists such that $x(x_0, t_i) \to x_i$, $\tau(t_i) \to \tau_i$ and $\dot{v}(x(x_0, t_i), t_j), \to 0$. Since $\dot{v}(x, \tau)$ is negative except the case $x_i \varepsilon Z$, $\tau \varepsilon T$, we obtain that $v(x_i, T) = 0$. For any convergent pair $(x, \tau) \to (\bar{x}, \bar{\tau})$ we must have $v(\bar{x}, \bar{\tau}) = v(x_i, \tau_i) = 0$. Finally, any convergent pair solves the maximin problem (19).

The presented convergence proof for autonomous system (20) implies the convergence of discrete version (21) (see [2]).

Interior Point Technique

Define the general interior penalty function for problem

(1)

$$H(x, \tau) = F(x) + \sum_{i=1}^{e} p^i g^i(x) + \tau^{-1} \sum_{i=1}^{c} \phi(h^i(x))$$

where $p \in E_e$, $\tau = \tau(t)$, $\phi = \phi(y)$ are scalar-valued functions of a single variable, defined for all $0 \leq t < \infty$, $-\infty < y < 0$ respectively and satisfies the following conditions

$$0 < \tau(t),\ 0 < \tau' = d\tau(t)/dt,\ \lim_{t\to\infty} \tau(t) = \infty \quad ,$$
$$0 < \phi(y) \leq -y\phi'(y) = -yd\phi(y)/dy,\ \lim_{y\to -0} \phi(y) = \infty \quad . \tag{22}$$

Using approach [1] and [7] consider the system which is described by the differential equation

$$\dot{x} = -H_x(x, \tau(t)) = -\left[F_x + \sum_{i=1}^{e} g_x^i p^i + \tau \sum_{i=1}^{c} \phi'(h^i) h_x^i\right] \tag{23}$$

We shall choose in such a way $p(t)$ so that function $g(x)$ would be a first integral of this system. Differentiating $g(x)$ along the solutions of (23) yields

$$\dot{g}^i = -(g_x^i, H_x) = 0 \quad , \quad i=1,2,\ldots,e \quad . \tag{24}$$

Let g_x be $n \times m$ matrix whose ij-th element is equal to $\delta g^j(x)/\delta x^i$.

We can assume without loss of generality that the matrix g_x has maximum rank e. Then the vector $p(t)$ can be found from linear system (24) of e equations in e unknowns. Substituting the solution obtained in (23), we get

$$\dot{x} = - N\left[F_x + \frac{1}{\tau} \sum_{i=1}^{c} \phi^{i}(h^{i}(x))\, h_x^{i}(x)\right],\ x(0)\ \varepsilon\ x_0 \tag{25}$$

where $N = I - g_x(g_x^T g_x)^{-1} g_x^T$, I is unit n x m matrix, superscript T denotes the transpose of a matrix, superscript -1 denotes the inverse of a matrix.

Theorem 3 If (1) is a convex programming problem, X_* and X_0 are non empty, X_* is a compact set, inequalities (22) hold, the matrix g_x has maximum rank e, then the method (25) converges on X_0 to the solution set X_*.

This theorem was proved in [2].

Consider a particular case when primal problem (1) has no equality constraints (e=0). Then for solving problem (1) we use the following modification of Newton's method

$$\dot{x} = - H_{xx}^{-1}(H_x + H_{x\tau}),\ x(0) = x_0\ \varepsilon\ X_0 \tag{26}$$

where H_{xx} is the Hessian

$$H_{xx} = F_{xx} + \tau^{-1} \sum_{i=1}^{c} \left[\phi''(h_x^i(x))\left[h_x^i(x)\right]^T + \phi'(h^i(x) h_{xx}^i(x)\right]$$

$$H_{x\tau} = \sum_{i=1}^{c} \phi'(h^i) h_x^i$$

Theorem 4 If (1) is a convex programming problem, e=0, functions F(x), h(x) and $\phi(y)$ are twice continuously differentiable, F(x) is strictly convex, X_* is a compact set, X_0 and X_* are non empty sets, then the method (26) converges on X_0 to X_*.

Because of our assumption that $F_{xx}(x)$ is a positive definite matrix, $h_x^i(h_x^i)^T$ and h_{xx}^i are positive semi-definite matrices. From conditions (22) it follows that $\phi''(y) > 0$ for all $y < 0$. Therefore $H_{xx}(x, \tau)$ is a positive definite matrix for any $x \in X_0$ and $\tau > 0$. Hence matrix $H_{xx}(x, \tau)$ has an inverse and a solution of system (26) exists at last for small t when the solution $x(x_0, t)$ remains in X_0. Since system (26) has a trivial first integral

$$H_x(x, \tau) = H_x(x_0, 0)e^{-t} \quad , \tag{27}$$

the norm of vector H_x is decreasing and solution $x(x_0, t)$ can never leave the feasible region X, since the norm of vector $H_x(x(x_0, t), \tau(t))$ would have infinity value there, contradicting the strictly monotonic decreasing property ensured by (27). Consequently, the solution of system (26) exists for any $t \geq 0$ and the set X_0 is invariant with respect to this system. Further proof proceeds in a manner similar to the proof of theorems 1 and 3.

Techniques for Solving a Set of Equations

The methods of exterior point can be used for solution of a set of equations. Suppose we have to find a feasible point $x \in X$ and this set is non empty and compact. Define function

$$P(x) = \sum_{i=1}^{e} \psi(|g^i(x)|) + \sum_{i=1}^{c} \psi(h_+^i(x)) \quad .$$

Assume that conditions (5) hold. Hence P(x) is a differentiable function and the set X coincides with the set of points that solve the equation $P_x(x) = 0$. That is, a primal problem is transformed to the problem of finding the stationary points of a function $P(x)$. Using the simplest gradient method yields the following differential equation

$$\dot{x} = - P_x = - \left[\sum_{i=1}^{e} \Psi' (|g^i(x)|) g_x^i + \sum_{i=1}^{c} \Psi' (h_+^i) h_x^i \right], \quad x(0) \in X_0 \tag{28}$$

Theorem 5: If $h(x)$ is convex, continuously differentiable function, $g(x)$ is affine, X is a non empty, compact set conditions (5) hold, then method (28) globally converges to the set X for any $x_0 \in E_n$.

Tw Liapunov functions can be used for the proof of this theorem

$$v_1(x) = \rho^2(x, X) \quad , \quad v_2(x) = P(x) \quad .$$

Taking into account convexity, we obtain that the total derivatives of v_1 and v_2 along the solution of (28) satisfy inequalities

$$\dot{v}_1 \leq - 2P(x) \leq 0 \quad , \quad \dot{v}_2 \leq - ||P_x||^2 \leq 0 \quad .$$

Proof of convergence follows immediately from these formulas.

In a particular case when $h(x)$ is affine, $\phi(y) = y^2$ this method coincides with the method suggested in [8]. If v_1 or v_2 satisfy Lipschitz condition then a discrete version of (28), similar to (21), also converges to the set X [2].

If $h(x)$, $g(x)$ $\phi(y)$ are twice differentiable functions then Newton's method can be used.

$$P_{xx}(x) \, \dot{x} = - P_x(x)$$

Proof of convergence is exactly the same as the proof of theorem 5.

Iterative Numerical Methods for Solving a Linear Programming Problem

Let us consdier the following linear programming problem

$$\text{minimize} \sum_{i=1}^{n} C^i x^i \tag{29}$$

subject to $x \in X = \{x | A\, x = b \quad , \quad x \geq 0\}$ where $C = (C^1, C^2, \ldots, C^n) \in E_n$, $b \in E_m$, A is m x n matrix.

The dual problem is

$$\text{maximize} \sum_{i=1}^{m} b^i y^i \tag{30}$$

subject to $y \in Y = \{y |\ C - A^T y\}$, where superscript T denotes the transpose of a matrix.

Let X_* and Y_* be the solution sets of problems (29) and (30) respectively. Suppose that they are non empty, compact.

The methods described above are applicable to these problems. For example, consider method (20). To simplify formulas we use the quadratic loss function to absorb the constraints and define penalty functions as

$$P(x, \tau) = C^T x + \tau \left[||Ax - b||^2 + ||x_-||^2 \right] / 2,$$

$$W(y, s) = b^T y - s||w_-||^2/2$$

where

$$w = C - A^T y, \ z_-^i = \max\, [0, -z^i] \geq 0, \ z_- = [z_-^1, z_-^2, \ldots, z_-^n] \quad .$$

Appyling (20) to problems (29), (30) yields

$$\dot{x} = - P_x = - C - \tau\left[A^T(Ax - b) + x_-\right],\ \dot{\tau} = (T - \tau)P_\tau \qquad (31)$$

$$\dot{y} = W_y = b - s\,Aw_-,\ \dot{s} = (s - T)W_s \qquad (32)$$

It is easy to show that the following evaluations hold

$$W(y, s) - \frac{1}{2s}||\,x_*||^2 \leq C^T x_* = b^T y_* \leq P(x, \tau) +$$

$$+ \frac{1}{2\tau}\left[||y_*||^2 + ||C - A^T y_*||^2\right],$$

$$C^T x_* - \frac{1}{2\tau}\left[||y_*||^2 + ||C - A^T y_*||^2\right] \leq \min_{x\varepsilon E_n} P(x, \tau) \leq C^T x_*,$$

$$b^T y_* + \frac{1}{2s}\,||x_*||^2 \geq \max_{y\varepsilon E_m} W\,(y, s) \geq b^T y_*$$

where $x_* \varepsilon X_*,\ y_* \varepsilon Y_*$.

Theorem 2 ensures the convergence of these methods and their discrete versions. Therefore these methods permit us to find an approximate solution for problem (29) or (30) with any required accuracy. Simplicity of calculations is the obvious advantage of these methods. Moreover the amount of computation is only slightly dependent on the dimensionality of the problem. But these methods can not be used for high precision calculations. This disadvantage is due to increasing penalty function coefficient.

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