Working Paper

Network Externalities and Path Dependent Consumer Preferences

Max Keilbach

WP-95-97
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Preface

The research project on *Systems Analysis of Technological and Economic Dynamics* at IIASA is concerned with modeling technological and organisational change; the broader economic developments that are associated with technological change, both as cause and effect; the processes by which economic agents – first of all, business firms – acquire and develop the capabilities to generate, imitate and adopt technological and organisational innovations; and the aggregate dynamics – at the levels of single industries and whole economies – engendered by the interactions among agents which are heterogeneous in their innovative abilities, behavioural rules and expectations. The central purpose is to develop stronger theory and better modeling techniques. However, the basic philosophy is that such theoretical and modeling work is most fruitful when attention is paid to the known empirical details of the phenomena the work aims to address: therefore, a considerable effort is put into a better understanding of the 'stylized facts' concerning corporate organisation routines and strategy; industrial evolution and the 'demography' of firms; patterns of macroeconomic growth and trade.

From a modeling perspective, over the last decade considerable progress has been made on various techniques of dynamic modeling. Some of this work has employed ordinary differential and difference equations, and some of it stochastic equations. A number of efforts have taken advantage of the growing power of simulation techniques. Others have employed more traditional mathematics. As a result of this theoretical work, the toolkit for modeling technological and economic dynamics is significantly richer than it was a decade ago.

During the same period, there have been major advances in the empirical understanding. There are now many more detailed technological histories available. Much more is known about the similarities and differences of technical advance in different fields and industries and there is some understanding of the key variables that lie behind those differences. A number of studies have provided rich information about how industry structure co-evolves with technology. In addition to empirical work at the technology or sector level, the last decade has also seen a great deal of empirical research on productivity growth and measured technical advance at the level of whole economies. A considerable body of empirical research now exists on the facts that seem associated with different rates of productivity growth across the range of nations, with the dynamics of convergence and divergence in the levels and rates of growth of income, with the diverse national institutional arrangements in which technological change is embedded.

As a result of this recent empirical work, the questions that successful theory and useful modeling techniques ought to address now are much more clearly defined. The theoretical work has often been undertaken in appreciation of certain stylized facts that needed to be explained. The list of these 'facts' is indeed very long, ranging from the microeconomic evidence concerning for example dynamic increasing returns in learning activities or the persistence of particular sets of problem-solving routines within business firms; the industry-level evidence on entry, exit and size-distributions – approximately log-normal – all the way to the evidence regarding the time-series properties of major economic aggregates. However, the connection between the theoretical work and the empirical phenomena has so far not been very close. The philosophy of this project is that the chances of developing powerful new theory and useful new analytical techniques can be greatly enhanced by performing the work in an environment where scholars who understand the empirical phenomena provide questions and challenges for the theorists and their work.

In particular, the project is meant to pursue an 'evolutionary' interpretation of technological and economic dynamics modeling, first, the processes by which individual agents and organisations learn, search, adapt; second, the economic analogues of 'natural selection' by which interactive environments – often markets – winnow out a population whose members have different attributes and behavioural traits; and, third, the collective emergence of statistical patterns, regularities and higher-level structures as the aggregate outcomes of the two former processes.

Together with a group of researchers located permanently at IIASA, the project coordinates multiple research efforts undertaken in several institutions around the world, organises workshops and provides a venue of scientific discussion among scholars working on evolutionary modeling, computer simulation and non-linear dynamical systems.

The research focuses upon the following three major areas:

1. Learning Processes and Organisational Competence.

2. Technological and Industrial Dynamics

3. Innovation, Competition and Macrodynamics

Network Externalities and Path Dependent Consumer Preferences

Max Keilbach*

Abstract

Commodities of high technological level play an increasingly important role in the economy. The market of these commodities can exhibit network externalities if different standards compete. This is due to the fact that the market of these technologies is linked to the market of its co-products. Network externalities engender positive feedback on the market i.e. the higher the market share of a certain technology, the higher the demand for it. The present paper suggests a flexible formal model of the dynamics of these markets. This approach allows for identification the dynamic behaviour of markets under different hypotheses concerning behaviour of producers and consumers. It makes explicit the role of network externalities on markets that exhibit increasing returns.

1 Introduction

Commodities of high technological level play an increasingly imporant role in the economy. This holds for consumption goods, where demand has increasingly shifted towards high-tech products, but also for investment goods where substitution leads to an increasing investment in complex production facilities. The markets of these investment goods and/or of durable consumer goods (let us denote it *complex technologies*) can exhibit different behaviour compared to markets

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of goods of daily consumption (goods with a low level of technology like e.g. food or clothing).[1] A main difference between these two types of goods being that the first very often require some kind of *complementary investment* that puts the technology to work. Think e.g. of training costs or some linked product like computer software. The market of complex technologies is linked to some extend to the market of co-products i.e. prices and ease of acces to co-products can be an important variable to influence choice if different technologies (different standards) of comparable performance coexist. Economically spoken the consequence is twofold:

- Once a buyer has chosen a certain technology (and realized some co-investment) it is probable that he will stick to it since the co-investment is experienced as *sunk costs.*
- The demand stucture of a potential byuer is – via the "co-market" (the market of complementary goods) – linked to the demand structure of others. Hence the market of complex technologies exhibits *network externalities.*

Hence we encounter phenomena that we know from another market, the market of telecommunication. But, unlike these markets, in the case of non-telecommunication complex technologies there is not a physical network but a "taciturn network". Let us call this an *investment network.* Such networks can exhibit the same characteristics as the – well studied – market of telecommunication[2]. If one technology (one standard) dominates the market its co-products can be expected to be cheaper and easier to obtain. Moreover we can expect that the variety among co-products is bigger and thus more attractive for a new buyer[3]. Hence there is an incentive to join this network since it can be expected to entail a higher user-value[4].

[1] *Arthur et al.* and *David* introduced this argument into the economic discussion. See e.g. [3, 5, 12]

[2] See *Callon* [8] or *Capello* [9] for a discussion of this issue. Research in the field of network externalities has been stimulated by a paper by *Rohlfs* [20]. It is probably due to this paper that the research in network economics has mainly focussed on telecommunication markets. Yet, what has been said with respect to these markets should also hold for investment networks. However a systematical analysis of markets where network externalities exist is still to be done.

[3] See also the dicussion in *Katz/Shapiro* [17]

[4] Katz/Shapiro [17, 18] dealt with the question of network exernalities when the market exhibits investment networks. Their studies were based on a comparative static approach

As an outcome of this "market failure", the dynamics of a market of such complex technologies can be expected to be fundamentally different to "classical markets". The following section describes the market dynamics more detailed and presents the model. In the subsequent section this approach is illustrated with for a limited number of goods.

2 Evolution of Markets when Network Externalities are present

2.1 Decision Process under Existence of Network Externalities

Suppose a market that is characterized by investment networks. On this market several types of technologies compete that can all fulfil the same task but have different technological characteristics, hence they work with *different standards*[5]. When a potential buyer is deciding which type of technology to purchase they look at its relative price, at its market share and at the availability of its co-products.

Suppose at a certain time instant the market is in a situation such that all competing technologies have the same market share and are sold at the same price. We would expect a potential buyer to be indifferent. Maybe he *is* indifferent, he then might make his decision simply by random. Or maybe he prefers one of the technologies, for which the reasons can be manifold: maybe he prefers a certain special characteristic, a friend might have recommended to do so because he uses the same technology etc. Due to this manifoldness of influences on his decision the outcome of this process (his choice) appears random to us[6].

The consequence of his decision is twofold. First, the market share of the product has changed. This increases the market of the co-products what makes them easier available. This again increases the probability that a next buyer chooses a unit of the same technology. Hence via the market of the co-products we can expect a *positive feedback* on the market of technologies. Second, the

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involving the assumption of rational expectations. The present paper suggests a different approach in that the model is inherently dynamic and the assumption of rationality is not involved.

[5]Think e.g. of different computer systems, of digital cassette recorders (DAT and DCC-systems) or – on another scale – different energy providing systems (see e.g. *R. Cowan* [10]).

[6]For a similar argument see *Arthur* [3]

producers that are now confronted with the new market share might change the product-price. Several reasons can play a role in that regard. Maybe they use their advantage on the market to increase the price of their products or they are confronted with different costs[7]. The way firms act influences the behaviour of the customers. If – due to the actions of the producers – the relative price of a product is changing, the propensity to buy that product is changing as well. Direction and extend of the change depends on the way the consumers react toward price change (the price-elasticity). Hence, with respect to prices, we can expect both, *negative* or *positive feedbacks*[8]. Of course in the case of network externalities this feedback might be traded off by the market share itself (see discussion below).

However, even if the market share of a product is higher and its price is lower, there is still a positive willingness to pay for other products since they differ in some characteristics. The next buyer, faced with the same decision problem as described above, might choose a product that is more expensive. This might be due to the reasons given above or simply because he does not like to do what the majority does[9]. Thus – again – the decision is of random nature.

2.2 The Model

Suppose a market where $K \geq 2$ technologies (standards) compete. Denote each of the possible choices $c_k, k = (1, 2, \ldots, K)$, where c_k denote technologies. Each technology c_k is produced by exactly one firm, hence we have K firms.

Suppose we have perfect correlation between the market of the technologies and the market of the co-products[10]. Hence we can limit ourselves to an analysis of the market share of the base-technology itself. Let n_k^t be the number of units of c_k in the market at t. Denote $s_k^t = n_k^t / \sum_{i=1}^{K} n_i^t$ the market share of c_k at time t, $t = (1, 2, \ldots)$. Suppose the price p_k^t of c_k at t is a function of its market share

$$p_k^t = f(s_k^t), \tag{1}$$

[7] See also the discussion in *Dosi/Kaniovski* [13] and *Dosi et al.* [14].

[8] This issue has been discussed in a number of papers. See e.g. *Arthur* [3, 5], *David* [11] or *Dosi/Kaniovski* [13].

[9] His motivation migh be a "search for diversity". See the discussion in *Dosi/Kaniovski*, [13].

[10] This implies that a certain technology cannot use co-products that fit a different standard. This assumption is straightforward for all types of technical co-products. The correlation can be less than one in the case of human skills.

where $f(\cdot)$ is a response function of firms with respect to market share.[11]. In the case of network externalities the demand of a potential consumer (the consumer that buys next, hence at $t+1$) depends on the price of the product at t – which is a function of the market share – *but also* on the market share at t itself. This is specified in the following demand function

$$D_k(s_k^t) = (s_k^t)^\sigma \cdot [f(s_k^t)]^\rho, \tag{2}$$

where t is not chronological time but defined by the sequential moments of buying. It should be noted, that a number of possible variables that can have an influence on the choice – like different technological characeristics or influences of friends – are not included in this demand function. Equation 2 is homogenous of degree σ in s_k^t and ρ in p_k^t and allows for substitution in terms of demand between market share and price. Given a certain market share at t we can determine the price via equation (1) (given that we know this function) thus, the absolute demand (or *revealed preference*) by the next buyer. The *relative* propensity of buying a certain product at time t is given by the *relative demand function* or *preference function* which is a vector function whose k-th coordinate is:

$$d_k(\mathrm{s}^t) = \frac{D_k(s_k^t)}{\sum_{i=1}^{K} D_i(s_i^t)}, \text{ where } \mathrm{s}^t = (s_1^t, s_2^t, \ldots, s_K^t). \tag{3}$$

Suppose that the probability that a potential buyer purchases a certain technology equals his relative preference for this product. Then function (3) specifies the conditional probabilities of choosing technology c_k given the current market shares of all technologies (i.e. given vector s^t)[12]. Define a K-dimensional random vector that is independent in t, $\boldsymbol{\beta}^t(\mathrm{s})$ of which the k-th coordinate $\beta_k^t(\mathrm{s})$ is 1 with probability $d_k(\mathrm{s})$, $k = (1, 2, \ldots, K)$. The evolution of the market shares can now be described as[13]

$$\mathrm{s}^{t+1} = \mathrm{s}^t + \frac{1}{n+t}\left[\boldsymbol{\beta}^t(\mathrm{s}^t) - \mathrm{s}^t\right], \tag{4}$$

where n is the initial number of goods in the market, i.e. $n = n_1^1 + n_2^1 + \ldots + n_K^1$. Let $\mathbf{d}(\mathrm{s}) := (d_1(\mathrm{s}), d_2(\mathrm{s}), \ldots, d_K(\mathrm{s}))$. Then, expanding equation (4) with $\mathbf{d}(\mathrm{s}^t)$

[11]This approach was also chosen by *Dosi/Kaniovski* [13] and *Dosi, Ermoliev and Kaniovski* [14]

[12]The concept of function 3 is very closely related to the notion of *allocation function* used by *Arthur et al.*. See [5, 13, 14].

[13]See [5, 13, 14].

yields

$$\mathbf{s}^{t+1} = \mathbf{s}^t + \frac{1}{n+t}\left[\mathbf{d}(\mathbf{s}^t) - \mathbf{s}^t\right] + \frac{1}{n+t}\left[\boldsymbol{\beta}^t(\mathbf{s}^t) - \mathbf{d}(\mathbf{s}^t)\right]. \tag{5}$$

Since $E(\boldsymbol{\beta}(\mathbf{s})) = \mathbf{d}(\mathbf{s})$ we have

$$E(\mathbf{s}^{t+1}|\mathbf{s}^t) = \mathbf{s}^t + \frac{1}{n+t}\left[\mathbf{d}(\mathbf{s}^t) - \mathbf{s}^t\right]. \tag{6}$$

Consequently, the system (5) on average shifts from a point $\mathbf{s}$ at t on $\frac{1}{n+t}\left[\mathbf{d}(\mathbf{s}) - \mathbf{s}\right]$, i.e. the limit points of the system (if any) belong to the roots of $\mathbf{d}(\mathbf{s}) - \mathbf{s}$.[14]

Once the choice is made the next agent is confronted with a new market share, hence he has a different preference structure and the same process applies iteratively. Section 3 will model this process explicitely.

3 Dynamics of markets under different pricing policies

So far the model has been formulated on a general level. In this section we are going to present an illustration of the market dynamics based on different specifications of the cost function. This will be done for three commodities ($K = 3$). We are going to illustrate the approach as follows:

- Prices depend on the pricing policies of the firms. We consider two cases: the case when firms sell their products at average costs and experience sinking average costs with market share and the case when firms are using price policy to increase profits.
- For each of the two cases we are going to show the relative demand function for three goods under two different (exemplary) price elasticities. Given these parameter constellations, the evolution of the market and the converging behaviour of market shares is discussed. This is done either formally or by means of simulation studies based on Polya-urn processes which are described in the following section.

[14]For a detailed analysis of the convergence behaviour of a system of this type see *Arthur, Ermoliev* and *Kaniovski* [7].

3.1 Polya-Urn Schemes as a Paradigm for Modelling the Dynamics of a Market

A particularly well suited stochastic model for our purposes are *Polya-urns.* Suppose an urn of infinite capacity with $K = 2$ types of balls. In $t = 1$ we have a certain positive number n_1^1, n_2^1 of balls of each type in the urn. The system evolves according to the rule: consider the proportion of balls in the urn. A new ball of a certain type is added to the urn with a probability that is a given function of the proportion of balls in the urn. Following [5, 13, 14] we call this function an *urn function.* In the simples case this function is an identity function, i.e. the probability that a new ball is of a certain type equals the proportion of this type.[15]. This case is referred to as the *standard Polya-process.*

The framework of Polya-urns has been extended in a number of papers[16] in that the urn function is not restricted to be the identity function but can take any shape and a number of arguments (i.e. types of balls) that can be greater than 2 under the condition that it maps the unit simplex into $[0, 1]$, i.e. its domain is $S_k = \{\mathbf{s} \in \mathbf{R}^K, s_i \geq 0, \sum_{i=1}^{K} s_i = 1\}$ and its range is $[0, 1]$. The function then defines the probability to add a ball of a certin type, given the vector of proportions $\mathbf{s}$. This approach is referred to as *generalized Polya-process.*

The evolution of the proportion in time (the trajectory) depends on random events and the lower the number of balls in the urn, the higher the influence a new ball added can have on the proportion. Thus, early random events can determine in which direction the system evolves. A new time step is defined by the fact that a ball is added to the urn. With increasing time the proportion of balls might exhibit strong convergence where the limit depends on past events. Hence the evolution of the trajectories is *path dependent.*

The application to markets with network externalities is straightforward. Instead of dealing with an urn we are considering a market and instead of balls we are considering technologies. A new buyer is making a decision according to the process described in section 2.1, i.e. he observes market shares and prices of the products. Then he decides in a way that appears random to us. With his decision he "adds a product to the market" and thus exerts an influence on the preference

[15]This is tha case that has been analyzed by Eggenberger and Polya in their paper of 1923. See [16].

[16]See *Arthur, Dosi, Ermoliev* and *Kaniovski*, [3, 5, 13, 14].

structure of the next buyer. The following section ilustrates this process.

3.2 Two hypothetical share-response functions

In Section 2.1 we argued that firms change the price of their products as market shares change. Let us assume that firms base their price settings on their average cost[17] such that the minimum price of the product equals its average costs. Moreover, assume that with increasing market share firms can extend their production capacity and hence they experience a sinking long-term average cost function[18]. Firms whose products have passed a certain market share might decide to use their market power to increase their prices and hence to obtain profits. If they do so, this price increase changes the behaviour of the feedback on the market.

To describe the price response to market share we suggest two *hypothetical share-response functions.*

$$p_k(s_k) = \frac{a}{s_k} \tag{7}$$

$$p_k(s_k) = \frac{b}{s_k} + c \cdot (s_k)^2, \tag{8}$$

where a, b, c are constants. Equation (7) describes the behaviour of the firm if they simply lower the price with market share, equation (8) describes the behaviour if firms increase the price beyond a critical market share. Figures 1 and 2 give a graphical representation of these functions. The following analyses will be based on these functions.

4 Emerging relative demand functions

Let us illustrate the discussion of section 2 with a hypothetical market where three goods compete. Since always

$$\sum_{k=1}^{K} s_k^t = 1$$

[17]Let us also assume that these average cost include "normal profit", i.e. the opportunity costs of production.

[18]The model to be presented in this paper is intrinsically dynamic. Since the production structure, hence costs, is subject to change with time we consider the ***long-term average cost function***. For a discussion of this function see e.g. *Varian* [21].

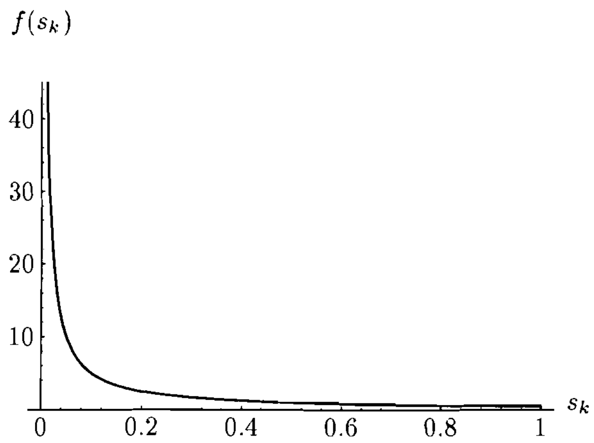

Figure 1: Plot of hypothetical share-response function (7) ($a = 0.5$)

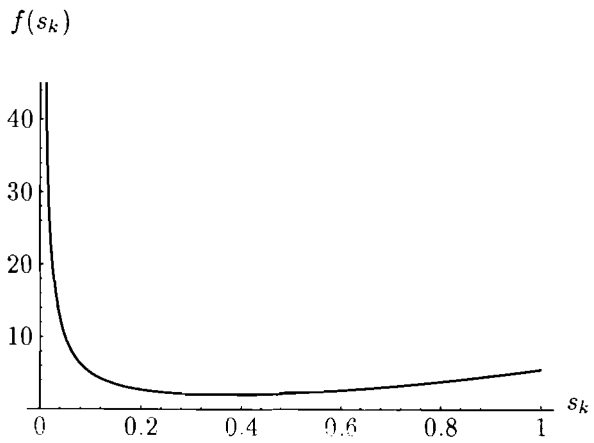

Figure 2: Plot of hypothetical share-response function (8) ($b = 0.5$, $c = 5$)

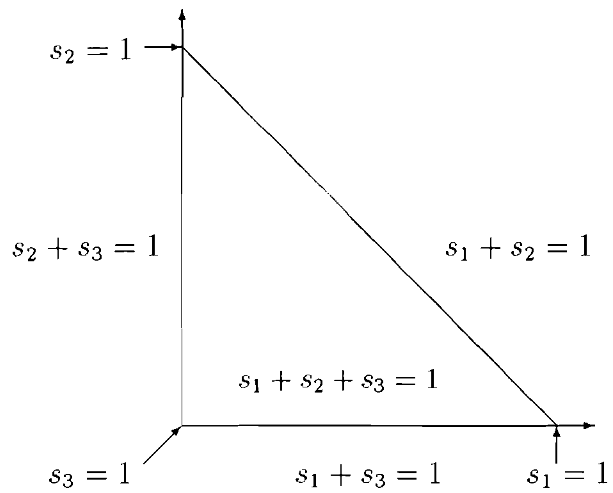

Figure 3: Graphical representation of the domain T_2 of a relative demand function of a market where three goods compete.

we can reduce the relative demand function (3) to a function of $K-1$ factors in $T_{K-1} = \{\mathbf{s} \in \mathbf{R}^{K-1}, s_i \geq 0, \sum_{i=1}^{K-1} s_i \leq 1\}$. In the case of three goods we can express function (3) e.g. as a function of s_1 and s_2. The function is defined for $0 \leq s_1 + s_2 \leq 1, s_1, s_2 \geq 0$. Figure 3 gives a graphical representation of the domain T_2.

4.1 Firms respond to increasing market share with decreasing prices

Let us assume that all firms behave as described in equation (7). Since the relative demand function (equation (3)) is now a function in T_2 we write

$$\tilde{d}_1(\mathbf{s}) = \frac{s_1^\sigma \left(\frac{a}{s_1}\right)^\rho}{s_1^\sigma \left(\frac{a}{s_1}\right)^\rho + s_2^\sigma \left(\frac{a}{s_2}\right)^\rho + (1 - s_1 - s_2)^\sigma \left(\frac{a}{1-s_1-s_2}\right)^\rho}, \tag{9}$$

where $\mathbf{s} = (s_1, s_2)$ and $0 \leq s_1 + s_2 \leq 1$. Rearranging yields

$$\tilde{d}_1(\mathbf{s}) = \frac{s_1^{\sigma-\rho}}{s_1^{\sigma-\rho} + s_2^{\sigma-\rho} + (1 - s_1 - s_2)^{\sigma-\rho}} \tag{10}$$

Figure 4 plots this function for $\sigma = 1, \rho = -1$. From (6) we conclude that the limit behaviour of the stochastic process (5) can be similar to the ones of the

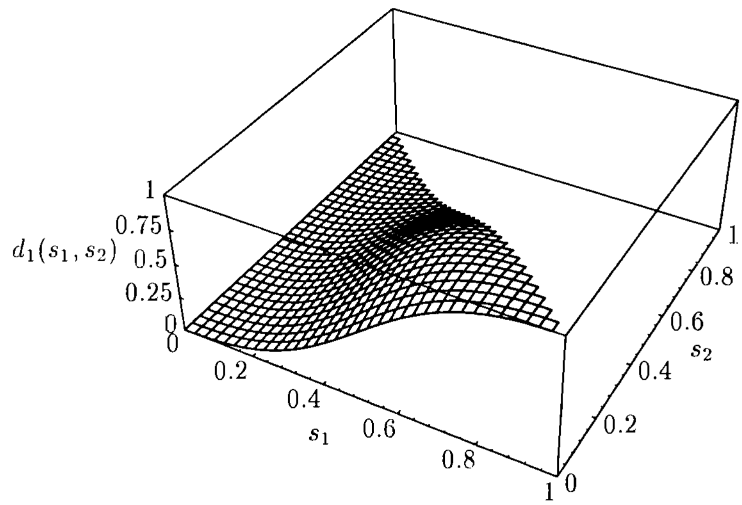

Figure 4: Emerging relative demand function given response function (7) and $\rho = -1, \sigma = 1$

following deterministic system of ordinary differential equations[19]

$$\dot{\mathrm{s}} = \tilde{\mathrm{d}}(\mathrm{s}) - \mathrm{s} \tag{11}$$

The possible attractors of this system are given by the solutions of the nonlinear equations

$$\tilde{d}_1(\mathrm{s}) - s_1 = 0, \; \tilde{d}_2(\mathrm{s}) - s_2 = 0. \tag{12}$$

Let s^* be a solution of the set of equations (12). We know from the results on unattainability of unstable points[20] that if s^* is a solution of (12) and the Jacobian $J(\mathrm{s}^*)$ of (11) has an eigenvalue with positive real part then the process (5) converges to s^* with probability 0.

To investigate the convergence of the market we need further assumptions. To clarify the impact of different parameter constelltions let us consider some particular cases. Suppose that $\sigma = 1$, i.e. a one per-cent increase in market share means that the demand for the product increases also at one per-cent[21]. We show that for different price elasticites ρ (namely $\rho = 0$, $\rho = -1$ and $\rho = 0.5$) the market dynamics exhibit different behaviour.

[19]See also *Arthur et al.* [5].

[20]See *Arthur, Ermoliev, Kaniovski* [6]

[21]So far we do not have any empirical studies that deal with the issue of response to market share and thus no estimates of σ. In any way, prevailing network externalities imply $\sigma > 0$.

Equilibrium points	*behaviour*
$s_1 = 1,\ s_2 = 0,\ (s_3 = 0)$	attracting
$s_1 = 0,\ s_2 = 1,\ (s_3 = 0)$	attracting
$s_1 = 0,\ s_2 = 0,\ (s_3 = 1)$	attracting
$s_1 = \frac{1}{2},\ s_2 = \frac{1}{2},\ (s_3 = 0)$	saddle
$s_1 = \frac{1}{2},\ s_2 = 0,\ (s_3 = \frac{1}{2})$	saddle
$s_1 = 0,\ s_2 = \frac{1}{2},\ (s_3 = \frac{1}{2})$	saddle
$s_1 = \frac{1}{3},\ s_2 = \frac{1}{3},\ (s_3 = \frac{1}{3})$	repelling

Table 1: Equilibrium points and their behaviour for a relative demand function for a market where three goods compete given decreasing price with increasing market share and $\rho = -1$

1. Suppose the consumers are indifferent with respect to prices, i.e. $\rho = 0$. Then the preference function degenerates to $d_k(\mathbf{s}) = s_k$. We know[22] that the shares converge with probaility one in this case. The limit has a *Dirichlet-distribution* with the density function

$$f_D(\mathbf{s}) = \begin{cases} c \cdot s_1^{n_1^1 - 1} s_2^{n_2^1 - 1} (1 - s_1 - s_2)^{n_3^1 - 1} & \text{for } \mathbf{s} \in T_2 \\ 0 & \text{else} \end{cases}$$

with $n_1^1, n_2^1, n_3^1 \geq 1$. n_1^1, n_2^1, n_3^1 are initial numbers of balls (technologies) and c is a normalizing constant that depends on this initial numbers. If $n_1^1 = n_2^1 = n_3^1 = 1$ the limit distribution is unform on T_2.

2. Suppose that a one per-cent increase in price of one of the products implies that the demand for this product decreases at one per-cent i.e. $\rho = -1$. Inserting ρ and solving the set of equations (12) we get the set of fixed points that are shown in Table 1 (once an equilibrium point (s_1^*, s_2^*) is computed, the corresponding value for s_3^* can be identified through the symmetricity of the functions $d_1(\cdot)$, $d_2(\cdot)$ and $d_3(\cdot)$).

To check the stability of these fixed-points we compute the eigenvalues of the Jacobi-matrix of each point (the second column of Table 1 shows the stability behaviour of each of the equilibrium points). The system possesses attracting equilibrium points at the corners (i.e where one $s_k = 1$),

[22]See *Arthreya* [2]

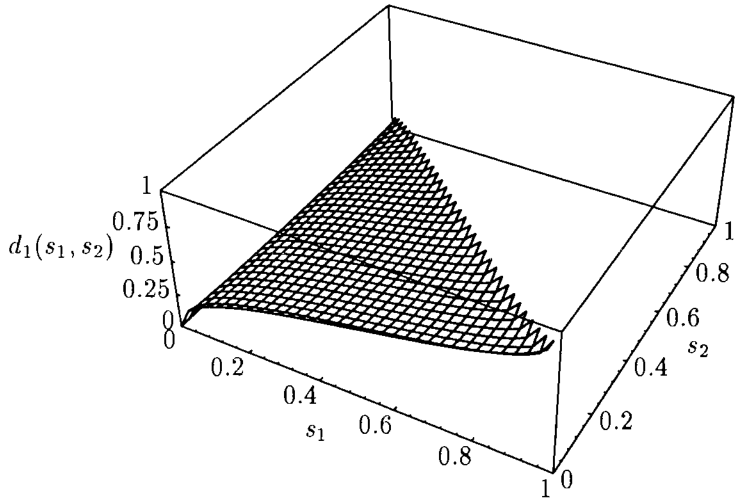

Figure 5: Emerging relative demand function given response function (7) and $\rho = 0.5, \sigma = 1$

it has a repelling equilibium point in $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and three saddle points in $(0, 0.5), (0.5, 0), (0.5, 0.5)$.

An identification of fixed points of the deterministic system is not sufficient to assess the limit behaviour of the stochastic system. However, we know[23] that if we can identify a Lyapunov function for the system, it converges with probability 1 to a random vector whose support belongs to the set of fixed points. A Lyapunov function for system (11) is given in the Appendix.

We conclude that given function (9) and $(\rho, \sigma) = (-1, 1)$, the market converges with probability one to a random vector taking at most three values (1,0,0), (0,1,0) and (0,0,1).[24] Furthermore, we can show[25] that each of these values is attained with positive probability for any initial set of numbers n_1^1, n_2^1, n_3^1.

3. Suppose that the demand is *increasing* by 0.5% if the price is increasing by 1%, hence $\rho = 0.5$. It might be not very probable that demand increases with price. Nevertheless we discuss this case since there is no empirical evidence on price response in markets of high-tech commodities. Figure 5

[23]See *Arthur et al.* [7]

[24]This follows from the eigenvalues of the Jacobi-matrix. For a proof see *Arthur, Ermoliev* and *Kaniovski* [7], p. 191-195

[25]For a proof See *Arthur, Ermoliev* and *Kaniovski* [4].

Equilibrium points	*behaviour*
$s_1 = 1,\ s_2 = 0,\ (s_3 = 0)$	repelling
$s_1 = 0,\ s_2 = 1,\ (s_3 = 0)$	repelling
$s_1 = 0,\ s_2 = 0,\ (s_3 = 1)$	repelling
$s_1 = \frac{1}{2},\ s_2 = \frac{1}{2},\ (s_3 = 0)$	saddle
$s_1 = \frac{1}{2},\ s_2 = 0,\ (s_3 = \frac{1}{2})$	saddle
$s_1 = 0,\ s_2 = \frac{1}{2},\ (s_3 = \frac{1}{2})$	saddle
$s_1 = \frac{1}{3},\ s_2 = \frac{1}{3},\ (s_3 = \frac{1}{3})$	attracting

Table 2: Equilibrium points and their behaviour for a relative demand function for a market where three goods compete given decreasing price with increasing market share and $\rho = 0.5$

gives a plot of the relative demand function (equation (3)) given $\sigma = 1$ and $\rho = 0.5$. Table 2 shows the equilibrium points in that case.

The equilibrium points are identical to those in the previous case. However their characteristics have changed. The only attracting equilibrium point of the deterministic system is given by $s_1 = s_2 = s_3 = \frac{1}{3}$. In this case (as for the case of $\rho \geq 0$ in general) we cannot identify a Lyapunov function, hence we have no proof that $\mathbf{s}^t$ converges with probability 1 to this point. We know however that each of the attracting equilibrium points of the deterministic system are attained with positive probability for any initial value n_1^1, n_2^1, n_3^1.[26] Also a simulation study based on Polya-urn processes suggests that the stochastic system converges to the attracting equilibrium point of the deterministic system, Thus, for the given parameter constellation only the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is attained in the limit with positive probability.[27]

4.2 Firms respond to increasing market share with a mixed strategy

Let us assume that all firms behave as described in equation (8), i.e they lower their prices with increasing market share when their market share is low but beyond a certain critical share they use their market power to increase prices

[26] For a proof See *Arthur, Ermoliev* and *Kaniovski* [4].

[27] Since we cannot exclude cycles in this case the probability of attaining this point can be less than 1.

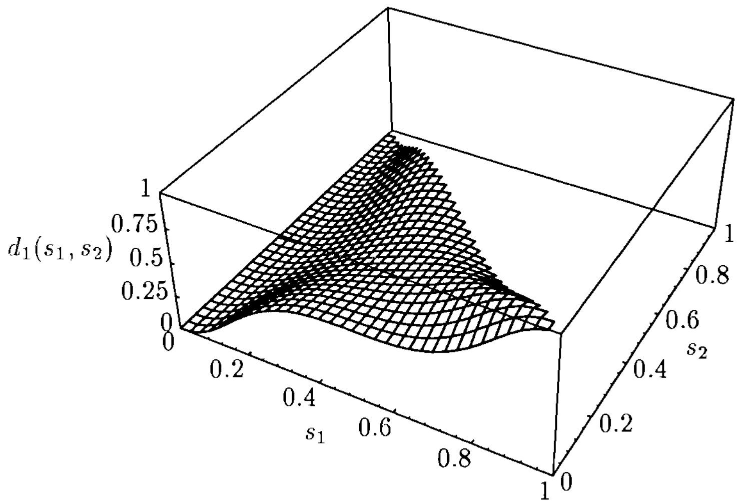

Figure 6: Emerging relative demand function given response function (8) and $\rho = -1, \sigma = 1$

and hence increase their profits. The relative demand function (3) in this case is of course more complex. Again, let us analyze the situation for three different values of ρ.

1. For $\rho = 0$ the situation is identical to the one discussed in the previous sector, i.e. the limit has a Dirichlet-distribution on the unit simplex.

2. Let $\rho = -1$. Figure 6 gives a graphical presentation of this function for $b = 0.5$ and $c = 3$. The equilibrium points of this function are given in Table 3. If the share-response function is more complex, the relative demand function becomes of course also more complex. For this case we could not find a Lyapunov function. Again, we can only state, that the attracting equilibrium points of the deterministic system are attained with positive probability. However, numerical simulations suggest that the stochastic system does not converge to limit cycles, i.e. it converges indeed to the attracting equilibrium points of the deterministic system. Figure 7 shows the outcome of this simulation study i.e. the distribution of market shares at $t = 100$.

   There are two interesting phenomena to observe if we compare these equilibrium points to those that emerge under function (7), given in Table 1. First, we obtain six further equilibrium points. These points are determined by the intersection points of $f(s_1)$ and $f(s_2)$ given $s_3 = 0$ (and

Equilibrium points			*behaviour*
$s_1 = 1,$	$s_2 = 0,$	$(s_3 = 0)$	attracting
$s_1 = 0,$	$s_2 = 1,$	$(s_3 = 0)$	attracting
$s_1 = 0,$	$s_2 = 0,$	$(s_3 = 1)$	attracting
$s_1 = \frac{1}{2},$	$s_2 = \frac{1}{2},$	$(s_3 = 0)$	attracting
$s_1 = \frac{1}{2},$	$s_2 = 0,$	$(s_3 = \frac{1}{2})$	attracting
$s_1 = 0,$	$s_2 = \frac{1}{2},$	$(s_3 = \frac{1}{2})$	attracting
$s_1 = \frac{1}{3},$	$s_2 = \frac{1}{3},$	$(s_3 = \frac{1}{3})$	repelling
$s_1 = 0.21132,$	$s_2 = 0,$	$(s_3 = 0.78867)$	saddle
$s_1 = 0.21132,$	$s_2 = 0.78867,$	$(s_3 = 0)$	saddle
$s_1 = 0,$	$s_2 = 0.21132,$	$(s_3 = 0.78867)$	saddle
$s_1 = 0.78867,$	$s_2 = 0,$	$(s_3 = 0.21132)$	saddle
$s_1 = 0.78867,$	$s_2 = 0.21132,$	$(s_3 = 0)$	saddle
$s_1 = 0,$	$s_2 = 0.78867,$	$(s_3 = 0.21132)$	saddle

Table 3: Equilibrium points their behaviour of a relative demand function for a market where three goods compete given share-response function (8) and $\rho = -1$

others symmetrically).[28] We see from the eigenvalues of the Jacobian that these points are *saddle points* of the system, i.e. the system might first evolve to these points and from there to attracting fixed points. Secondly, the point $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and its permutatitions are now *attracting equilibrium points*. This implies that they are now attained in the limit with positive probability (e.g. from a situation with initially equal market shares i.e. from the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ which is unstable). Hence the system converges with positive probability to a state where either two technologies share the market or one technology takes the whole market.

3. Let $\rho = 0.5$. As in the case of the simple share-response function the equilibrium points are the same, but change their character with changing sign of the pice elasticity ρ, i.e. repelling fixed points become attracting and vice versa. Saddle points keep their behaviour. Hence, we can identify the dynamics of the system through Table 3, inverting the behaviour of the equilibrium points. Figure 8 gives a plot of the relative demand function

[28]Let $s_3 = 0$ then $f(s_1)$ and $f(s_2)$ have three intersection points: one at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and two more which depend on the parameters b and c. By symmetry we achieve similar results for $s_1 = 0$ of $s_2 = 0$.

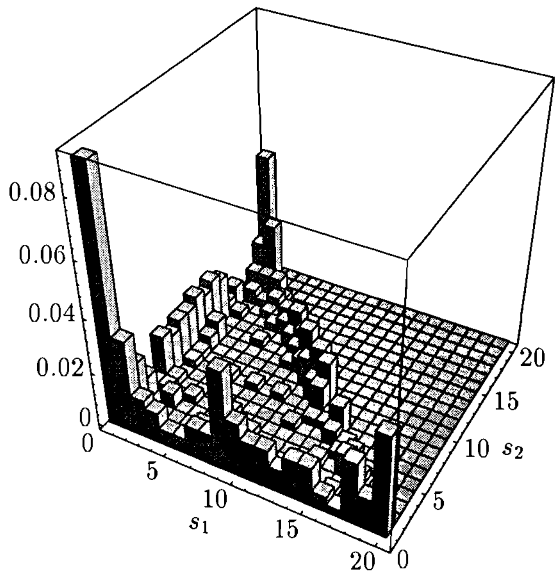

Figure 7: Emerging distribution of market share at $t = 100$ as outcome of 450 simulation runs. The share-function is (8), $\rho = -1, \sigma = 1$ and $n_1^1 = n_2^1 = n_3^1 = 1$

for $\sigma = 1$ and $\rho = 0.5$. We see that only the vector $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is attained in the limit with positive probability. Figure 9 illustrates the emerging distribution of market shares by means of simulation study.

We see that the dynamics of the system is now inverse compared to the case where $\rho = -1$ (see Figure 7). The system converges with positive probability to $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, i.e. a situation where the market is divided at equal share among the three goods.

5 Summary and Outlook

This paper is dealing with markets where network externalities prevail. To do so, we specified a demand function and defined the conditional probabilities of buying a certain technology via a *relative demand function*. We then made different hypotheses on the price setting behaviour of the firms with respect to the market share of their product. This approach allows us to identify the dynamics and limit states of these markets. We illustrated this for a market where three goods compete under different constellations of parameters that determine the demand for these goods. We see from that exercise that network externalities are a sufficient condition for increasing returns.

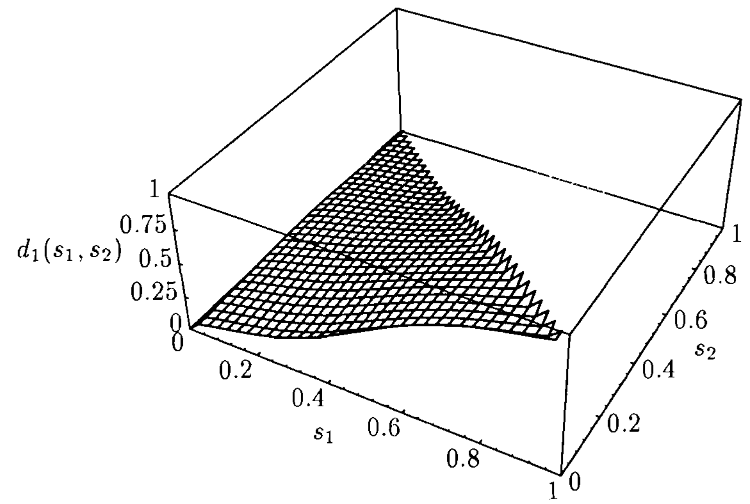

Figure 8: Emerging relative demand function given response function (8) and $\rho = 0.5, \sigma = 1$

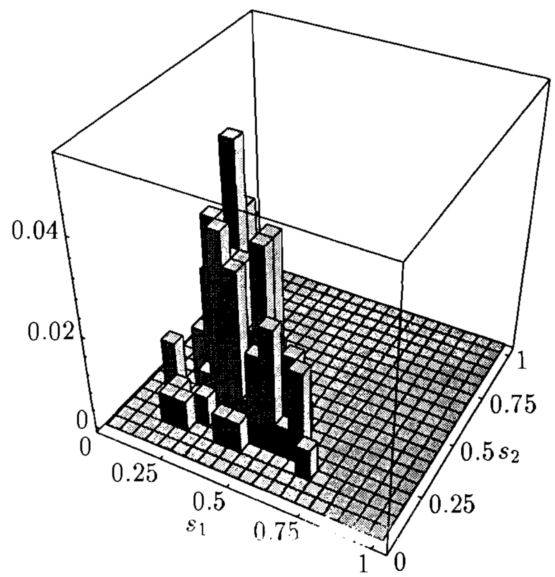

Figure 9: Emerging distribution of market share at $t = 100$ as outcome of 450 simulation runs. The share-function is (8), $\rho = 0.5, \sigma = 1$ and $n_1^1 = n_2^1 = n_3^1 = 1$

The specification of the demand function and the relative demand function should allow for empirical investigation of markets where network externalities prevail. An econometric estimation of ρ and σ of a certain market could be a step towards an investigation of the evolution of shares in this market.

Appendix: A Lyapunov function for system (11)

A function $L = L(s_1, s_2, \ldots, s_{K-1})$ is a *Lyapunov function* of a system of differential equations $\dot{s}_k = \tilde{d}_k(s_1, s_2, \ldots, s_{K-1})$, $k = (1, 2, \ldots, K-1)$ defined on an open set T_{K-1} if

$$\sum_{k=1}^{K} \frac{\partial L(\mathbf{s})}{\partial s_k} \tilde{d}_k(\mathbf{s}) > 0 \ \forall s_k \in T_{K-1} \text{ and } \mathbf{s} = (s_1, s_2, \ldots, s_{K-1}) \in T_{K-1}. \tag{13}$$

For system(11) given relative demand function (10) we define the following Lyapunov function:

$$L(\mathbf{s}) = (s_1)^2 + (s_2)^2 + (1 - s_1 - s_2)^2 \tag{14}$$

To prove that this function is a Lyapunov function, we set

$$f_1(s_1, s_2) = \tilde{d}_1(s_1, s_2) - s_1 = \frac{s_1^2}{s_1^2 + s_2^2 + (1 - s_1 - s_2)^2} - s_1 \tag{15}$$

$$f_2(s_1, s_2) = \tilde{d}_2(s_1, s_2) - s_2 = \frac{s_2^2}{s_1^2 + s_2^2 + (1 - s_1 - s_2)^2} - s_2. \tag{16}$$

If we multiply the right side of the differential equation (11) with a positive term its phase portrait does not change[29]. Thus, to obtain a function easier to handle we achieve by multiplying the right side by $(s_1^2 + s_2^2 + (1 - s_1 - s_2)^2)$

$$\bar{f}_1(s_1, s_2) = s_1^2 - s_1(s_1^2 + s_2^2 + (1 - s_1 - s_2)^2) \tag{17}$$

$$\bar{f}_2(s_1, s_2) = s_2^2 - s_2(s_1^2 + s_2^2 + (1 - s_1 - s_2)^2). \tag{18}$$

Inserting this functions in equation (13) yields a polynomial $g(s_1, s_2)$ of degree 4 in s_1 and s_2. A plot of $g(s_1, s_2)$ suggests that this polynomial has roots for all equilibrium points identified in Table 1 (see Figure 10), hence that condition (13) holds on the simplex T_{K-1}. To proof this we show

[29]See [15], section 12.4.

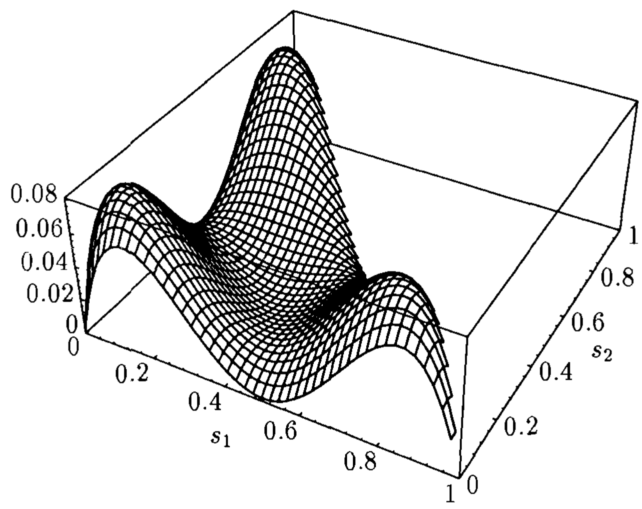

Figure 10: Graphical representation of condition (13) given Lyapunov function (14)

i) that (13) holds on the boundary of the simplex, i.e. for $s_1 = 0$ or $s_2 = 0$ or $(1 - s_1 - s_2) = 0$.

ii) that it holds for all local minima in the interior.

ad i) Consider the case $s_1 = 0$. Then $g(s_1, s_2)$ reduces to a polynomial in s_2. It is zero at the unique local minimum $(0.5, 0.5)$ as well as in the boundary points $s_2 = 0$ and $s_2 = 1$. Hence condition (13) holds if $s_1 = 0$. The other cases follow by symmetry.

ad ii) A straighforward calculation shows that $g(s_1, s_2)$ has only one local minimum (s_1^*, s_2^*) in the interior of the simplex and that $g(s_1^*, s_2^*) = 0$.

Thus, since $g(s_1, s_2) \geq 0$ holds for all points on the boundary of the simplex and for all local minima in the interior, it holds on the whole simplex.

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