BOUNDARIES OF STABILITY:
A POTPOURRI OF DYNAMIC PROPERTIES

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Boundaries of Stability

(A Potpourri of Dynamic Properties)

OR

Is Resilience Resilient?

THE STRATEGIC PROBLEM

There is more to a system than its equilibrium points. Associated with every stable equilibrium point (or stable limit cycle) is a region of state-space such that any unperturbed trajectory initiated in the region will stay within that region. This is called the region of stability. The boundaries of stability separate contiguous stability regions. An important property of system behavior near these boundaries is that a very small perturbation can move the state of a system across a boundary and transfer the system entirely from one region to another. The system's state cannot move back across the boundary without a subsequent outside perturbation.

The performance of systems near their equilibrium points has been the focus of a considerable amount of investigation. Considerations of optimization, maximization, stable states are examples. The properties of systems far from equilibrium, and particularly near regions of instability (i.e., the boundaries) are not well known.

The significant strategic problem that this paper hopes to address is to locate these boundaries and to determine system dynamics near them. On a tactical level, some approaches are suggested and their usefulness discussed.

SYSTEM DESCRIPTION

The state variable description of a system specifies the state of a system at any instant by a collection of variables:

$$\underline{x} = (x_1, x_2, \ldots, x_n). \tag{1}$$

For a dynamic system $\underline{x}$ is a function of time,

$$\underline{x} = \underline{x}(t), \tag{2}$$

that develops temporally by a relation such as

$$\underline{x}(t_2) = \underline{f}(\underline{x}(t_1), t_1). \tag{3}$$

This relation is true for all systems--from real ecological ones to formal mathematical abstractions. The function $\underline{f}(\underline{x}, t)$ is the set of all 'rules' that cause $\underline{x}$ to change through time from $\underline{x}$ at t_1 to $\underline{x}$ at t_2. The rules may be divine guidance, a complex FORTRAN program, or a Lotka-Volterra equation. Stochastic processes are included in the function.

The "solution" of Equation (3) is a record of $\underline{x}$ over time. The nature of the system will dictate how the solution is obtained. It might be from field observation, laboratory experiment, simulation, or analytic integration. Obviously the solution for all initial conditions may be hard to find because of time, expense, or analytic intractibility.

In lieu of solutions to (3), what can we infer from the instanteous rules $\underline{f}(\underline{x},t)$ about the qualitative behavior of the system over time? Specifically, what can be said about boundaries?

If we have defined our system, we can find $\underline{f}(\underline{x},t)$ for at least a certain number of points. In some sense the set of rules $\underline{f}(\underline{x},t)$ is the system. In a field experiment, the points $\underline{x}$ where $\underline{f}(\underline{x},t)$ is known are specified for us. In the laboratory we can select $\underline{x}$ and measure $\underline{f}$. The same applies to a simulation model. In the analytic case $\underline{f}(\underline{x},t)$ is known explicitly.

For simple (2-dimensional) systems, the "solution" will likely be easy enough to find, at least in approximation. But for larger systems this will be the exception. For example, the most efficient way to locate boundaries in the predator/prey simulation seems to be to plot trajectories on a phase plane and locate them by eye.

In this paper I will deal with rather simple systems. Not because they are fundamental or even realistically interesting, but because if I can't find ways of looking at these simple systems first, then it won't be fruitful to try the general case.

The first simplification is to restrict ourselves to continuous, autonomous systems. In this case Equation (3) becomes

$$\underline{\dot{x}} = \underline{f}(\underline{x}) \tag{4}$$

where $\underline{\dot{x}} = \frac{d}{dt}\underline{x}$ is a consequence of continuity. The system is autonomous because $\underline{f}$ does not explicitly contain t -- the rules do not change with time -- and the variables $\underline{x}$ "drive themselves".

Most of the formalism to follow applies to the n-dimensional case. Frequently, the 2-dimensional system

$$\dot{x} = P(x,y)$$
$$\dot{y} = Q(x,y) \qquad (5)$$

will be used for illustration.

The "solution" of (5) is the set of all trajectories from all starting points (x_o,y_o):

$$x(t) = x(t;x_o,y_o) \quad \text{and} \quad y(t) = y(t;x_o,y_o). \qquad (6)$$

This is what is being done in the simulation models by picking initial points and plotting the subsequent trajectory.

PHYSIOMORPHISM

Physiomorphism is the attribution of the notions of physics to things that are not basically physical. It is a phenomenon that is frequently observed among systems ecologists. Rather than reject it outright, let's consider some of the possibilities.

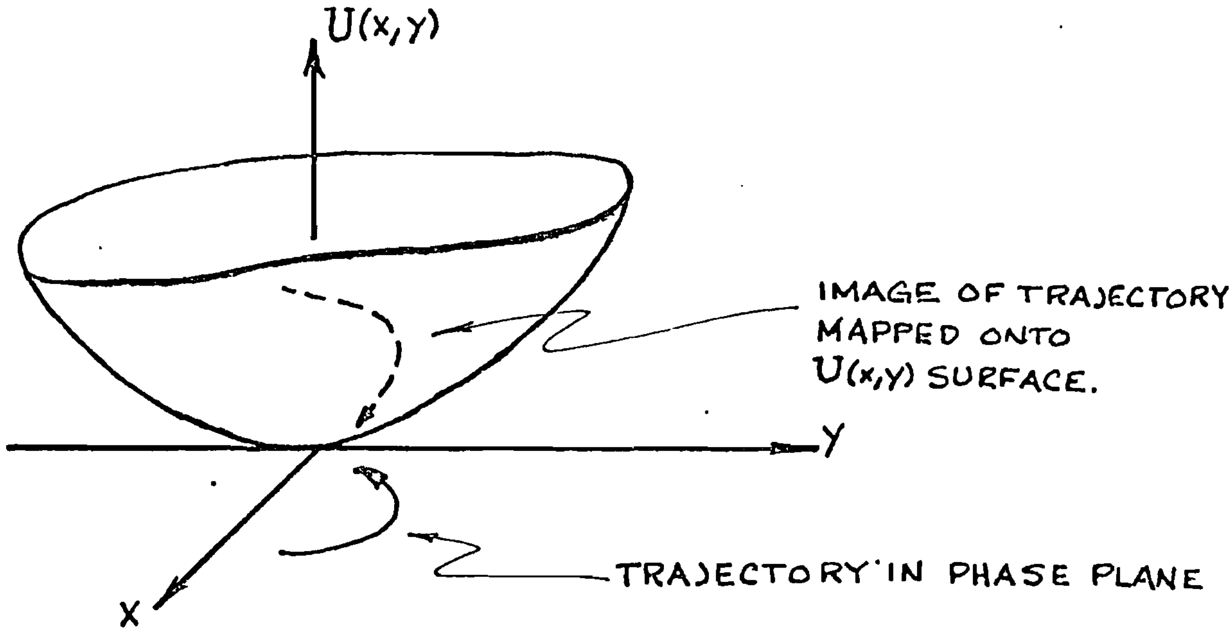
A. We would like to develop some measure throughout the state-space that will indicate where the trajectories will go and at what rates. In two dimensions this is some function U(x,y) where the shape of its surface implies the system dynamics. What is suggested here is some equivalent to potential energy.

Before proceeding, we shift the state-space axes so that the origin is at an equilibrium point. We can assume for the moment that there is a finite region of stability surrounding this point. Several conditions on the function U(x,y) can be specified:

a) U(x,y) should be a minimum at the origin. We can set U(0,0) = 0 without loss of generality.

b) U(x,y) should increase as we go outward from the origin to the boundary. That is, U(x,y) is bowl-shaped and centered at the origin. (Formally, U(x,y) is positive-definite within the region of stability.)

c) Points and trajectories are mapped one-to-one from the x,y-plane to the surface U(x,y).



d) Mapped trajectories follow a path from higher U to lower U. A by-product of this is that the boundaries between regions of stability would be relative high points.

e) Since x = x(t), y = y(t), the function U(x,y) can be considered to be U = U(t) along trajectories, and

$$\frac{d}{dt}U(t) = \dot{U} = \nabla U(\underline{x}) \cdot \frac{d\underline{x}}{dt} = \nabla U \cdot \underline{f} \qquad (7)$$

where we have used the vector notation of Equation (4).

B. We have not yet specified what it is that U measures. For this function to have any use, the contours $U(x,y) = \text{constant}$ must have some relation to the system or its dynamics.

Consider the following: Let

$$\dot{U} = -\lambda, \tag{8}$$

that is, U decreases at a constant rate λ. In this case, the contours U = constant are "isoclines of time", i.e., all trajectories take the same amount of time to travel between contour lines.

When (8) is applied to (7), we get the partial differential equation

$$\nabla U \cdot \underline{f} = -\lambda \tag{9}$$

with the condition $U(0) = 0$. In 2-dimensions,

$$\frac{\partial U}{\partial x} P(x,y) + \frac{\partial U}{\partial y} Q(x,y) = -\lambda, \quad U(0,0) = 0. \tag{10}$$

In general, we cannot expect (9) or (10) to be easier to solve than the trajectories of the original Equation (4). These isoclines could be easily established once $x(t)$ and $y(t)$ are known. At this point, there doesn't appear to be much future for this interpretation of U.

C. Next consider a velocity vector

$$\begin{aligned} \underline{A} &= \underline{\dot{x}} = \underline{f}(\underline{x}) \\ &= \dot{x}\hat{i} + \dot{y}\hat{j} = P(x,y)\hat{i} + Q(x,y)\hat{j} \end{aligned} \tag{11}$$

where $\hat{i}$ and $\hat{j}$ are unit vectors along the x- and y-axes, respectively. $\underline{A}$ is the instantaneous rate of change of the state of the system. If we say that trajectories "go down U(x,y) like a ball rolling down a hill", then $\underline{A}$ will be co-linear with the direction of steepest descent (the fall line) of the surface U(x,y). The line of steepest descent is a vector called the gradient of U(x,y),

$$\text{Gradient } U(x,y) = \nabla U(x,y).$$

If $\underline{A}$ is co-linear with ∇U, then

$$\nabla U = k\underline{A} \tag{12}$$

However, it is a property of vectors that the curl of a gradient is zero, i.e.,

$$\nabla \times \nabla U = 0 \tag{13}$$

but $\nabla \times k\underline{A} \neq 0$ in general.

Thus we can eliminate this interpretation of U.

D. We have defined $\underline{A}$ as the velocity vector of our system. The product $\underline{A} \cdot \underline{A} = \underline{A}^2$ is a measure of the speed of the system motion. If $U(\underline{x}) = \underline{A}^2$, then U is a measure of speed. If we are willing to suspend reality momentarily, we will note that $\underline{A}^2 = (\dot{\underline{x}})^2$ is very like the kinetic energy of a mechanical system ($\frac{1}{2}mv^2$). Further, in a conservative system, $(\dot{\underline{x}})^2$ is a linear function of the potential energy. With these tenuous links, we try

$$\begin{aligned} U(\underline{x}) &= (\dot{\underline{x}})^2 = \underline{A} \cdot \underline{A} = \underline{A}^2 \\ &= P^2 + Q^2. \end{aligned} \tag{14}$$

We have automatically $U(\underline{0}) = 0$ because $\dot{\underline{x}} = 0$ at the origin.

The gradient of U is

$$\nabla U = 2\left\{ (PP_x + QQ_x)\hat{i} + (PP_y + QQ_y)\hat{j} \right\}, \tag{15}$$

where the partial derivatives are

$$P_x = \frac{\partial P}{\partial x}, \quad P_y = \frac{\partial P}{\partial y}, \quad Q_x = \frac{\partial Q}{\partial x}, \quad Q_y = \frac{\partial Q}{\partial y}.$$

From (7), we have

$$\begin{aligned} \dot{U} &= \nabla U \cdot \dot{\underline{x}} = \nabla U \cdot \underline{A} \\ &= 2\left\{ P^2 P_x + Q^2 Q_y + PQ\,(Q_x + Q_y) \right\}. \end{aligned} \tag{16}$$

We require that $\dot{U} \leq 0$. It is not obvious that this requirement will be satisfied by (16), and in fact examples can be found that violate this condition.

E. The conditions that we have set for $U(\underline{x})$ in Section A. above are equivalent to the conditions used to determine stability by Liapunov's Direct Method. This procedure says that if one can find <u>some</u> function $V(\underline{x}) > 0$ $(|\underline{x}| > 0)$ with $\dot{V}(\underline{x}) \leq 0$, then the equilibrium point is stable. However, this function need not have any other significance -- it is not a measure of what is going on (besides establishing stability). The conditions that we have put on $U(\underline{x})$ mean that it would qualify for a Liapunov function, $V(\underline{x})$. Unfortunately, since Liapunov developed this test in 1892, no general method has been found to construct $V(\underline{x})$. The prospects of finding $U(\underline{x})$ appear smaller than we'd like.

F. Before leaving our fling with physiomorphism, we should examine the idea of force. In mechanical systems, force is proportional to the gradient of a potential energy.

The concept of a force in a general system is about as realistic as calling $(\dot{\underline{x}})^2$ an energy as we did earlier. Therefore, we may as well continue to use $U(\underline{x}) = \underline{A} \cdot \underline{A}$ as a potential energy even though it did not provide us with our earlier objective. We propose as a force

$$\begin{aligned} \underline{F} &= -\nabla U = -\nabla(\underline{A} \cdot \underline{A}) = -2(\underline{A} \cdot \nabla)\underline{A} + 2\underline{A} \times (\nabla \times \underline{A}) \\ &= -2\left\{(PP_x + QQ_x)\hat{i} + (PP_y + QQ_y)\hat{j}\right\}. \end{aligned} \tag{17}$$

from (15) above. We leave this one at this point for now.

G. An alternate candidate of force comes straight from Newton's Second. Namely, force = mass x acceleration, or

$$F \simeq \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right). \tag{18}$$

In our system, $\left(\frac{dx}{dt}\right) = \underline{A}$. Thus,

$$\begin{aligned} \frac{d}{dt}\underline{A} &= \frac{d}{dt}P\hat{i} + \frac{d}{dt}Q\hat{j} = (PP_x + QP_y)\hat{i} + (PQ_x + QQ_y)\hat{j} \\ &= (\underline{A} \cdot \nabla)\underline{A} \\ &= \tfrac{1}{2}\nabla(\underline{A} \cdot \underline{A}) - \underline{A} \times (\nabla \times \underline{A}) \\ &= \tfrac{1}{2}\nabla U + (\nabla \times \underline{A}) \times \underline{A}. \end{aligned} \tag{19}$$

The pseudoforce that we have derived is the sum of two terms: (a) $\frac{1}{2}\nabla(\underline{A} \cdot \underline{A})$, the gradient of a scalar potential; and (b) $(\nabla \times \underline{A}) \times \underline{A}$, which suggests the existence of a vector potential.

The scalar potential ($\underline{A} \cdot \underline{A}$) was discussed above. What meaning can be attached to $(\nabla \times \underline{A}) \times \underline{A}$? The factor $(\nabla \times \underline{A})$ is the curl (or rotation) of the vector $\underline{A}$. It can be thought of as a measure of the curvature of the trajectories. The second term is a vector product which reorients $(\nabla \times \underline{A})$ back into the phase plane and makes the units equivalent to the A^2 found in the scalar potential.

In our 2-dimensional example (5), we have

$$\begin{aligned}(\nabla \times A) \times A &= (P_y - Q_x)\ (Q\hat{i} - P\hat{j}) \\ &= (QP_y - QQ_x)\hat{i} + (PQ_x - PP_y)\hat{j} \\ &= -(\nabla \underline{B})\underline{B},\end{aligned} \qquad (20)$$

where

$$\underline{B} = Q\hat{i} - P\hat{j}$$

is the vector $\underline{A}$ rotated clockwise by 90°. The factor $\nabla \underline{B}$, the divergence of B, is a scalar. Therefore $(\nabla \times \underline{A}) \times \underline{A} = -(\nabla \underline{B})\underline{B}$ is a vector pointing at right angles to the trajectory. Thus, the pseudoforce is the vector sum of the gradient of a potential and a vector perpendicular to the motion of the trajectory.

A very obvious exact physical analogy comes to mind -- the motion of a charged particle in an electromagnetic field, where our pseudoforce is isomorphic with the Lorentz force. We therefore have a consistent formulation of a force with

$$\begin{aligned}\underline{A} \cdot \underline{A} &= \text{scalar potential} \\ \underline{A} &= \text{vector potential}\end{aligned} \qquad (21)$$

In practice, the pseudoforce $\frac{d}{dt}\underline{A} = \underline{\mathring{A}}$ is easy to approximate as

$$\underline{\mathring{A}} \simeq \frac{\underline{A}(t + \Delta t) - \underline{A}(t)}{\Delta t} \tag{22}$$

RESILIENCE

Our development of a force has lead to $\underline{\mathring{A}}$, which is nothing more than the second time derivative of the state vector. The vectors $\underline{A}$ and $\underline{\mathring{A}}$ do not cause the state vector to move, they merely describe how the state does move. By the same manner, do the 'rules', $\underline{f}(\underline{x})$, cause $\underline{x}$ to change or do they describe the change? There is no right answer to the question. Cause and effect are linked cyclically in the system:

$$f(x) \rightarrow x \rightarrow f(x) \rightarrow x \rightarrow f(x) \rightarrow x \rightarrow \text{etc.}$$

The motion causes the 'force' just as much and the 'force' causes the motion.

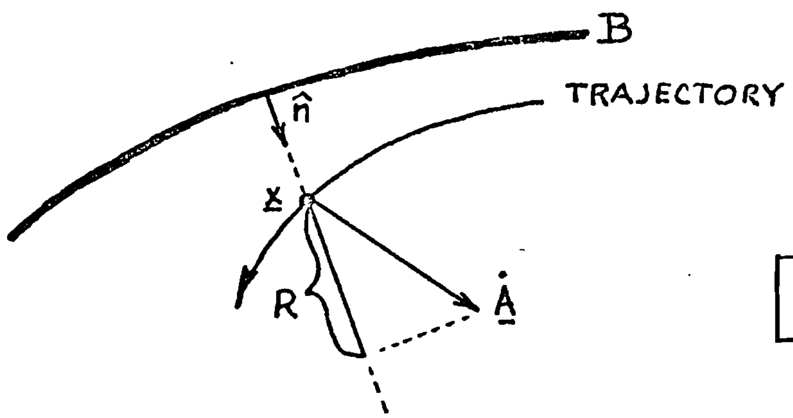
The dynamic characteristics of the system state $\underline{x}$ are enough of a description of the system without the artificial addition of 'force' and 'energy'. The acceleration vector $\underline{\mathring{A}}$ provides the performance ascribed to force without its physical connotations. This does not negate our reason for developing a U-function, as long as that function is some measure of the dynamics of the system.

Insofar as perturbations can be considered as rates, the

$\dot{\underline{A}}$ is a measure of the system's ability to remain within its region of stability.

We can intuitively define resistance as the opposition of a perturbation by the motion of state. The greater the projection of $\dot{\underline{A}}$ in a direction opposite to the perturbation, the greater the resistance.

Resilience can be defined as the amount of resistance offered to a perturbation of the state toward a boundary. In the figure, B is the boundary to a stable region below it. The vector $\hat{n}$ is a unit vector normal to the boundary. $\dot{\underline{A}}$ is the acceleration of $\underline{x}$ at a nearby point. The projection of $\dot{\underline{A}}$ onto $\hat{n}$ is the resilience R.



$$R = \dot{\underline{A}} \cdot \hat{n} \tag{23}$$

Resistance and resilience are properties of points within the region of stability rather than properties of the region as a whole.

A distinction should be made between resistance and resilience. Resistance relates the unperturbed motion of the system to the direction of an applied perturbation. Resilience, on the other hand, relates the motion of the unperturbed system to a particular location in the state-space -- the boundary of stability. This interpretation of resilience becomes ambiguous at points far from the boundary.-- distance being measured in terms of the size of the stable region and the size of distortions in the boundary. It should be noted that both resistance and resilience can be negative. The interpretation of negative values is that a perturbation is reinforced by the system dynamics rather than impeded.

Resistance and resilience, as defined above, have not been operationally tested as measures of system stability under perturbation. Some combination of $\underline{A}$ and $\dot{\underline{A}}$ may prove to be a more advantageous device for judging system response to change.

POINTS, PATHS, AND PERTURBATIONS

Perturbations have been used in an intuitive sense only. Before we can judge system response to change, these perturbations must be related to the state dynamics. There are two basic categories of perturbations: (a) Those that directly change the components of the state vector $\underline{x}$, and (b) Those that change the 'rules' of the system. A simple example will illustrate the distinction.

Consider the one-dimensional system

$$\dot{x} = bx. \tag{24}$$

This is the same type of system as Equation (4). Perturbation type (a) would change (24) to

$$\dot{x} = bx + a(t), \tag{25}$$

where a(t) adds directly to $\dot{x}$. The term a(t) is commonly called a forcing function or a driving variable. Type (b) perturbations are of the form

$$\dot{x} = b(t)x. \tag{26}$$

Here x changes indirectly through changes in the parameter b(t). The perturbation a(t) can be added directly to the state velocity

$$A = bx \tag{27}$$

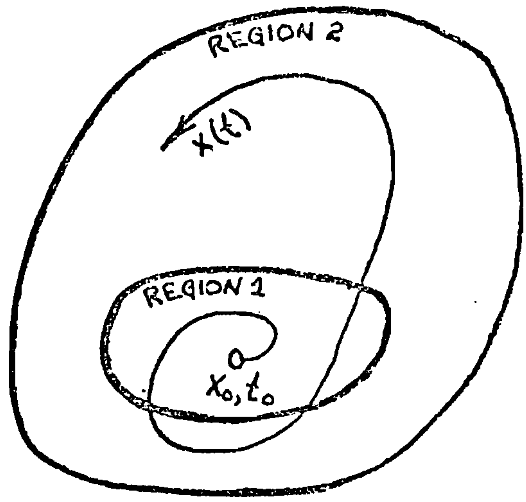
to give A' = A + a(t).

Stability under a(t) is determined by the magnitude of $\dot{A}$ of the unperturbed system. A perturbation of type (b) clearly changes the system geometry and the boundary location and the meaning of resilience becomes unclear.

Implicit in our approach to instability is that perturbations act over a short amount of time. If the perturbation has a long or continuous duration, then the entire system is time-varying and the boundaries change with time. The system becomes nonautonomous, i.e.,

$$\underline{\dot{x}} = \underline{f}(\underline{x}, t). \tag{28}$$

A region of stability would be defined as that area (Region 1) where for all starting points and times within it the subsequent trajectories remain within some finite region (Region 2) for all time.



Although complexity increases when we go from $f(x) \rightarrow f(x,t)$, the problem can be handled by methods similar to what we have employed here. In fact, $f(x,t)$ can be made autonomous by adding the additional state variable time: the system becomes

$$\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, t)$$

$$\frac{dt}{dt} = 1. \qquad (29)$$

By our old standards, this system is unstable because one of the state variables (time) goes to infinity. The projection of all trajectories onto the $\underline{x}$ (hyper)plane would provide the required regions of stability.