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ON RELATIONSHIPS BETWEEN OPTIMAL CONTINUOUS-TIME AND DISCRETE-TIME CONTROLS OF DISTRIBUTED PARAMETER SYSTEMS

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Abstract

In this paper, for a distributed parameter system described by a partial differential equation of parabolic type, two optimal control problems are investigated. From the engineering standpoint on the construction of control devices, we assumed that both distributed and boundary controls are respectively concentrated spatially onto some spatial domains or onto some parts of the boundary. First, for a performance index in quadratic form, an optimal control problem with continuous-time control is considered. Applying the technique of dynamic programming, a non-linear integro-partial differential equation analogous to the Riccati equation has been obtained. Second, using Green's function representation, the same optimal control problem with discrete-time control is discussed, and the recurrence relationships to determine the optimal control policy have been derived. Lastly, relationships between the above-obtained optimal continuous-time and discrete-time controls have been discussed.

1. Introduction and Problem Statement

Recently, control theory for distributed parameter systems has been developing very rapidly, and we can notice this trend from the excellent survey presented by A.C. Robinson [8]. In the development of this field of research, one of the basic approaches is to extend the accepted theories for lumped parameter control systems to distributed parameter control systems. Furthermore, we must investigate the peculiarities which only distributed parameter control systems show. It is also important to investigate the obtained results from more general viewpoints and to establish a unified control theory within the framework of distributed parameter systems, and to present rational criteria for approximation. For example, in the optimal control theory for linear lumped parameter systems with quadratic performance indices, the relationships between optimal continuous-time and discrete-time controls already have been well investigated. For the cases of distributed parameter systems, the same optimal control problems have been researched by many scientists, for example, P.K.C. Wang [9], H. Erzberger and M. Kim [1,2], and J. Lions [5]. However, the relationships between optimal continuous-time and discrete-time controls have not been investigated thoroughly.

The purpose of this paper is to make clear the relationships between optimal continuous- and discrete-time controls for a distributed parameter system described by a linear partial differential equation of parabolic type, and to establish rational criteria for approximation. The performance index is assumed to be in quadratic form, and, as it is very difficult in practice to construct control devices which can change the intensity of control inputs continuously with respect to space variable, we assumed that both distributed and boundary controls are concentrated spatially onto some spatial domains or onto some parts of the boundary. The latter assumption is admittable from the practical engineering point of view (see Porter [7]).

From now on, let us mathematically describe the problem in more detail. The dynamic behavior of the distributed parameter control system considered in this paper is described by the following partial differential equation of the parabolic type:

$$\frac{\partial u(t,x)}{\partial t} = Au(t,x) + F(t,x), \ x \in D \subset R^m, \ t \in (t_o, t_f] \quad , \tag{1}$$

where the bounded spatial domain D is an open, connected subset of an m-dimensional Euclidean space R^m. The function F(t,x) represents a distributed control, and A denotes a linear partial differential operator defined by

$$A = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2} - \gamma(x_1, \ldots, x_m) \cdot \quad , \tag{2}$$

where $\gamma(x)$ shows, for instance, the ratio of calorific power which is lost by heat radiation.

For Eq. (1), the boundary condition

$$\alpha(\xi)u(t,\xi) + \{1-\alpha(\xi)\}\frac{\partial u(t,\xi)}{u} = \alpha(\xi)G(t,\xi) .$$

$$\xi \in S, \ t \in (t_o, t_f], \ 0 \leqq \alpha(\xi) \leqq 1 \quad , \tag{3}$$

is imposed, where $\partial u(t,\xi)/\partial n$ denotes the differentiation of $u(t,\xi)$ along the outward directed normal from the boundary S of D, and $G(t,\xi)$ is a boundary control function. The initial state of the system is given a priori as

$$\lim_{t \downarrow t_o} u(t,x) = u_o(x) \quad . \tag{4}$$

Let us impose the following restrictions to the above-mentioned distributed and boundary controls $F(t,x)$ and $G(t,\xi)$:

1) $F(t,x)$ and $G(t,\xi)$ are spatially concentrated respectively onto some finite number of spatial domains $D_1,\ldots,D_{k_1}$ in D and onto some finite parts of S, say $S_1,\ldots,S_{k_2}$.

2) $F(t,x)$ and $G(t,\xi)$ are constants with respect to x and ξ at each $D_1,\ldots,D_{k_1}$, and $S_1,\ldots,S_{k_2}$, respectively.

Let us define the following characteristic functions $\phi_i(x)$ and $\psi_i(\xi)$ to each D_i and S_i, respectively; i.e.

$$\Phi_i(x) = \begin{cases} 1 : x \in D_i \subset D \\ 0 : x \notin D_i \end{cases} \quad (i = 1,\ldots,k_1) \quad , \tag{5a}$$

$$\psi_i(\xi) = \begin{cases} 1 : \xi \in S_i \subset S \\ 0 : \xi \notin S_i \end{cases} \quad (i = 1,\ldots k_2) \quad . \tag{5b}$$

Then, we can write the control functions $F(t,x)$ and $G(t,\xi)$ respectively as

$$F(t,x) = \sum_{i=1}^{k_1} f_i(t)\phi_i(x) \quad , \tag{6}$$

$$G(t,\xi) = \sum_{i=1}^{k_2} g_i(t)\psi_i(\xi) \quad . \tag{7}$$

Particularly, when the domain D_i and/or boundary S_i is concentrated into some respective points, say d_i and/or s_i, each characteristic function must be defined as

$$\phi_i(x) = \delta_m(x - d_i), \quad (i = 1,\dots,k_1) \quad , \tag{8a}$$

and/or

$$\psi_i(\xi) = \delta_{m-1}(\xi - s_i), \quad (i = 1,\dots k_2) \quad , \tag{8b}$$

where δ_m and δ_{m-1} are respectively m- and (m - 1)-dimensional Dirac's delta functions (see Wiberg [10]).

Moreover, we introduce the vector valued functions as

$$\phi(x) = (\phi_1(x),\dots,\phi_{k_1}(x))' \quad , \tag{9a}$$

$$\psi(\xi) = (\psi_1(\xi),\dots,\psi_{k_2}(\xi))' \quad , \tag{9b}$$

and

$$f(t) = (f_1,\dots,f_{k_1}(t))' \quad , \tag{10a}$$

$$g(t) = (g_1(t),\dots,g_{k_2}(t))' \quad , \tag{10b}$$

where the prime denotes the transpose. Then, two control functions of (6) and (7) can be respectively represented as

$$F(t,x) = \phi'(x)f(t) = f'(t)\phi(x) \quad , \tag{11}$$

$$G(t,\xi) = \psi'(\xi)g(t) = g'(t)\psi(\xi) \quad . \tag{12}$$

Summing up the foregoing assumptions, we consider finally the distributed parameter system governed by

$$\frac{\partial u(t,x)}{\partial t} = Au(t,x) + \phi'(x)f(t) \quad , \tag{13}$$

and

$$\alpha(\xi)u(t,\xi) + \{1-\alpha(\xi)\}\frac{\partial u(t,\xi)}{\partial n} = \alpha(\xi)\psi'(\xi)g(t) \quad . \tag{14}$$

As the performance criterion function, we introduce the quadratic one as

$$J_c = \int_{t_o}^{t_f} \left\{ \int_D \int_D u(t,x)q(t,x,y)u(t,y)dxdy + f'(t)K_1(t)f(t) + g'(t)K_2(t)g(t) \right\} dt + \int_D \int_D u(t_f,x)r(x,y)u(t_f,y)dxdy \quad , \tag{15}$$

where

$q(t,x,y)$, $r(x,y)$: scalar valued, symmetric kernels defined on $D \times D$, which are positive semidefinite, i.e. $\int_D \int_D v(x)q(t,x,y)v(y)dxdy \geqq 0$ for all square-integrable function v; and (16)

$K_1(t)$, $K_2(t)$: $k_1 \times k_1$ and $k_2 \times k_2$ positive definite symmetric matrices, respectively.

Now, we consider the following optimal control problem: given the system equation (13), the boundary condition (14), and the initial condition (4), find the optimal control functions, $f(t) = f^*(t)$ and $g(t) = g^*(t)$, which minimize the performance criterion function (15). We also consider the same optimal control problem for discrete-time control policy, and the relationships between the foregoing two optimal control policies. These problems will be explained in more detail in the following sections.

2. Derivation of the Riccati Equation

In this section, we shall use the technique of dynamic programming to solve the problem stated in the previous section, and as a result, the Riccati equation to determine the optimal control law, which is a nonlinear integro-partial differential equation, is derived. First, let us introduce the minimum error function defined by

$$P(t,u(t,x)) = \min_{\substack{f(\tau),\ g(\tau) \\ t \leqq \tau < t_f}} \left[\int_t^{t_f} \left\{ \int_D \int_D u(\tau,x) q(\tau,x,y) u(\tau,y)\, dx dy \right.\right.$$
$$\left. + f'(\tau) K_1(\tau) f(\tau) + g'(\tau) K_2(\tau) g(\tau) \right\} d\tau$$
$$\left. + \int_D \int_D u(t_f,x) r(x,y) u(t_f,y)\, dx dy \right] . \tag{17}$$

It is easy to show that it holds the relation as

$$P(t_o,u(t_o,x)) = \min_{\substack{f(t),\ g(t) \\ t_o \leqq t < t_f}} J_c , \tag{18}$$

and at time $t = t_f$, we get the terminal condition

$$P(t_f,u(t_f,x)) = \int_D \int_D u(t_f,x) r(x,y) u(t_f,y)\, dx dy . \tag{19}$$

The next step is to apply the dynamic programming to the minimization of the error functional given by (17). Invoking the principle of optimality, it follows that

$$P(t,u(t,x)) = \min_{\substack{f(\tau),\ g(\tau) \\ t \leqq \tau < t+\Delta t}} \left[\int_t^{t+\Delta t} \left\{ \int_D \int_D u(\tau,x) q(\tau,x,y) u(\tau,y)\, dx dy \right.\right.$$
$$\left. + f'(\tau) K_1(\tau) f(\tau) + g'(\tau) K_2(\tau) g(\tau) \right\} d\tau$$
$$\left. + P(t + \Delta t, u(t + \Delta t, x)) \right] . \tag{20}$$

The method of solving the functional equation (20) is similar in principle to the method of solving the equation used for lumped parameter systems; that is, the equation for lumped parameter systems consists of assuming a specific form for P which is then substituted into Eq. (20) in order to verify its correctness. Here, by the analogical inference from lumped parameter systems, P is taken to be the form

$$P(t,u(t,x)) = \int_D \int_D u(t,x)p(t,x,y)u(t,y)dxdy \quad . \tag{21a}$$

Simultaneously, we assume that $p(t,x,y)$ is symmetric with respect to x and y because of the assumptions that $q(t,x,y)$, $r(x,y)$, $K_1(t)$ and $K_2(t)$ are all symmetric; i.e.

$$p(t,x,y) = p(t,y,x) \quad . \tag{21b}$$

To solve the relation (20), we must expand the functional $P(t + \Delta t,u(t + \Delta t,x))$ with respect to Δt. Because of the system equation (13), for sufficiently small Δt, it follows that

$$u(t + \Delta t,x) \approx u(t,x) + \frac{\partial u(t,x)}{\partial t} \Delta t = u(t,x) + \{Au(t,x) + \phi'(x)f(t)\}\Delta t \quad , \tag{22}$$

and at the same time, we get

$$p(t + \Delta t,x,y) \approx p(t,x,y) + \frac{\partial p(t,x,y)}{\partial t} \Delta t \quad . \tag{23}$$

Then, from Eqs. (21), (22) and (23), we can derive an expansion such as

$$P(t + \Delta t,u(t + \Delta t,x)) = \int_D \int_D u(t + \Delta t,x)p(t + \Delta t,x,y)u(t + \Delta t,y)dxdy$$

$$\approx \int_D \int_D u(t,x)p(t,x,y)u(t,y)dxdy$$

$$+ \left[\int_D \int_D \left\{Au(t,x) + \phi'(x)f(t)\right\}p(t,x,y)u(t,y)dxdy\right.$$

$$+ \int_D \int_D u(t,x)p(t,x,y)\left\{Au(t,y) + \phi'(y)f(t)\right\}dxdy$$

$$\left.+ \int_D \int_D u(t,x)\frac{\partial p(t,x,y)}{\partial t} u(t,y)dxdy\right]\Delta t \quad . \qquad (24)$$

After substituting Eqs. (21) and (24) into Eq. (20), and dividing both sides of this equation by Δt, let Δt tend to zero; then we obtain

$$0 = \min_{f(t),g(t)}\left[\int_D \int_D u(t,x)q(t,x,y)u(t,y)dxdy + f'(t)K_1(t)f(t)\right.$$

$$+ g'(t)K_2(t)g(t) + \int_D \int_D \left\{Au(t,x) + \phi'(x)f(t)\right\}p(t,x,y)u(t,y)dxdy$$

$$+ \int_D \int_D u(t,x)p(t,x,y)\left\{Au(t,y) + \phi'(y)f(t)\right\}dxdy$$

$$\left.+ \int_D \int_D u(t,x)\frac{\partial p(t,x,y)}{\partial t} u(t,y)dxdy\right] \quad . \qquad (25)$$

Our next step is to set up the procedure to transform Eq. (25) with the help of Green's formula [3] given by

$$\int_D \left\{Av(x) \cdot w(x) - v(x) \cdot Aw(x)\right\}dx = \int_S \left\{\frac{\partial v(\xi)}{\partial n} w(\xi) - v(\xi)\frac{\partial w(\xi)}{\partial n}\right\}d\xi \qquad (26)$$

Using Eq. (26), we get

$$\int_D \int_D Au(t,x) \cdot p(t,x,y)u(t,y)dxdy = \int_D u(t,y)dy\left[\int_D u(t,x)\right.$$

$$\cdot A_x p(t,x,y)dx + \int_S \left\{\frac{\partial u(t,\xi)}{\partial n} p(t,\xi,y)\right.$$

$$\left.\left. - u(t,\xi)\frac{\partial p(t,\xi,y)}{\partial n}\right\}d\xi\right] \quad , \qquad (27)$$

where the symbol A_x denotes the operator defined by (2) taken with respect to the x variable of $p(t,x,y)$.

Next, we divide the boundary S into two parts, say σ_1 and σ_2, in the following manner;

$$\left.\begin{array}{l} S = \sigma_1 \cup \sigma_2 \\ \alpha(\xi) > 0 \quad \text{on } \sigma_1 \\ \alpha(\xi) = 0 \quad \text{on } \sigma_2 \end{array}\right\} . \tag{28}$$

Then, from the boundary condition (14), we get

$$u(t,\xi) = -\frac{1-\alpha(\xi)}{\alpha(\xi)} \frac{\partial u(t,\xi)}{\partial n} + \psi'(\xi)g(t) \quad \text{on } \sigma_1 , \tag{29}$$

and

$$\frac{\partial u(t,\xi)}{\partial n} = 0 \quad \text{on } \sigma_2 . \tag{30}$$

Substituting these two relations into Eq. (27), we can derive

$$\begin{aligned} &\int_D \int_D Au(t,x) \cdot p(t,x,y)u(t,y)\,dxdy = \int_D \int_D u(t,x)A_x p(t,x,y) \\ &\cdot u(t,y)\,dxdy + \int_D u(t,y)\,dy \int_{\sigma_1} \frac{1}{\alpha(\xi)} \frac{\partial u(t,\xi)}{\partial n} \Big\{ \alpha(\xi)p(t,\xi,y) \\ &+ (1-\alpha(\xi))\frac{\partial p(t,\xi,y)}{\partial n} \Big\} d\xi - \int_D u(t,y)\,dy \int_{\sigma_1} \psi'(\xi)\frac{\partial p(t,\xi,y)}{\partial n}\, d\xi g(t) \\ &- \int_D u(t,y)\,dy \int_{\sigma_2} u(t,\xi)\frac{\partial p(t,\xi,y)}{\partial n}\, d\xi . \end{aligned} \tag{31}$$

In the same way, we also get the relation

$$\int_D \int_D u(t,x)p(t,x,y) \cdot Au(t,y)dxdy = \int_D \int_D u(t,x)A_y p(t,x,y)$$

$$\cdot u(t,y)dxdy + \int_D u(t,x)dx \int_{\sigma_1} \frac{1}{\alpha(\xi)} \frac{\partial u(t,\xi)}{\partial n} \left\{\alpha(\xi)p(t,x,\xi)\right.$$

$$\left. + (1-\alpha(\xi))\frac{\partial p(t,x,\xi)}{\partial n}\right\} d\xi - \int_D u(t,x)dx \int_{\sigma_1} \psi'(\xi)\frac{\partial p(t,x,\xi)}{\partial n} d\xi g(t)$$

$$- \int_D u(t,x)dx \int_{\sigma_2} u(t,\xi)\frac{\partial p(t,x,\xi)}{\partial n} d\xi \quad . \tag{32}$$

Substituting Eqs. (31) and (32) into Eq. (25), it follows that

$$0 = \min_{f(t),\ g(t)} \left[\!\left[f'(t)K_1(t)f(t) + g'(t)K_2(t)g(t) + 2\int_D \int_D \phi'(x)\right.\right.$$

$$\times p(t,x,y)u(t,y)dxdyf(t) - 2\int_D \int_{\sigma_1} \psi'(\xi)\frac{\partial p(t,\xi,y)}{\partial n} u(t,y)d\xi dyg(t)$$

$$+ \int_D \int_D u(t,x)\left\{q(t,x,y) + \frac{\partial p(t,x,y)}{\partial t} + (A_x + A_y)p(t,x,y)\right\}u(t,y)dxdy$$

$$+ \int_D u(t,y)dy \times \left[\int_{\sigma_1} \frac{1}{\alpha(\xi)} \frac{\partial u(t,\xi)}{\partial n} \left\{\alpha(\xi)p(t,\xi,y) + (1-\alpha(\xi))\right.\right.$$

$$\left.\left. \frac{\partial p(t,\xi,y)}{\partial n}\right\} d\xi - \int_{\sigma_2} u(t,\xi)\frac{\partial p(t,\xi,y)}{\partial n} d\xi\right] + \int_D u(t,x)dx$$

$$\times \int_{\sigma_1} \frac{1}{\alpha(\xi)} \frac{\partial u(t,\xi)}{\partial n} \left\{\alpha(\xi)p(t,x,\xi) + (1-\alpha(\xi))\frac{\partial p(t,x,\xi)}{\partial n}\right\} d\xi$$

$$\left.\left. - \int_{\sigma_2} u(t,\xi)\frac{\partial p(t,x,\xi)}{\partial n} d\xi \right]\!\right] \quad . \tag{33}$$

The optimal controls $f^*(t)$ and $g^*(t)$, which minimize the right hand side of Eq. (33), are found by setting the functional derivative of Eq. (33) with respect to $f(t)$ and $g(t)$ to zero, respectively. The resulting expressions can be given by

$$f^*(t) = -K_1^{-1}(t) \int_D \left\{ \int_D \phi(x) p(t,x,y) dx \right\} u(t,y) dy \quad , \tag{34}$$

and

$$g^*(t) = K_2^{-1}(t) \int_D \left\{ \int_{\sigma_1} \psi(\xi) \frac{\partial p(t,\xi,y)}{\partial n} d\xi \right\} u(t,y) dy \quad . \tag{35}$$

Substituting these optimal control functions into Eq. (33), we obtain the equation that p(t,x,y) must satisfy; i.e.

$$\begin{aligned}
0 = \int_D \int_D u(t,x) \Bigg[& q(t,x,y) + \frac{\partial p(t,x,y)}{\partial t} + (A_x + A_y) p(t,x,y) \\
& - \int_D \phi'(z) p(t,z,x) dz K_1^{-1}(t) \int_D p(t,z,y)_\phi(z) dz \\
& - \int_{\sigma_1} \psi'(\xi) \frac{\partial p(t,\xi,x)}{\partial n} d\xi K_2^{-1}(t) \int_{\sigma_1} \frac{\partial p(t,\xi,y)}{\partial n} \psi(\xi) d\xi \Bigg] u(t,y) dx dy \\
& + \int_D u(t,y) dy \Bigg[\int_{\sigma_1} \frac{1}{\alpha(\xi)} \frac{\partial u(t,\xi)}{\partial n} \Big\{ \alpha(\xi) p(t,\xi,y) \\
& + (1 - \alpha(\xi)) \frac{\partial p(t,\xi,y)}{\partial n} \Big\} d\xi - \int_{\sigma_2} u(t,\xi) \frac{\partial p(t,\xi,y)}{\partial n} d\xi \Bigg] \\
& + \int_D u(t,x) dx \Bigg[\int_{\sigma_1} \frac{1}{\alpha(\xi)} \frac{\partial u(t,\xi)}{\partial n} \Big\{ \alpha(\xi) p(t,x,\xi) \\
& + (1 - \alpha(\xi)) \frac{\partial p(t,x,\xi)}{\partial n} \Big\} d\xi - \int_{\sigma_2} u(t,\xi) \frac{\partial p(t,x,\xi)}{\partial n} d\xi \Bigg] \quad .
\end{aligned} \tag{36}$$

Eq. (36) must be satisfied for any state u. Therefore, the coefficients of integrands multiplied by the same function must themselves be zero. However, as it is possible to change u in the interior of D without changing it on the boundary, it follows that terms with different regions of integration are independent of each other and therefore must be equated to zero separately. From Eq. (36), these observations yield the relations as

$$- \frac{\partial p(t,x,y)}{\partial t} = q(t,x,y) + (A_x + A_y)p(t,x,y)$$

$$- \int_D \phi'(z)p(t,z,x)dzK_1^{-1}(t) \int_D p(t,z,y)\phi(z)dz$$

$$- \int_S \psi'(\xi)\frac{\partial p(t,\xi,x)}{\partial n} d\xi K_2^{-1}(t) \int_S \frac{\partial p(t,\xi,y)}{\partial n}\psi(\xi)d\xi \quad , \tag{37}$$

$$\alpha(\xi)p(t,x,\xi) + \left\{1 - \alpha(\xi)\right\}\frac{\partial p(t,x,\xi)}{\partial n} = 0 \quad \text{on } \sigma_1 \quad , \tag{38a}$$

$$\alpha(\xi)p(t,\xi,y) + \left\{1 - \alpha(\xi)\right\}\frac{\partial p(t,\xi,y)}{\partial n} = 0 \quad \text{on } \sigma_1 \quad , \tag{38b}$$

$$\frac{\partial p(t,x,\xi)}{\partial n} = \frac{\partial p(t,\xi,y)}{\partial n} = 0 \quad \text{on } \sigma_2 \quad . \tag{38c}$$

Recalling that $\alpha(\xi) = 0$ on σ_2, we can express the boundary conditions of Eqs.(38) more concisely as

$$\alpha(\xi)p(t,x,\xi) + \left\{1 - \alpha(\xi)\right\}\frac{\partial p(t,x,\xi)}{\partial n} = 0 \quad \text{on } S \quad , \tag{39a}$$

$$\alpha(\xi)p(t,\xi,y) + \left\{1 - \alpha(\xi)\right\}\frac{\partial p(t,\xi,y)}{\partial n} = 0 \quad \text{on } S \quad . \tag{39b}$$

From Eq. (19), the terminal condition for $p(t,x,y)$ can be given by

$$p(t_f,x,y) = r(x,y) \quad . \tag{40}$$

Eq. (37) with boundary conditions (39) is a nonlinear integro-partial differential equation analogous to the Riccati equation, which has never been studied before in the vast literature on partial differential euqations (see Erzberger and Kim [1,2]). If we can solve the equation (37) with the terminal condition given by (40), the optimal control policies, $f^*(t)$ and $g^*(t)$, are given by (34) and (35) respectively, and at the same time, the optimal error functional $P(t,u(t,x))$ can be calculated from

Eq. (21). The resulting controls are obviously of the state feedback type.

3. Discrete-Time Control Policy

In this section, we consider the same optimal control problem stated in Section 1 but impose the following restriction to the control functions: both the distributed and boundary control functions, $F(t,x)$ and $G(t,\xi)$, are stepwise functions with respect to time. In other words, for $t_{j-1} \leqq t < t_j$, let the respective control functions $f(t)$ and $g(t)$ in Eq. (10) be

$$f(t) = f_{j-1} = (f_{1,j-1},\ldots,f_{k_1,j-1})' \quad , \tag{41a}$$

$$g(t) = g_{j-1} = (g_{1,j-1},\ldots,g_{k_2,j-1})' \quad , \tag{41b}$$

where $j = 1,\ldots,N$ and N is the total number of sampling stages. For the convenience of consideration, we shall choose all sampling intervals to be equal to each other; namely

$$t_j = t_o + jT, \quad (j = 1,\ldots,N), \quad T = (t_f - t_o)/N, \quad t_N = t_f \quad . \tag{42}$$

We shall consider the performance criterion function of the following discrete form, which corresponds to the one given by Eq. (15);

$$J_d = \sum_{j=1}^{N} \left\{ \int_D \int_D u_j(x) q_j(x,y) u_j(y) dxdy + f'_{j-1} K_{1,j-1} f_{j-1} + g'_{j-1} K_{2,j-1} g_{j-1} \right\} T + \int_D \int_D u(t_f,x) r(x,y) u(t_f,y) dxdy \quad , \tag{43}$$

where

$$u_j(x) = u(t_j,x) \quad , \tag{44}$$

$q_j(x,y)$, $r(x,y)$: scalar valued, positive semidefinite kernels, which are symmetric on $D\times D$,

$K_{1,j-1}, K_{2,j-1}$: $k_1 \times k_1$ and $k_2 \times k_2$ positive definite symmetric matrices, respectively.

From now on, let us determine the sequences of optimal control policy $\{f^*_{j-1}\} = \{f^*_o, \ldots, f^*_{N-1}\}$ and $\{g^*_{j-1}\} = \{g^*_o, \ldots, g^*_{N-1}\}$ which minimizes the performance index J_d given by (43) under the conditions of (13), (14) and (4). The above-mentioned optimal control problem has already been investigated by the author (see Ito [4] and Matsumoto and Ito [6]). Therefore, let us here briefly explain only the derived results.

Let the function $U(t,x,y)$ be the Green's function associated with the homogeneous system of Eqs. (1) and (3). Then, the response of the inhomogeneous system can be written as (see Friedmann [3])

$$u(t,x) = \int_D U(t-t_o, x,y)u_o(y)dy + \int_{t_o}^{t} d\tau \int_D U(t-\tau,x,y)F(\tau,y)dy + \int_{t_o}^{t} d\tau \int_S U_B(t-\tau,x,\xi)\alpha(\xi)G(\tau,\xi)d\xi \quad , \tag{45}$$

where

$$U_B(t,x,\xi) = U(t,x,\xi) - \frac{\partial U(t,x,\xi)}{\partial n} \quad . \tag{46}$$

Particularly when we consider the control functions given by (41), we can get the following relation from Eq. (45); that is

$$u_j(x) = \mathcal{L}u_{j-1}(x) + h_1'(x)f_{j-1} + h_2'(x)g_{j-1}, \quad (j = 1,\ldots,N) \quad , \tag{47}$$

where $\mathcal{L}$ is an integral operator defined as

$$\mathcal{L}u(x) = \int_D U(T,x,y)u(y)dy \quad , \tag{48}$$

and both $h_1(x)$ and $h_2(x)$ are vector valued functions of the form

$$h_1(x) = \begin{pmatrix} \int_0^T d\tau \int_{D_1} U(T-\tau,x,y)dy \\ \cdot \\ \cdot \\ \cdot \\ \int_0^T d\tau \int_{D_{k_1}} U(T-\tau,x,y)dy \end{pmatrix}, \qquad (49a)$$

and

$$h_2(x) = \begin{pmatrix} \int_0^T d\tau \int_{S_1} U_B(T-\tau,x,\xi)\alpha(\xi)d\xi \\ \cdot \\ \cdot \\ \cdot \\ \int_0^T d\tau \int_{S_{k_2}} U_B(T-\tau,x,\xi)\alpha(\xi)d\xi \end{pmatrix}. \qquad (49b)$$

From this place, for the convenience of mathematical description, we adopt vectors and a matrix as follows;

$$\bar{f}_{j-1} = \begin{pmatrix} f_{j-1} \\ g_{j-1} \end{pmatrix}, \qquad (50a)$$

$$h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix}, \qquad (50b)$$

$$K_{j-1} = \begin{pmatrix} K_{1,j-1} & 0 \\ 0 & K_{2,j-1} \end{pmatrix}. \qquad (50c)$$

Then, Eqs. (43) and (47) can be written respectively

$$J_d = \sum_{j=1}^{N} \left\{ \int_D \int_D u_j(x) q_j(x,y) u_j(y) dxdy + \bar{f}'_{j-1} K_{j-1} \bar{f}_{j-1} \right\} T$$

$$+ \int_D \int_D u(t_f,x) r(x,y) u(t_f,y) dxdy \quad , \qquad (51)$$

$$u_j(x) = \mathcal{L} u_{j-1}(x) + h'(x) \bar{f}_{j-1} \quad . \qquad (52)$$

We shall now solve the optimal control problem stated above by using dynamic programming technique. Let the error functional $P_j(u_j(x))$ be defined as

$$P_j(u_j(x)) = \min_{\substack{\bar{f}_i \\ i=j,\ldots,N-1}} \left[\sum_{i=j+1}^{N} \left\{ \int_D \int_D u_i(x) q_i(x,y) u_i(y) dxdy \right.\right.$$

$$\left. + \bar{f}'_{i-1} K_{i-1} \bar{f}_{i-1} \right\} T$$

$$\left. + \int_D \int_D u(t_f,x) r(x,y) u(t_f,y) dxdy \right] \quad . \qquad (53)$$

Now, making an assumption that the error functional of (53) be of the form

$$P_j(u_j(x)) = \int_D \int_D u_j(x) p_j(x,y) u_j(y) dxdy \quad , \qquad (54)$$

and let the sequence $\left\{ \bar{f}^*_{j-1} \right\} = \left\{ \bar{f}^*_o, \ldots, \bar{f}^*_{N-1} \right\}$ be the optimal control policy. Then the resulting form of the optimal control law can be written as

$$\bar{f}^*_{j-1} = -\int_D s_{j-1}(x) u_{j-1}(x) dx, \quad (j = 1,\ldots,N) \quad , \qquad (55)$$

where

$$s_{j-1}(x) = \left[TK_{j-1} + \int_D \int_D h(x)\left\{Tq_j(x,y) + p_j(x,y)\right\} h'(y)\,dxdy\right]^{-1} \times \int_D \mathcal{L}_x \left\{Tq_j(x,y) + p_j(x,y)\right\} h(y)\,dy, \quad (j = 1,\ldots,N) \quad . \tag{56}$$

Furthermore the function $p_j(x,y)$ must satisfy the recurrence relationship as

$$p_{j-1}(x,y) = \mathcal{L}_x \cdot \mathcal{L}_y \left\{Tq_j(x,y) + p_j(x,y)\right\} - s'_{j-1}(x)\left[TK_{j-1} + \int_D \int_D h(x)\left\{Tq_j(x,y) + p_j(x,y)\right\} h'(y)\,dxdy\right] s_{j-1}(y),$$

$$(j = 1,\ldots,N) \quad . \tag{57}$$

As a result, starting with the terminal condition

$$p_N(x,y) = r(x,y) \quad , \tag{58}$$

we solve the recurrence functional relationships of (56) and (57) with respect to $s_{j-1}(x)$ and $p_{j-1}(x,y)$. Then, the sequence of the optimal control policy $\{\bar{f}^*_{j-1}\}$ and the error functional $P_j(u_j(x))$ can be determined by (55) and (54) respectively. The control policy is also given as a feedback control.

4. Relationships Between Optimal Continuous-Time and Discrete-Time Control Policies

In this section, let us investigate the relationships between the results derived in the preceding sections. We shall show that we can derive the Riccati equation of (37) from the recurrence relationships of (56) and (57) when the sampling interval T tend to zero. To begin with, as a preparation of the following investigation, let us enumerate some properties which the Green's function U(t,x,y) satisfies (see Friedmann [3]), i.e.

$$1)\quad \frac{\partial U(t,x,y)}{\partial t} = A_x U(t,x,y) \quad , \tag{59a}$$

$$\frac{\partial U(t,x,y)}{\partial t} = A_y U(t,x,y) \quad , \tag{59b}$$

$$2)\quad \alpha(\xi)U(t,\xi,y) + \left\{1 - \alpha(\xi)\right\}\frac{\partial U(t,\xi,y)}{\partial n} = 0 \quad , \tag{60a}$$

$$\alpha(\xi)U(t,x,\xi) + \left\{1 - \alpha(\xi)\right\}\frac{\partial U(t,x,\xi)}{\partial n} = 0 \quad , \tag{60b}$$

$$3)\quad \lim_{t\downarrow 0}\int_D U(t,x,z)p(z,y)dz = p(x,y) \quad . \tag{61}$$

The next step is to substitute Eq. (56) into Eq. (57) and it follows that

$$\begin{aligned} p_{j-1}(x,y) &= \mathcal{L}_x \cdot \mathcal{L}_y\left\{Tq_j(x,y) + p_j(x,y)\right\} \\ &\quad - \int_D \mathcal{L}_x\left\{Tq_j(x,z) + p_j(x,z)\right\}h'(z)dz\Big[TK_{j-1} \\ &\quad + \int_D\int_D h(x)\left\{Tq_j(x,y) + p_j(x,y)\right\}h'(y)dxdy\Big]^{-1} \\ &\quad \times \int_D \mathcal{L}_y\left\{Tq_j(y,z) + p_j(y,z)\right\}h(z)dz \quad . \end{aligned} \tag{62}$$

Then, let us consider the case of (62) in the limit as the sampling interval T tends to zero. First, using the relations of (59) to (61) and Green's formula given by (26), we get

$$\begin{aligned} \mathcal{L}_x p_j(x,y) &= \int_D U(T,x,z)p_j(z,y)dz \approx \int_D \Big\{U(0,x,z) \\ &\quad + \frac{\partial U(t,x,z)}{\partial t}\Big|_{t=0} T\Big\}p_j(z,y)dz = p_j(x,y) \\ &\quad + T\int_D A_z U(0,x,z) \cdot p_j(z,y)dz = p_j(x,y) \\ &\quad + \Big[\int_D U(0,x,z) \cdot A_z p_j(z,y)dz \end{aligned}$$

$$+ \int_S \left\{ \frac{\partial U(0,x,\xi)}{\partial n} p_j(\xi,y) - U(0,x,\xi) \frac{\partial p_j(\xi,y)}{\partial n} \right\} d\xi \Bigg] T = p_j(x,y)$$

$$+ A_x p_j(x,y) T + \left[\int_{\sigma_1} \frac{1}{\alpha(\xi)} \frac{\partial U(0,x,\xi)}{\partial n} \left\{ \alpha(\xi) p_j(\xi,y) + (1 - \alpha(\xi)) \right. \right.$$

$$\left. \left. \times \frac{\partial p_j(\xi,y)}{\partial n} \right\} d\xi - \int_{\sigma_2} U(0,x,\xi) \frac{\partial p_j(\xi,y)}{\partial n} d\xi \right] T \quad . \tag{63}$$

Equation (63) must be satisfied for the arbitrary Green's function $U(0,x,z)$. As mentioned in Section 2, since it is possible to change $U(0,x,z)$ with respect to z in the interior of D without changing them on the boundary, it follows that terms with different regions of integration are independent of each other and therefore must be equated to zero separately. At the same time, considering that Eq. (63) must be satisfied for the arbitrary values of both x and y, we get

$$\mathcal{L}_x p_j(x,y) \approx p_j(x,y) + A_x p_j(x,y) T \quad , \tag{64}$$

and

$$\alpha(\xi) p_j(\xi,y) + \left\{ 1 - \alpha(\xi) \right\} \frac{\partial p_j(\xi,y)}{\partial n} = 0 \text{ on } \sigma_1 \quad , \tag{65a}$$

$$\frac{\partial p_j(\xi,y)}{\partial n} = 0 \text{ on } \sigma_2 \quad . \tag{65b}$$

Now let us tend the sampling interval T to zero and write $p_j(x,y)$ as $p(t,x,y)$; then the boundary conditions of (65) can be written as

$$\alpha(\xi) p(t,\xi,y) + \left\{ 1 - \alpha(\xi) \right\} \frac{\partial p(t,\xi,y)}{\partial n} = 0 \text{ on } S \quad . \tag{66}$$

In the same way, it follows that

$$\mathcal{L}_y p_j(x,y) \approx p_j(x,y) + A_y p_j(x,y) T \quad , \tag{67}$$

and we get another boundary condition for $p(t,x,y)$ as

$$\alpha(\xi) p(t,x,\xi) + \left\{ 1 - \alpha(\xi) \right\} \frac{\partial p(t,x,\xi)}{\partial n} = 0 \text{ on } S \quad . \tag{68}$$

Let us continue the same procedures, and we get the expressions

$$h_1(x) \approx \phi(x)T \quad , \tag{69a}$$

$$h_2(x) \approx T\int_S \left\{U(0,x,\xi) - \frac{\partial U(0,x,\xi)}{\partial n}\right\}\alpha(\xi)\psi(\xi)d\xi \quad , \tag{69b}$$

$$\mathcal{L}_x \cdot \mathcal{L}_y \left\{Tq_j(x,y) + p_j(x,y)\right\} \approx Tq_j(x,y)$$

$$+ p_j(x,y) + A_x p_j(x,y)T + A_y p_j(x,y)T \quad , \tag{70}$$

$$\left[TK_{j-1} + \int_D\int_D h(x)\left\{Tq_j(x,y) + p_j(x,y)\right\}h'(y)dxdy\right] \approx TK_{j-1} \quad , \tag{71}$$

$$\int_D \mathcal{L}_x\left\{Tq_j(x,z) + p_j(x,z)\right\}h_1(z)dz \approx T\int_D p_j(x,z)\phi(z)dz \quad , \tag{72a}$$

$$\int_D \mathcal{L}_x\left\{Tq_j(x,z) + p_j(x,z)\right\}h_2(z)dz \approx -T\int_S \frac{\partial p_j(x,\xi)}{\partial n}\psi(\xi)d\xi \quad . \tag{72b}$$

After substituting Eqs. (64), (67) and (69) to (72) into Eq. (62), we divide the both sides of this relation by T, and tending T to zero, then we get the relation that p(t,x,y) must satisfy, which is equal to the Riccati equation of (37); i.e.

$$-\frac{\partial p(t,x,y)}{\partial t} = q(t,x,y) + (A_x + A_y)p(t,x,y)$$

$$-\int_D \phi'(z)p(t,z,x)dzK_1^{-1}(t)\int_D p(t,z,y)\phi(z)dz$$

$$-\int_S \psi'(\xi)\frac{\partial p(t,\xi,x)}{\partial n}d\xi K_2^{-1}(t)\int_S \frac{\partial p(t,\xi,y)}{\partial n}\psi(\xi)d\xi \quad . \tag{73}$$

Applying the same procedure to $s_{j-1}(x)$ of Eq. (56), we get the optimal control policy of continuous-time form; that is

$$f^*(t) = -K_1^{-1}(t) \int_D \left\{ \int_D \phi(x)p(t,x,y)dx \right\} u(t,y)dy \quad , \qquad (74)$$

$$g^*(t) = K_2^{-1}(t) \int_D \left\{ \int_{\sigma_1} \psi(\xi) \frac{\partial p(t,\xi,y)}{\partial n} d\xi \right\} u(t,y)dy \quad . \qquad (75)$$

Obviously, the terminal condition of $p(t,x,y)$ can be given from Eq. (58) as

$$p(t_f,x,y) = r(x,y) \quad . \qquad (76)$$

All these results are just the same as those of Section 2.

5. Concluding Remarks

Two optimal control problems have been discussed for a linear distributed parameter system governed by a partial differential equation of the parabolic type. We imposed a restriction on both distributed and boundary control functions such that these controls are concentrated spatially onto some parts of a spatial domain from the standpoint of control device construction.

In Section 2, the optimal control problem with continuous-time control was considered. The performance criterion function of quadratic form was evaluated by using dynamic programming technique, and the Riccati equation was derived. In Section 3, the same optimal control problem with discrete-time control functions was investigated by using Green's function representation, and recurrence formulae for determining the optimal control policy were obtained. Finally, the relationships between the optimal continuous- and discrete-time control policies were discussed in Section 4.

We can develop the foregoing discussions up to the problem where the closed spatial domains $D_1,\ldots,D_{k_1}$ and boundaries $S_1,\ldots,S_{k_2}$ move with respect to time within the spatial domain. In this case, the characteristic functions $\phi_i(x)$ and $\psi_i(\xi)$ and the vector valued function $h(x)$ become the functions of time t. It is notable that when we disregard the assumption that control functions are concentrated with respect to space variables as shown by Eqs. (6) and (7), then it becomes impossible to derive recurrence formulae (56) and (57).

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