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EXTRAPOLATING TRENDING GEOLOGICAL BODIES

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ABSTRACT

An attempt is made to structure the heuristic process of extrapolating trending geologic bodies in the analytic framework of Bayesian inference. The approach models spatial properties of trending bodies rather than geological processes, and includes components of uncertainty arising out of trend model selection. Inclusion of several components of uncertainty leads to rapid dispersion of the probability density of predicted location away from the region of observations, in conformity with the intuitive notion of valid distances of prediction. The philosophical foundations of exploration and the role of probabilistic predictions in decision-making are briefly discussed.

I. INTRODUCTION

The process of geological exploration often encounters formations or bodies which might be described as "linearly trending." Here, the word trending is not used in the sense of so-called trend surface analysis, but rather as a description of bodies whose planar shape can be approximated by lines or low-order curves. Examples are shoestring sands, buried reefs, some mineralizations, and high-permeability

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channels of subsurface flow (the last being of importance in civil construction)(Figure 1). The problem addressed in this paper is how the location of trending bodies in regions yet to be explored might be predicted on the basis of known locations in adjacent regions. In particular, an attempt is made to structure a rule-of-thumb approach on a rigorous foundation in the philosophy of exploration.

During the past twenty years, contributions have been made to the literature of decision analysis, search theory, and operations research generally which allow us to allocate exploration effort in ways which maximize the information we can expect to obtain. However, these methods require quantitative predictions: they require that predictions be encoded in probabilistic terms so that questions of the sort, "How much more probable is it that an ore body lies at point A than B?" can be answered. Certainly, evaluation of such probabilities is the foundation of exploration and the only reason geologists are the ones who carry it out. The purpose of the present paper is to attempt a quantification of predictions associated with one type of formation.

A Rule-of-Thumb Approach

It's dangerous to characterize rule-of-thumb approaches too narrowly; geologists tend to be independent sorts and there are many procedures for handling any specific problem. Nonetheless, a typical one for linear extrapolation is to assume that for short distances a body centerline may be approximated by a line or low-order curve, and to extend this line into unobserved regions as the likely continuation of the body (Figure 2). Clearly, the faith one puts in this extrapolation diminishes the further it's extended. In a decision sense this line represents the locus of most probable locations of the body as one moves away from the observations, and is the line along which further exploration would initially take place.

The line or curve fitted to observed locations depends on the geologist's experience and his understanding of fundamental geological processes. While this heuristic approach is not based directly on geological theory (i.e., it is not a random process model), knowledge of the theory leads one to intuitively suspect certain forms of spatial behavior over others, and thus the approach does represent informed geological opinion.

The relationship between informed geological opinion, uncertainty, observations, and spatial modelling is largely neglected in the literature. So, before proceeding to quantification, the philosophical basis of exploration upon which the present work is predicated should be discussed.

II. EXPLORATION PHILOSOPHY

Conclusions drawn from the results of exploration contain much more than the physical records themselves. Patterns recognized, maps drawn, similarities inferred, these all transcend the observations actually made. Hypotheses are the product of exploration. Exploration uncertainties manifest themselves in the degree to which hypotheses either are or are not confirmed by exploration data. Therefore, geological mapping is not merely a faithful reporting of instrumental observations, but is an interpretive, inductive task reflecting currently held concepts of geological structure (see, e.g., Harrison, 1963).

Hypotheses arise and are given initial credibility through a process which is entirely subjective. They are generated by a process of discovery (much discussed in the philosophy-of-science literature), and assigned *a priori* degrees-of-confirmation based primarily on extra-evidential factors[1]. In entirety this process is simply inductive reasoning. Although *a priori* degrees-of-confirmation are subsequently modified as new data become available, their foundation is always and purely subjective. Thus, as

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[1]Some of these are the simplicity or aesthetic appeal of a hypothesis, its conformity to larger sets of hypotheses, and lack of better hypotheses. A review of inductive philosophy is given by Salmon (1966).

the uncertainties of exploration are predicated on the subjective process of inductive reasoning, they too are fundamentally subjective. One has experience and knowledge of geology which causes one to suspect conditions not directly manifested in exploration data, and the uncertainties one associates with these hypotheses cannot be objectively derived from the records of exploration.

As data from exploration accrue, initial degrees-of-confirmation are modified by the extent to which the predictions following from each hypothesis are consistent with observation. A method for doing this analytically is Bayes' Theorem. Let there be some set of alternative hypotheses, $H_1,\ldots,H_n$, with respect to subsurface conditions at a site; and assume that the a priori degree-of-confirmation assigned to each is $p^o(H_i)$ (i.e., the probability of hypothesis H_i being correct). Given a set of observations $\underline{z}$, by Bayes' Theorem the a posteriori degree-of-confirmation of each hypothesis is

$$p'(H_i|\underline{z}) = \frac{p^o(H_i)\ L(\underline{z}|H_i)}{\sum_{i=1}^{n} p^o(H_i)\ L(\underline{z}|H_i)} \quad , \qquad (1)$$

in which $L(\underline{z}|H_i)$ is the likelihood of the observations, $\underline{z}$, conditioned on H_i (i.e., the probability of observing $\underline{z}$ were hypothesis H_i correct), and the denominator is simply a normalizing constant. Clearly, as the number of times this process is iterated increases, the importance of $p^{o}(H_i)$ in establishing $p'(H_i|\underline{z})$ decreases. The degree-of-confirmation, given a hypothesis, comes to depend more and more on observations alone.

Subjective Probability

Structuring inductive tasks in terms of Bayes' Theorem indicates that we are approaching exploration problems from a degree-of-belief perspective on probability; our description of interpreting exploration data indicates that we are approaching them subjectively. The task here is not to repeat arguments for and against belief and frequency--these are voluminously argued in other places (e.g., Savage, et al., 1962)--but there are operational arguments as well as philosophical ones for adopting a subjectivist approach, and these may provide justification to those more skeptical of "Bayesian" analysis.

First, subjective approaches include the prior feelings and intuition of the exploration geologist directly in the analytical model. These feelings are important sources of information which other approaches do not consider analytically.

Second, subjective approaches allow the inclusion of components of uncertainty (e.g., model selection) which otherwise must be dealt with judgementally. They provide rigorous procedures for aggregating uncertainties from several sources in evaluating total uncertainty.

Third, predictions which result from subjectivist models are expressed in terms of probabilities of hypotheses or events and can be directly incorporated in decision-making. This allows use of sophisticated methodologies developed in decision analysis, search theory, and other branches of operations research.

Fourth, geological structure is a highly complex phenomenon of which we have random process models for only the simplest cases[2]. A subjectivist approach allows us to employ heuristic models and assign levels of credibility to them within the analytical framework. Also, empirical evidence in other fields (e.g., see Murphy and Winkler,

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[2]What we have here called random process models are often called structural models. That is, they are models based on first principles of the physical system. We use the first name to avoid confusion with "structural" geology, however.

1974) indicates that subjective forecasts may even be more accurate than the best random process models in treating certain types of predictions.

Lastly, in subjective theory probability is defined with respect to the individual. Recent work (Morris, 1974) allows us to coalesce the feelings of more than one geologist into *a priori* probabilities, and thus both allocate initial effort and make predictions on a broad expert base which has been rigorously aggregated.

Accepting the subjective approach for quantitative analyses of exploration requires placing numbers on *a priori* feelings: quantifying *a priori* subjective probabilities. This quantification does not imply objectivity; it is merely a process of scaling subjective feelings on a rigorously based metric so that feelings may be analytically combined with other parts of exploration.

The theory of subjective probability and techniques for assessment are topics which cannot be adequately presented here. The literature of statistical decision analysis and behavioral decision theory, however, contains extensive work on these topics, and Grayson (1960) has presented a well-known discussion of subjective probability within the context of oil and gas drilling.

Models and Model Selection

The selection of models with which to analyze geological data and make predictions is, like exploration itself, a subjective task. The geologist reviews his experience with geologically similar formations and assigns (explicitly or implicitly) degrees of appropriateness to each of several models he might employ. He applies the models deemed most appropriate to the existing data, and then reassesses the weight attached to each by how well it "fits" the data. The process is the same as for evaluating alternative hypotheses. In making subsequent predictions one evaluates uncertainty by compounding uncertainties in the validity of the model with uncertainties in its predictions. In other words, the probability of an event, $\mathcal{E}$, becomes

$$Pr[\mathcal{E}] = \sum_i Pr[\mathcal{E}|M_i] \; Pr[M_i] \quad , \qquad (2)$$

in which $Pr[\mathcal{E}|M_i]$ is the probability of the event as predicted by the i^{th} model and $Pr[M_i]$ is the probability of the i^{th} model being correct (assuming the $Pr[M_i]$ independent).

Models applied to predicting spatial properties may be based either on an understanding of fundamental geological processes (e.g., the physics of sedimentation) or on heuristic rules inferred from experience. When quantified as stochastic relationships the former are referred to as random process models, while the latter will be referred to here simply as heuristic models. Random process models stem from theories of geological processes which lead deductively to spatial properties; heuristic models stem from no identifiable geological theory and are justified only in that they adequately fit (and predict) observations. This should not be taken to mean that random process models are universally preferred, because operationally heuristic models may be more useful.

Random process models require that geological processes be well understood, and that the set of controlling variables be both identifiable and small. In practice, these conditions are not often met, and geologists themselves are generally unable to formulate conceptual models in terms of first principles (Krumbein, 1970). Practical limitations of random process models are that the mathematics of the models rapidly become intractable, and controlling variables are often unmeasurable. In matters of scientific inquiry, models based on first principles are clearly preferable to heuristic ones, but in exploration this is not necessarily the case: models which work (i.e., which yield valid predictions for whatever reason), and are simple enough to apply, are favored.

The degree-of-belief one has in the validity of particular models, just as the degree-of-confirmation he assigns to hypotheses, is a complex function of evidential and extra-evidential factors. On the one hand, the better the performance of a given model with past data, the more faith one places in it; while on the other hand, the more compatible a model is with larger sets of geological theories, the more faith one places in it. These tendencies sometimes pull in opposing directions. The stability of one's belief in a model clearly relates to its foundation in theory. Heuristic models are quickly discarded when they do not fit data in new situations; for random process models this is not the case.

The tendency in fitting heuristic models, particularly for trend extrapolation, is to make them as simple as possible; this means as low-order as possible. Linear or quadratic trends are usually preferred to 10- or 12-degree trends. Simplicity is not merely a prejudice of geologists, but reflects experience (i.e., it is evidential). High-order curves and surfaces have sufficient flexibility that the probability distributions of their predictions decay more rapidly than experience suggests they should: we appear to be able to make more confident and further-extended predictions than high-order trends imply. Thus one generally avoids high-order trends as having little *a priori* validity or usefulness in practical problems.

Summary

We have tried to present a short discussion of the logic of inference in exploration. In particular, we have tried to emphasize the following points:

1. Exploration is an inductive rather than deductive undertaking whose results transcend the physical record of explorations.
2. Uncertainties in the conclusions drawn from exploration are of subjective origin, and should be treated by subjectivist probability theory.
3. There is a fundamental difference between models which predict spatial properties based on heuristic reasoning and those which do so by modelling geological processes.

III. QUANTITATIVE ANALYSIS

The present approach to predicting the location of trending bodies is an analytical formulation of the heuristic centerline extrapolation technique; it is not a random process model of the spatial properties of geological bodies based on genetic concepts of sedimentation, implacement, etc. Therefore, it is not so much a model of geology as it is a model of spatial relationship based on empirical experience with other similar formations. However, the model does provide an accounting of uncertainties from various sources and thus provides insight into the dispersion of certainty with which predictions can be made away from observations.

We assume that on the basis of previous drilling and exploration some region within which the trending body lies has been explored, and that from this exploration two types of information are available. First, we know that the body exists at several discrete points in the horizontal plane (Figure 3); second, we have information on which some subjective feeling for the orientation of the body, exclusive of boring locations, can be based. (For example, we may have relevant geomorphological information, cross-bedding orientations in core samples, grain-size changes at progressive locations, etc.)

Based on information of the latter type, a priori feelings about the trend and width of a body may be evaluated using techniques of subjective probability theory. Then, using known locations of the body as data, the probabilities both of the centerline trend and of the width are updated to give a posteriori probabilities from which predictions can be made.

Given that probability distributions on centerline trend and width have been updated, probabilities that the body exists at unobserved locations can be evaluated by a procedure shown schematically in Figure 4. Let the probability density function of the intersection of the centerline with the line $x = x_o$ be $f(y'|x_o)$. The conditional probability that the body exists at some point (x_o,y_o) is simply the probability that the distance between (x_o,y_o) and the centerline is less than half the body width,

$$\Pr[x_o, y_o | y') = \Pr\left[|y_o - y'| \leq \frac{w}{2}\right] \quad . \tag{3}$$

But as the centerline location is itself uncertain, the probability must be weighted and integrated over possible centerline locations, or

$$\Pr[x_o, y_o] = \int_{y'} \Pr\left[|y_o - y'| \leq \frac{w}{2}\right] f(y'|x = x_o)\, dy' \quad . \tag{4}$$

So, once the probability functions of width and centerline location are determined, probabilistic predictions of body location on the basis of any particular trend model can be generated by equation 4.

Probability density functions (pdf's) of centerline location and body width can be evaluated for a particular trend model by performing a (Bayesian) regression on known locations. Once this is done, model uncertainty can be accounted for by evaluating the posterior probability of each model and forming a so-called composite Bayesian model.

Centerline Distribution: Bayesian Regression

Let the known locations of the body be represented by the set of data points $(\underline{x}, \underline{y})$, and the trend model be

$$\underline{y} = \underline{\beta}\underline{\underline{X}} + \underline{e} \quad . \tag{5}$$

Here, $\underline{y}$ is the vector of y-components of the data set, $\underline{\underline{X}}$ is a matrix of functions of the x-components, $\underline{\beta}$ is the vector of regression coefficients, and $\underline{e}$ is an error term with zero-mean and variance σ^2. For example, for the model,

$$y_i = \beta_1 + \beta_2 x_i + \beta_3 x_i^2 + \cdots + e_i \quad , \qquad (6)$$

$$\underline{y} = (y_1, y_2, \ldots, y_n)$$

$$\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)$$

$$\underline{\underline{X}}^t = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{k-1} \end{bmatrix} .$$

By the Bayesian argument, prior probabilities on $\underline{\beta}$ and σ are updated to yield posterior probabilities on the basis of the likelihood of observations conditioned on $\underline{\beta}$ and σ. That is, probabilities are updated on the basis of conformity between observations and predictions. If we let $f^o(\underline{\beta},\sigma)$ be the prior joint pdf of the regression parameters, then the posterior joint pdf of $\underline{\beta}$ and σ by Bayes' Theorem is

$$f'(\underline{\beta},\sigma|\underline{\underline{X}},\underline{y}) \propto f^o(\underline{\beta},\sigma)\ L(\underline{\underline{X}},\underline{y}|\underline{\beta},\sigma) \quad . \qquad (7)$$

If for the prior distribution $f^o(\underline{\beta},\sigma)$ we use the so-called "uninformed" prior[3],

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[3]We have chosen here to use "uninformed" or flat priors simply for convenience of presentation. In reality, the geologist's opinion of local geological structure would enter the analysis through $f^o(\underline{\beta},\sigma)$. Informed priors are discussed in Appendix C.

$$f^{o}(\underline{\beta},\sigma) \propto \frac{1}{\sigma} \qquad (8)$$

(i.e., $\underline{\beta}$ and ln σ uniformly distributed), and if for the error term, $\underline{e}$, we assume a zero-mean normally distributed random variable, then one can show (Zellner, 1971) that the posterior distributions of $\underline{\beta}$ and σ and simple functions of $\underline{\beta}$ and σ belong to well-known families of distributions (Appendix A). In particular, the distribution of interest in extrapolation is the centerline pdf. From equation 5, centerline location conditioned on x is simply a weighted sum of the random variables $\underline{\beta}$, and can be shown to be distributed as a univariate Student t (Zellner, 1971):

$$f(y'|x_o) \propto \left\{\nu + \frac{(y - \hat{y})^2}{S^2 c}\right\}^{-(\nu+1)/2} , \qquad (9)$$

in which ν is degrees-of-freedom, S^2 is a squared error term from the data set, and c is a constant depending on values of the data set and x_o. The term $\hat{y}$ is the expected location of the centerline.

Model Uncertainty

Beyond uncertainties inherent in estimating model parameters there are also uncertainties in which model of centerline trend to fit. For example, should a linear trend be used, or is some low-order curve a better representation? The importance of including model uncertainty in prediction is that it is a substantial component of total uncertainty, and that this increased uncertainty leads to an increased rate of decay in the probability density of predicted location (i.e., a more rapid "broadening" of the pdf), and thus shortens the length to which extrapolations can be made.

The approach to model uncertainty used here is that suggested by Benjamin and Cornell (1970) and by Wood (1974), in which a weighted sum of the prediction of each model is formed using posterior model probabilities as weights.

Adopting the "linear" model of equation 6 to predict centerline trend, the shape of the extrapolation is described by k, the order of polynomial. Allowing the prior belief in the validity of k_i to be $p^o(k_i)$, posterior probabilities are updated in the normal way:

$$f'(\underline{\beta},\alpha,k_i|\text{data}) \propto f^o(\underline{\beta},\alpha,k_i)\ L[\text{data}|\underline{\beta},\alpha,k_i] \quad . \qquad (10)$$

Then

$$f'(\underline{\beta},\alpha|\text{data}) = \sum_{i=1}^{3} f'(\underline{\beta},\alpha,k_i|\text{data})$$

$$= \sum_{i=1}^{3} f'(\underline{\beta},\alpha|k_i,\text{data})\ p'(k_i|\text{data}) \quad ; \qquad (11a)$$

where

$$p'(k_i|\text{data}) \propto p^o(k_i) \int_{\underline{\beta}} \int_{\alpha} L[\text{data}|\underline{\beta},\alpha,k_i]\ f'(\underline{\beta},\alpha|k_i)\ d\underline{\beta}\ d\alpha \quad . \qquad (11b)$$

Width Distribution

For convenience, we assume that the probability density of body half-width is distributed as the Maxwell distribution,

$$f(w|\sigma) = \frac{\sqrt{2}\, w^2}{\sigma^3 \sqrt{\pi}} \exp\,(-w^2/2\sigma^2) \quad , \qquad \text{for } w \geq o \tag{12}$$

with parameter σ.

If we assume that the known locations of the body are randomly (i.e., uniformly) distributed across the width of the body, then the pdf of the "error" term away from the body centerline (Figure 5a) is

$$f(e|w) = \frac{1}{w} \quad , \qquad \text{for } o \leq e \leq w \quad . \tag{13}$$

The marginal distribution of e is (Appendix A)

$$f(e) = \int f(e|w)\; f(w)\; dw = \frac{\sqrt{2}}{\sigma\, \sqrt{\pi}}\; e^{-w^2/2\sigma^2} \quad , \tag{14}$$

which can be seen to decay as a one-sided normal distribution with variance σ^2. This is, of course, our justification for using the Maxwell distribution to begin with. With this distribution on width, "error" about the centerline is normally distributed and the results of Normal Bayesian regression can be directly employed.

Body Length

The model proposed thus far does not account for the finite length of geological bodies; it assumes them to be infinite. Therefore, probabilities which result from this model must be modified.

From past experience one has some idea of the distribution of lengths of similar bodies, and this information can be modeled by a probability density function, $f(\ell)$. Here, we will assume $f(\ell)$ to be lognormal, as this distribution family adequately fits many geometric properties of geological formations. Since we know that the body whose location is being predicted is at least of length l_o (Figure 5b), by Bayes' rule the conditional probability density of it being of length ℓ' is

$$f'(\ell'|\ell_o) = \begin{cases} \dfrac{f(\ell)}{\int_{\ell_o}^{\infty} f(\ell)\, d\ell} & \text{for } \ell \geq \ell_o \\ 0 & \text{otherwise} \end{cases} .$$

Assuming that the planar shape of the body is independent of its length (an assumption which may be questionable), the probability of its being located at any point is simply the product of the model prediction and the probability of its extending to or beyond the point in question,

$$\Pr[x_o, y_o] = \Pr[x_o, y_o | \ell \geq \ell'] \int_{\ell'}^{\infty} f'(\ell' | \ell_o) \, d\ell \quad .$$

IV. EXAMPLE PREDICTIONS

The present procedure for extrapolation was applied to the data shown in Figure 6 (a second data set and prediction is shown in Appendix B). Three simple trend models were fitted (linear, quadratic, cubic), assuming equal *a priori* model probabilities and Maxwell-distributed width.

Figure 7 shows the marginal posterior distribution of the regression coefficients $\underline{\beta}$ for the linear model (those for the quadratic and cubic models being harder to plot here), and Figure 8 shows the marginal posterior distributions of width for each of the models. Extrapolation predictions for each model are shown in Figure 9a, b, c, respectively; and the composite prediction, in Figure 10. Figure 11 shows the composite prediction modified by consideration of finite length.

The most striking feature of these extrapolations is how rapidly the certainty of location prediction decays away from the data set, an observation which is not so clearly demonstrated when non-quantified approaches are used.

V. LIMITATIONS AND ERRORS

This model is an attempt to quantify the sorts of spatial predictions which exploration geologists routinely make on the basis of observations. The model is not refined, and it clearly suffers the limitations of the heuristic technique on which it is based. On the other hand, the model has analytical shortcomings as well as geological ones, and these are what we turn attention to here.

To begin with, the model assumes that the x-components of known locations are independent, but in reality this is not so. Just as one considers present information when locating the next well or observation, so one considered it in the past. Thus, observations are biased toward lying on a straight line or low-order curve; that is precisely the way they were sequentially placed by whoever was making the decisions.

Second, the analysis neglects part of the location information we have. While reconnaissance information (e.g., geophysical data, etc.) can be included in establishing prior probabilities, "dry wells" or locations where the body is known not to exist are neglected. This causes the model to generate predictions which are too diffuse.

Third, the distribution model for width is inadequate because it also neglects information and because account should be taken of width-model uncertainty as well as centerline-model uncertainty. Box and Tiao (1973) suggest a way of doing this. Since, as before, we have information on where the body isn't, the width distribution should have an upper bound--which the Maxwell distribution does not.

Next, there is no reason to believe that the body has constant width. The assumption makes regression easier, and this is the reason it is made, but real formations have varying widths. As long as there is no trend of width with length, however, the assumption of uniformity is probably not too bad. If there is a trend--and finite length means that at some point there must be--the predictions may be substantially in error.

Finally, the procedure for updating model probabilities still requires thought. Here, likelihoods were calculated on the basis of "fit" of a centerline trend to observations. For a large number of observations (i.e., relative to the trend order, k), higher-order curves will always fit better than lower-order ones. Yet, as we said earlier, empirically we know that high-order curves are too flexible and their predictions overly diffuse. When the number of data points, n, is small, this problem doesn't necessarily occur because the degrees-of-freedom

($\nu = n - k$) is substantially affected by changes in k (e.g., Appendix C).

VI. LOCATION PREDICTIONS AND DECISION-MAKING

In this last section, as an addendum, we will briefly discuss the place of quantified predictions in decision-making for geological exploration. In general, there are two types of decisions which might be made with spatial predictions. One is the exploitation decision: where should a new producing well be placed, or where should a well point be located to drain a pervious stratum? The second is the allocation of exploration effort: where should observations be made, or how closely spaced should geophysical traverses be placed? These decisions have different objectives and do not necessarily lead to similar optimizations. For example, the optimal location of an exploitation well might not be the same as the optimal location for gathering information on structure. Here we will describe the exploitation decision as it is analytically simpler, yet highlights the role of quantified predictions.

Assume that the decision to be made is where to place a well for production of some resource or for dewatering a construction site; and assume that this is a one-stage decision (i.e., information gathering has already been finished). A decision tree for this decision is shown in

Figure 12. Let the cost of drilling, c, be independent of location, and let the value of hitting oil, water, or whatever, be a function, g(d), of the distance from the closest "producing" well. (In other words, assume that two wells in proximity draw on the same volume of resource and thus have lower individual yields than two more distant wells; see Figure 13.)

Taking the predictions of Section IV, let the locus of points of maximum probability of the body's location away from the data set be represented by line J (Figure 14). The decay in probability along this line is shown in Figure 15. Using expected value as the criterion of decision[4], we can graph the objective function over distance along J as

$$E[\text{value}] = (-c) + g(d) \times E[\text{value of resource}] \times \Pr[\text{hitting body with resource}] \quad .$$

Combining yields a maximum at d_*, which if greater than zero would be the optimal location for drilling based on model predictions.

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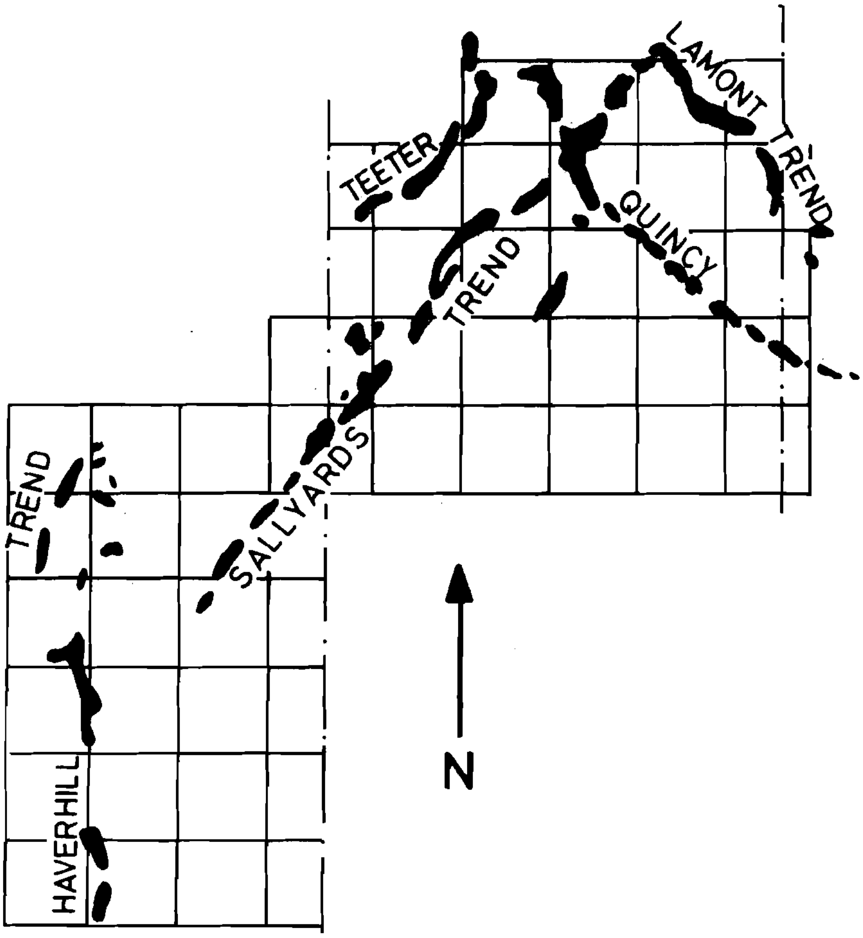
[4] This of course assumes a linear objective function which is expected monetary value. Clearly this may not always be the case. But the problem may be overcome by introducing utility functions, which Grayson (1960) has discussed in a geological context.

VII. CONCLUSIONS

We have presented an analytical model for quantifying location predictions of linearly trending geological bodies, so that these predictions might be included in larger decision models for exploration. The model requires further refinement, but illustrates how the geologist's subjective judgement may be included in quantified approaches to optimizing exploration strategies. The model also sheds light on traditional questions in exploration, such as how far trends may be extrapolated away from observations. Our hope in presenting this work is that it will contribute to the larger task of the development of a general theory of rational exploration based on the subjective judgements of geologists.



SHOESTRING-SAND POOLS OF KANSAS
(AFTER LEVORSEN, 1954)

FIGURE 1

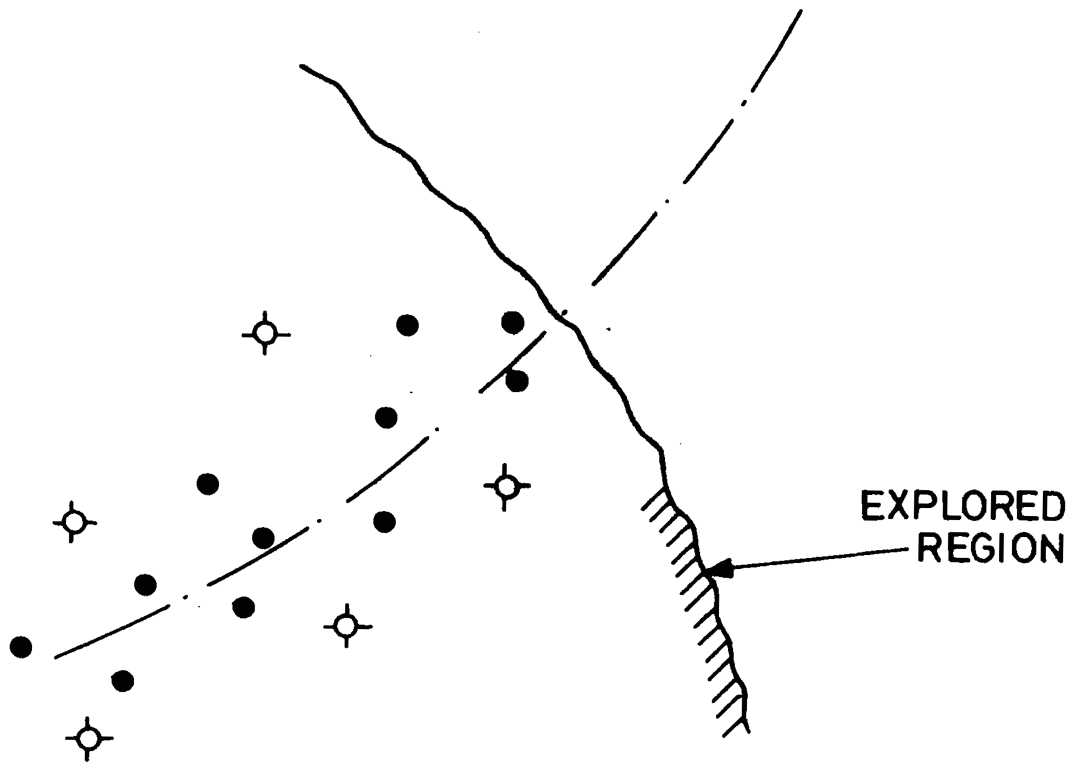


FIGURE 2

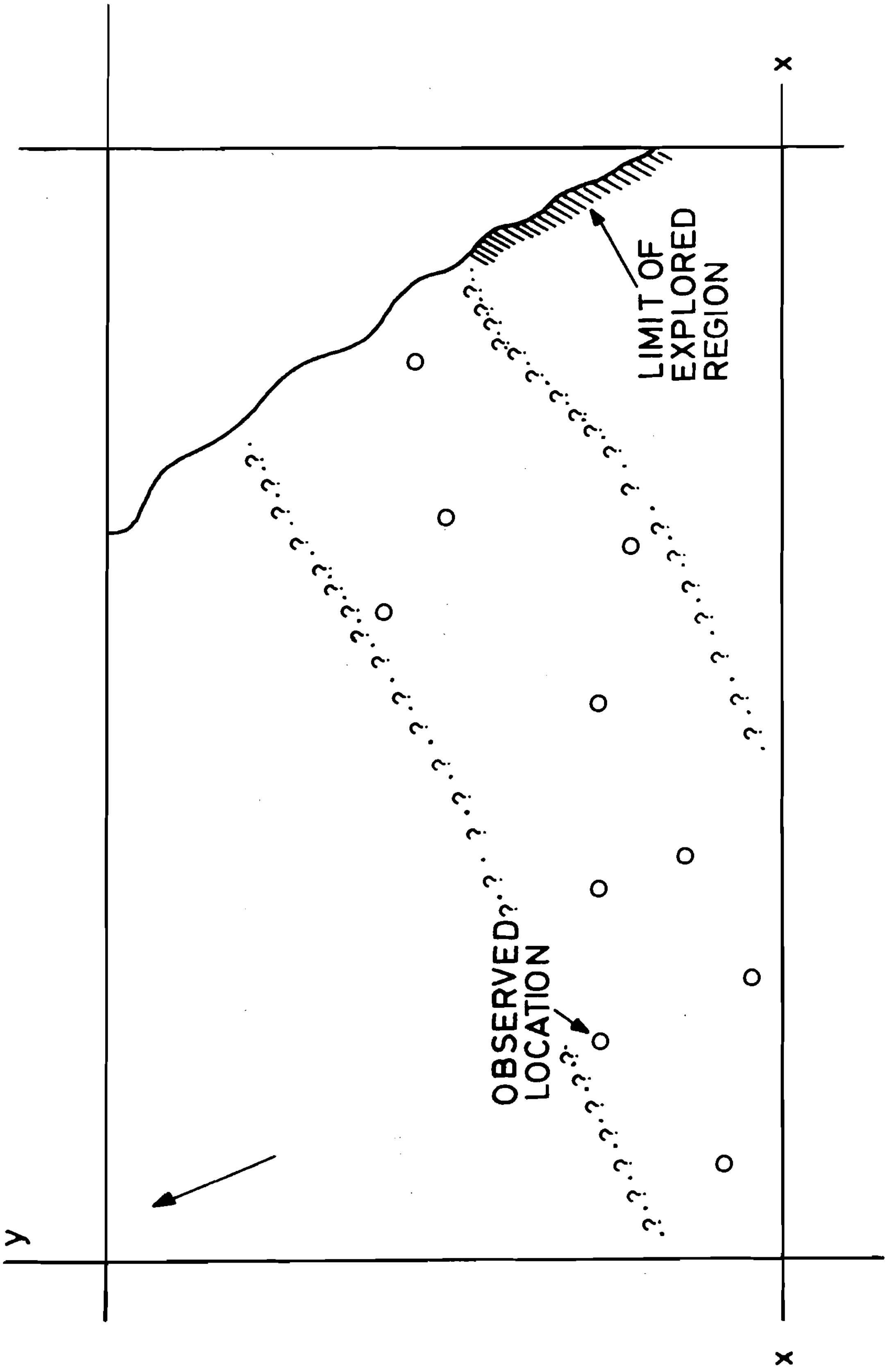


FIGURE 3

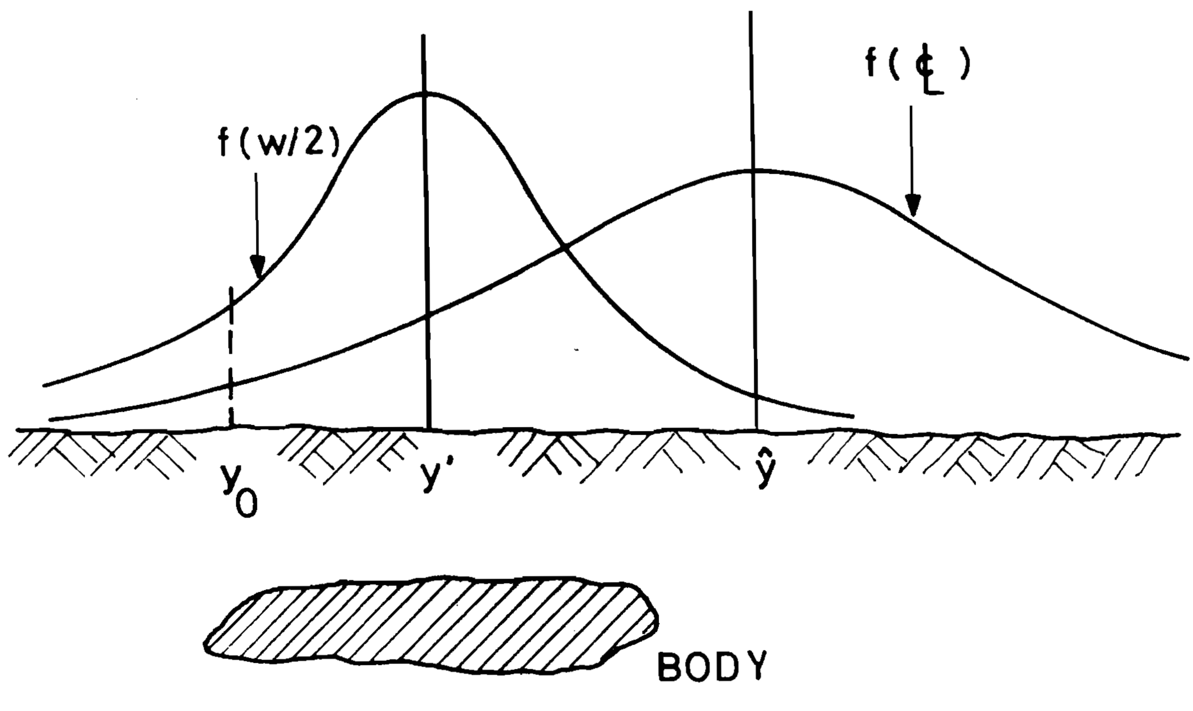


FIGURE 4

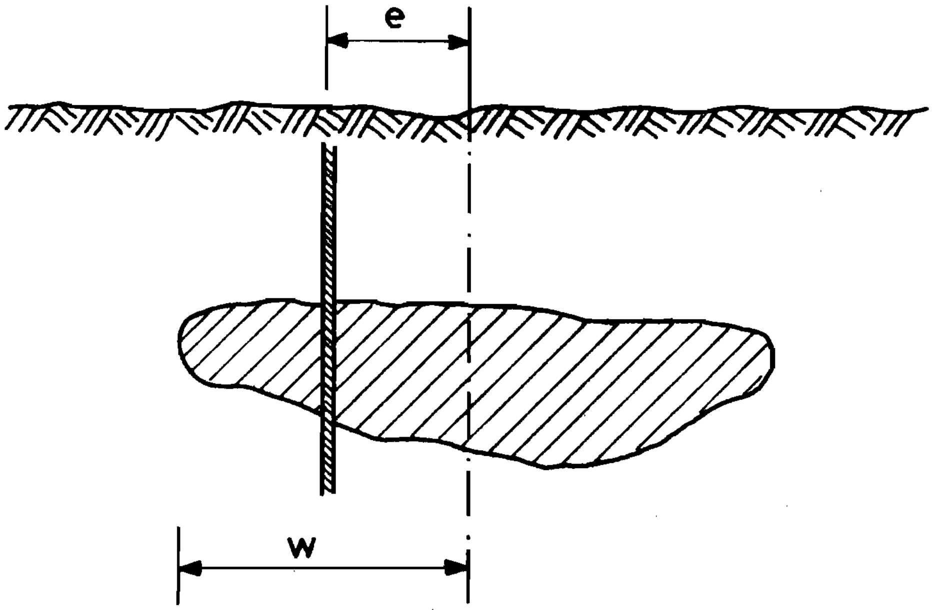


FIGURE 5a

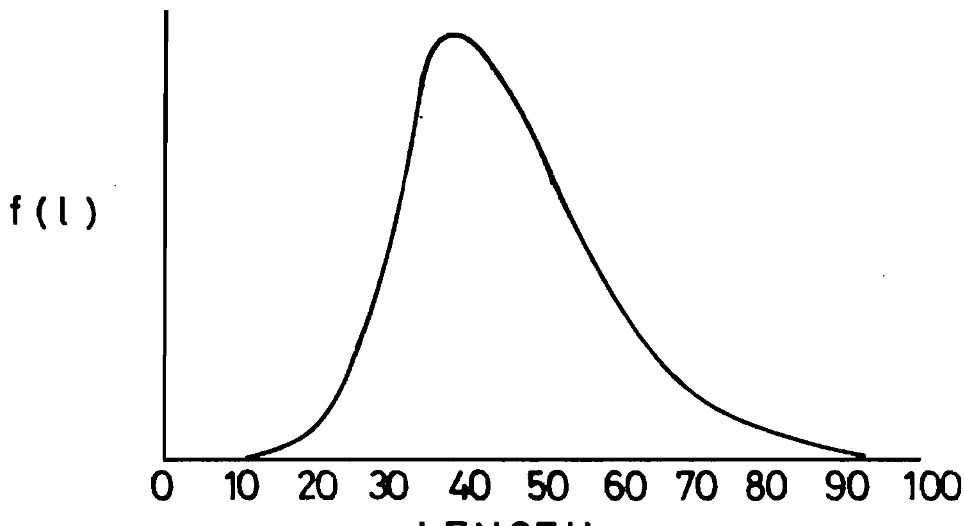
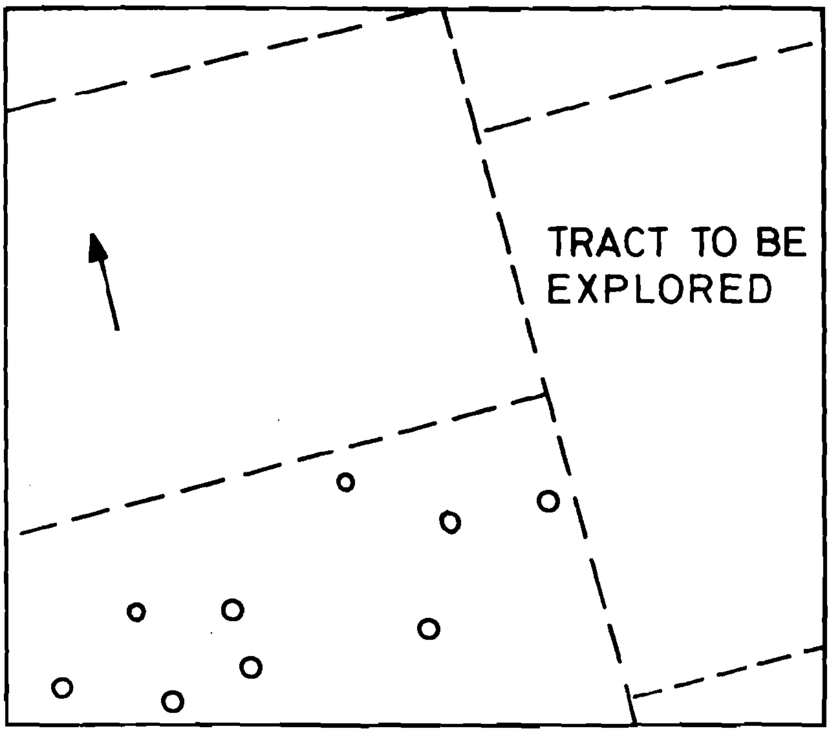
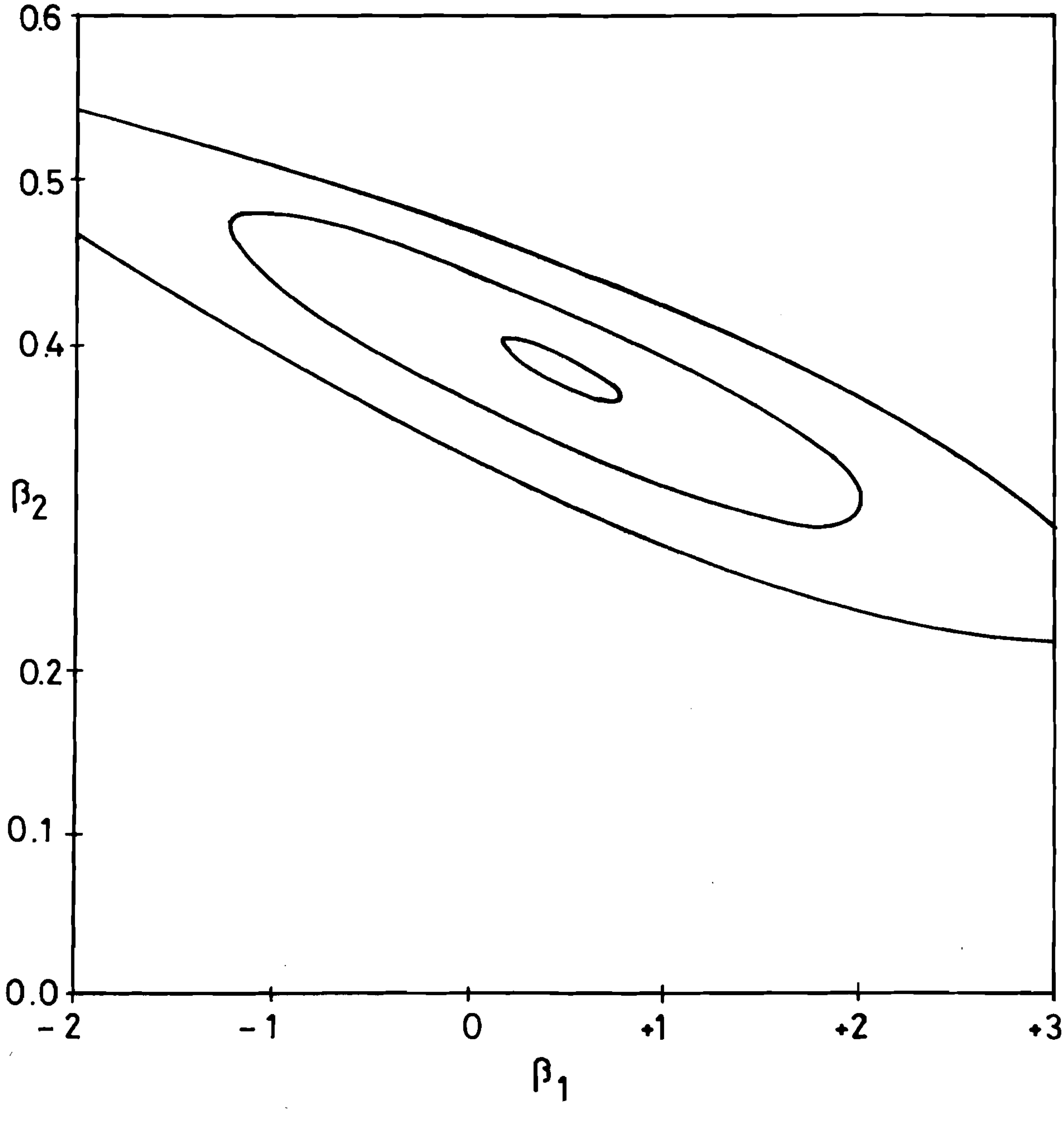


FIGURE 5b



KNOWN LOCATIONS OF BODY

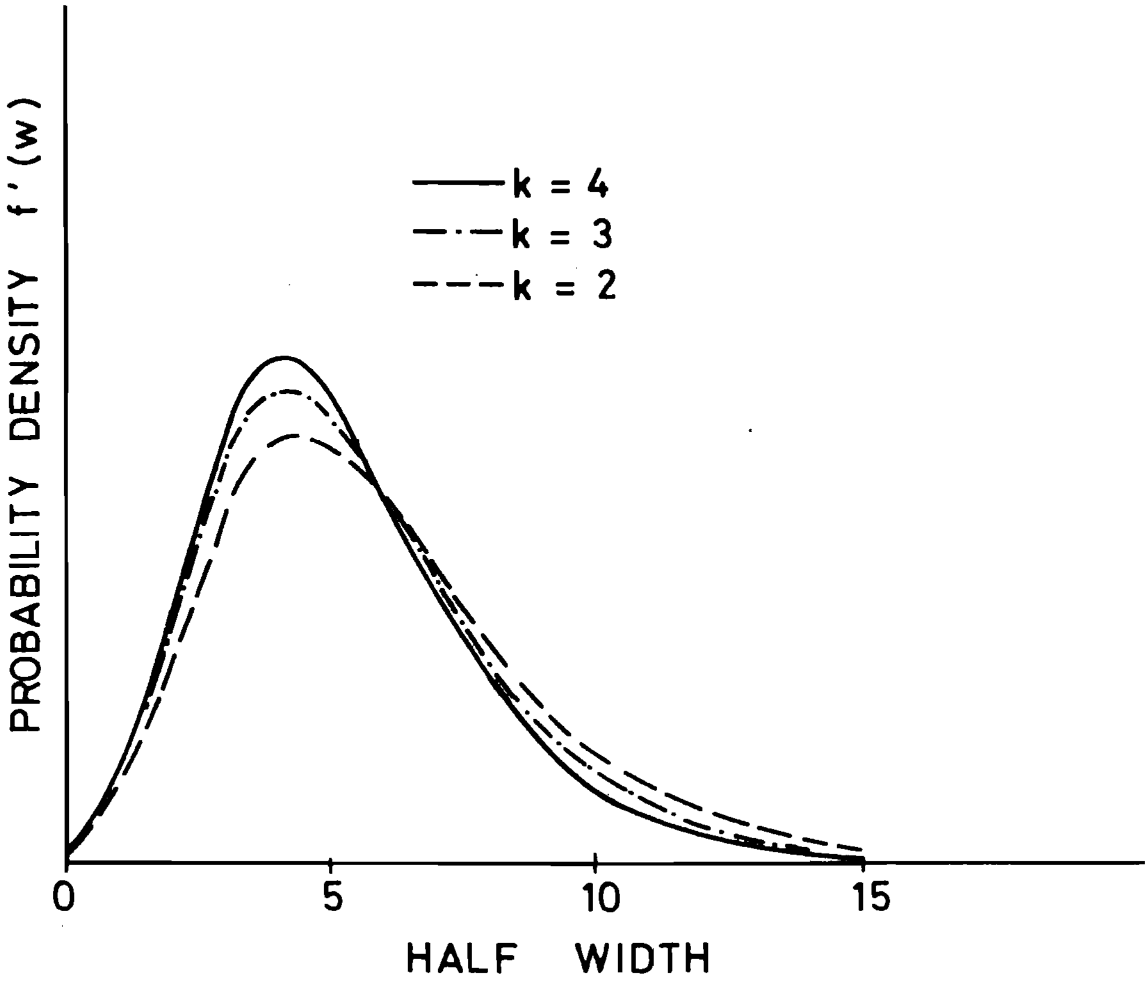
FIGURE 6



POSTERIOR DISTRIBUTION OF $\underline{\beta}$

LINEAR MODEL

FIGURE 7



POSTERIOR DISTRIBUTION OF HALF WIDTH

FIGURE 8

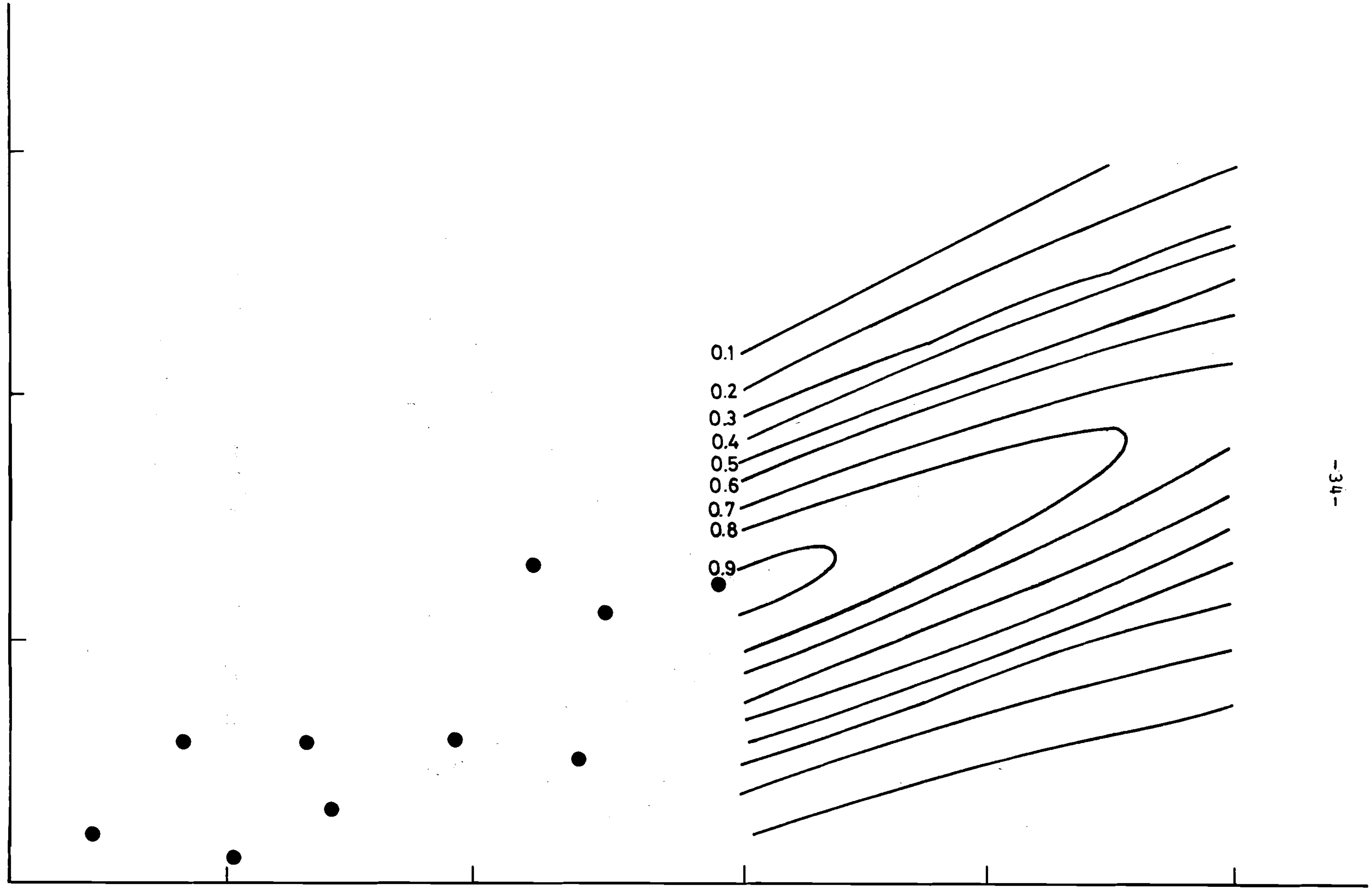


FIGURE 9a: LINEAR MODEL

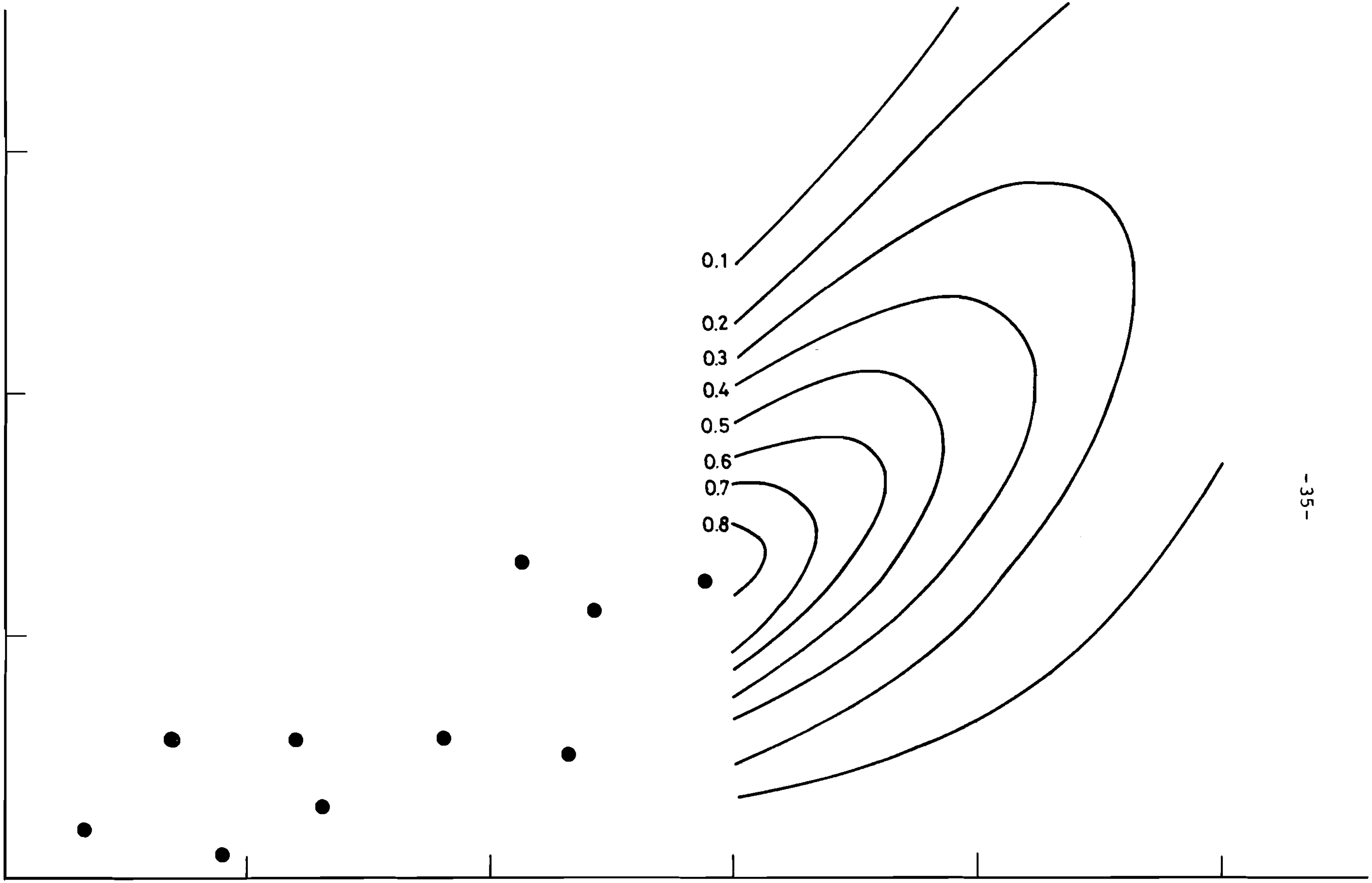


FIGURE 9b : QUADRATIC MODEL

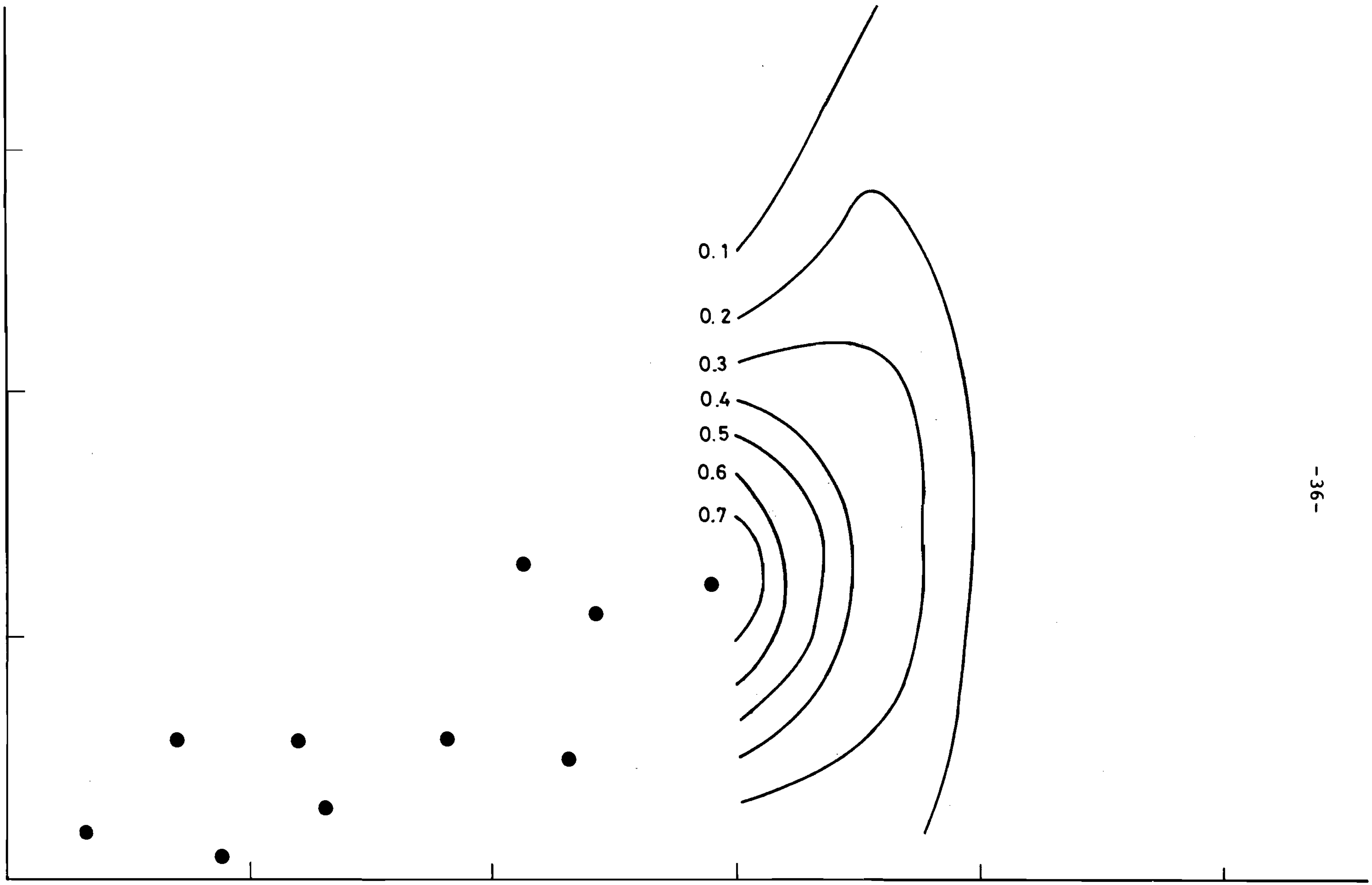


FIGURE 9c: CUBIC MODEL

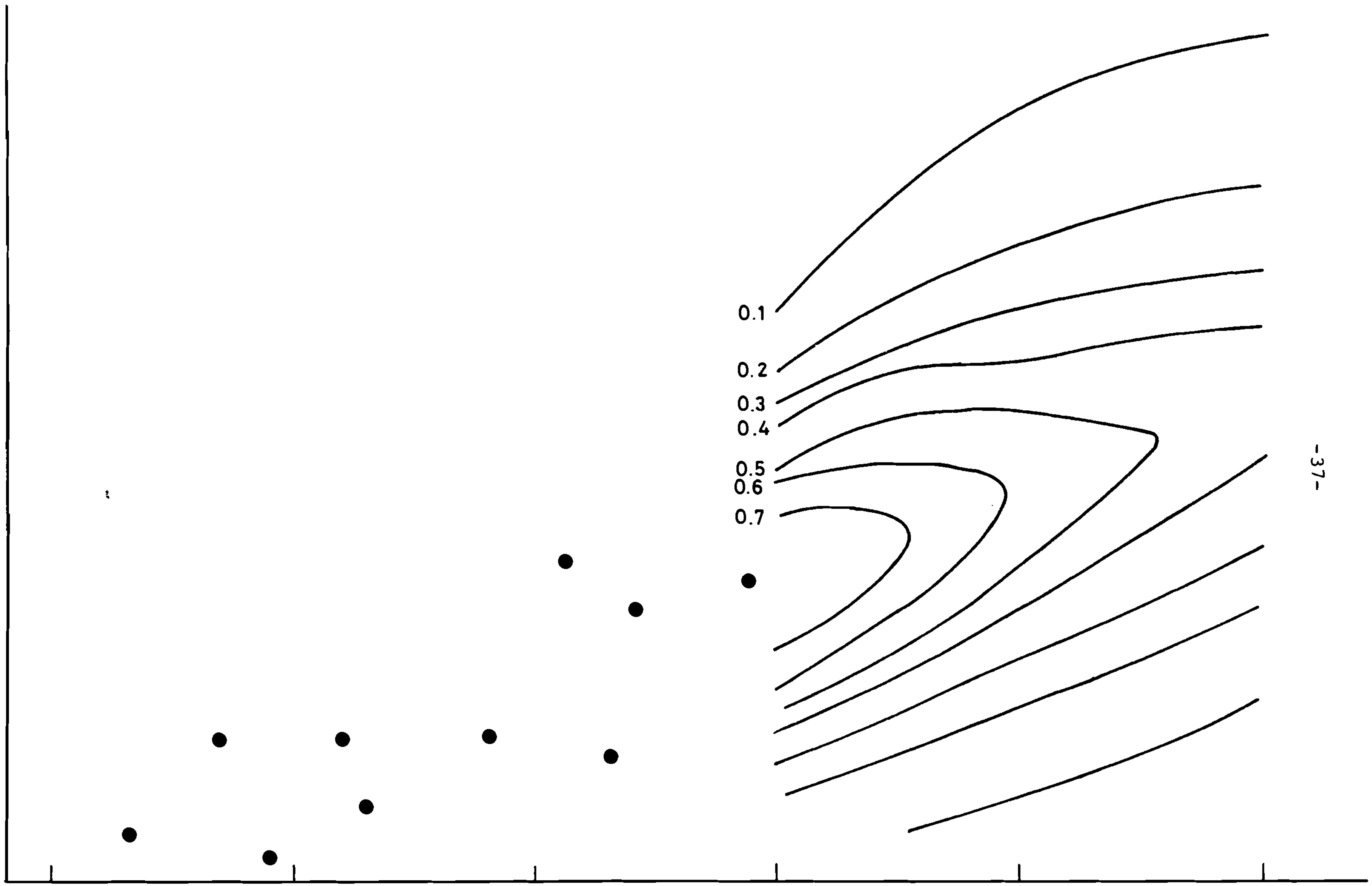


FIGURE 10: COMPOSITE

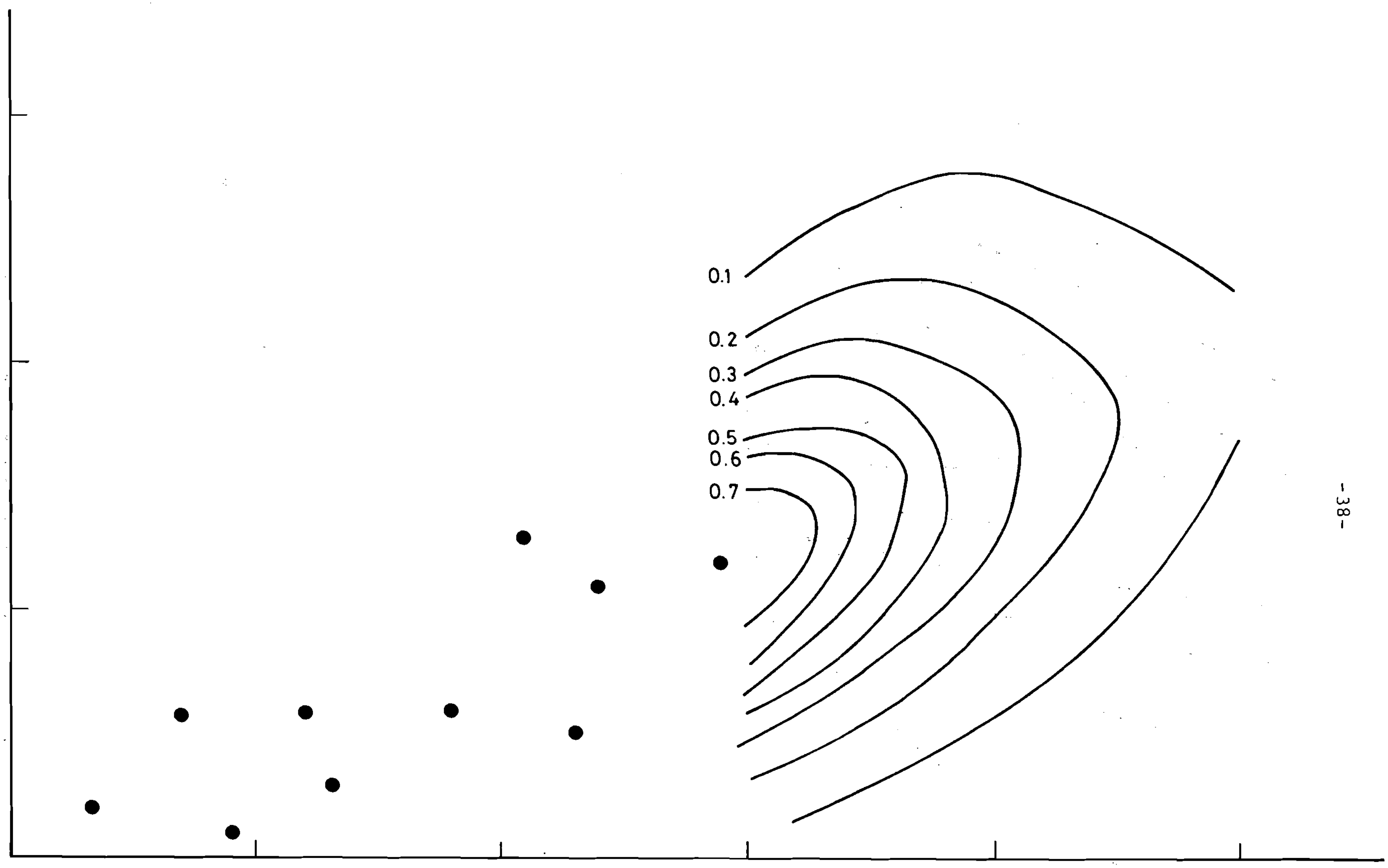


FIGURE 11 :COMPOSITE WITH LENGTH CORRECTION

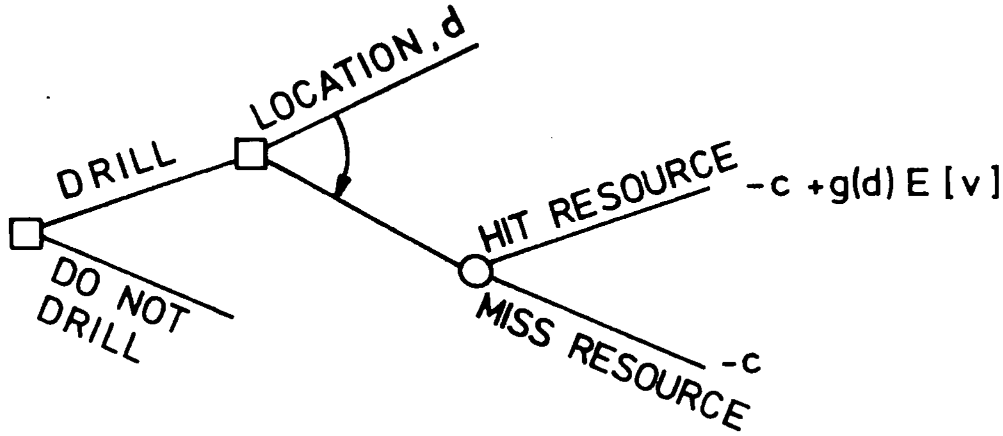


FIGURE 12

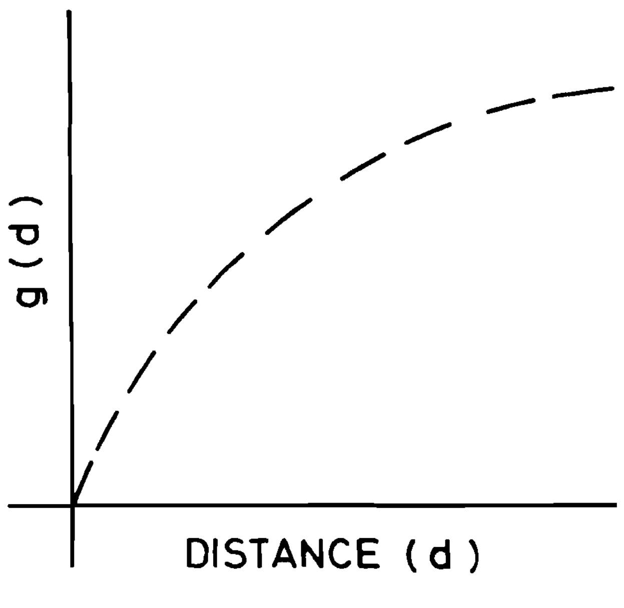


FIGURE 13

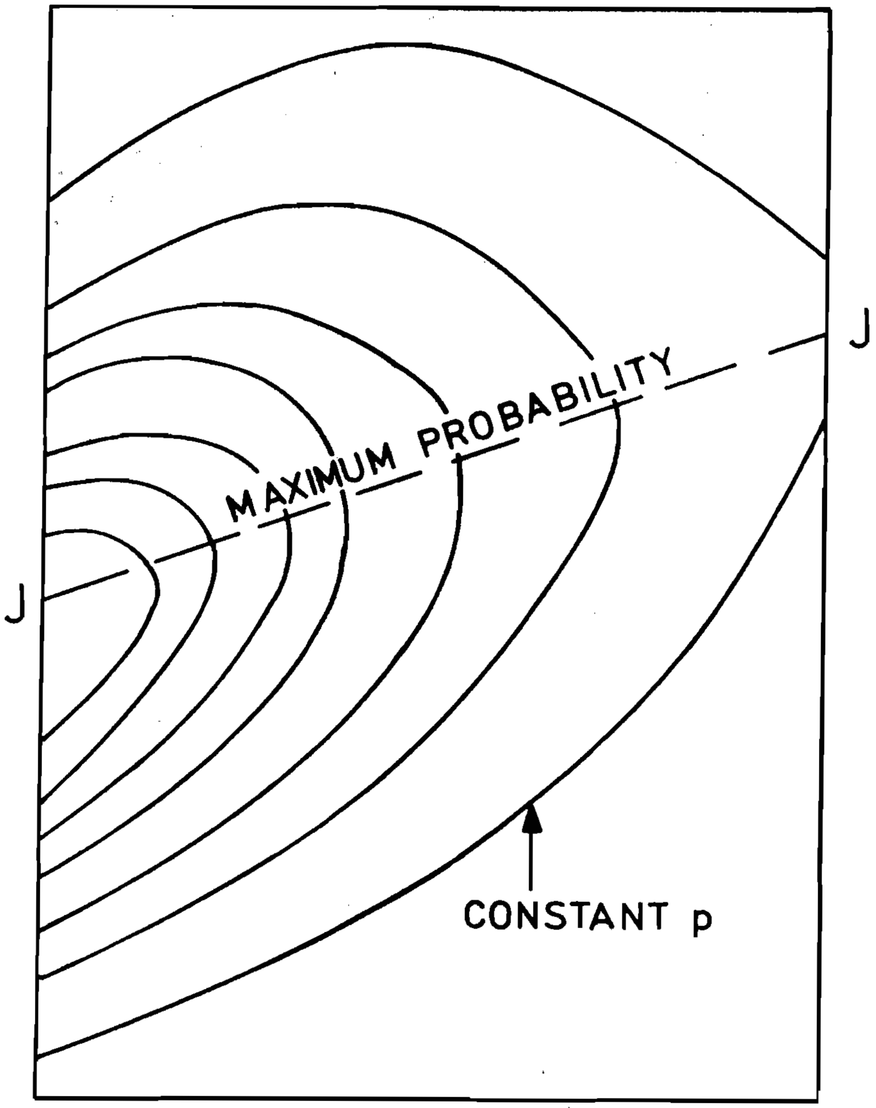


FIGURE 14

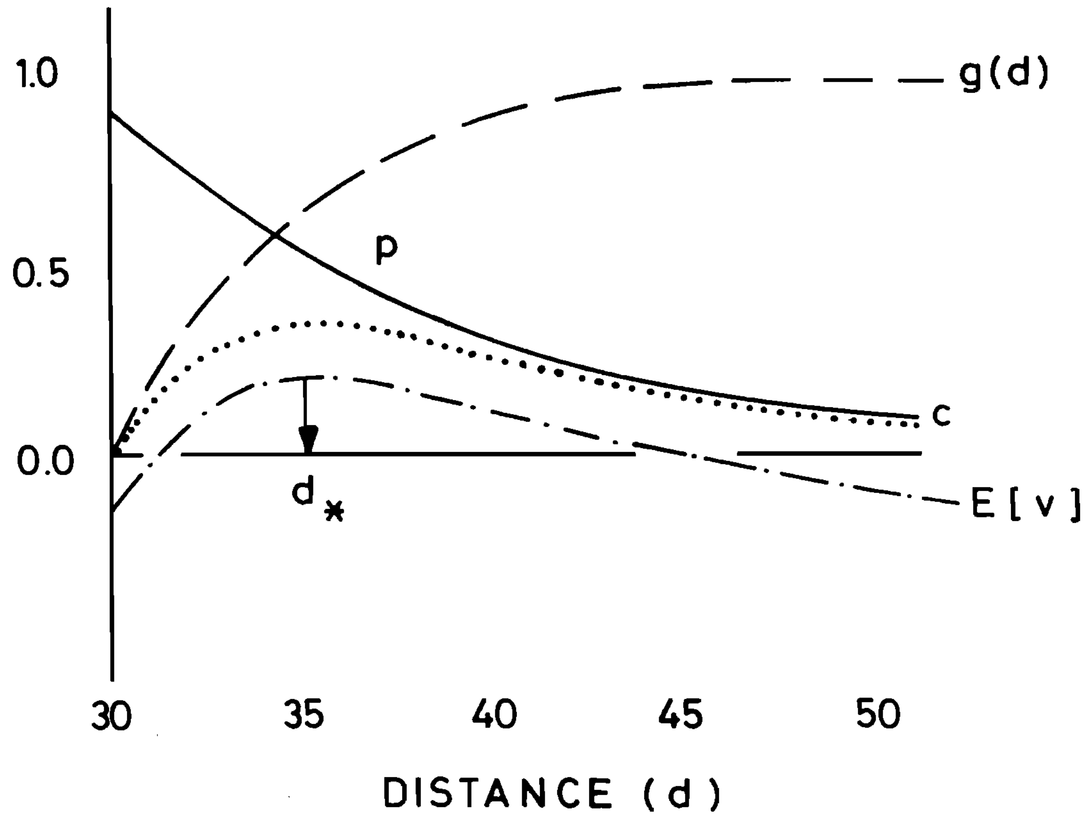


FIGURE 15

APPENDIX A

Mathematical Structure of Analysis

Width Distribution

The probability density of width is assumed to be distributed as the Maxwell distribution with unknown parameter σ

$$f(w|\sigma) = \frac{\sqrt{2}\ w^2}{\sigma^3\ \sqrt{\pi}} \exp\ [-w^2/2\sigma^2] \quad . \tag{A1}$$

Further, assuming that borings intersecting the body are randomly distributed across the body width, the probability density function of the distance, e, between a boring and the centerline is

$$f(e|w) = \frac{1}{w} \qquad o \leq e \leq w \quad . \tag{A2}$$

Hence,

$$f_e(e) = \int f(e|w)\ f(w)\ dw \tag{A3}$$

$$= \int_{w=e}^{\infty} \frac{1}{w}\ \frac{\sqrt{2}\ w^2}{\sigma^3\ \sqrt{\pi}}\ \exp\ [-w^2/2\sigma^2]\ dw$$

$$= \frac{\sqrt{2}}{\sigma\ \sqrt{\pi}}\ \exp\ [-w^2/2\sigma^2] \quad , \tag{A4}$$

which decays as a (one-sided) normal distribution. This distribution corresponds to that of the error in locating the centerline.

A Bayesian regression to estimate parameters of the centerline is performed on known locations of the body using a zero-mean normally distributed error term with variance σ^2, and an *a priori* pdf on the regression parameters,

$$f^o(\underline{\beta},\sigma) \propto \frac{1}{\sigma} \quad . \tag{A5}$$

This is the so-called "uninformed" prior based on uniform distribution of $\underline{\beta}$ and $\ln \sigma$. Using the notation of Section III, the posterior distribution of the parameters $(\underline{\beta},\sigma)$ is

$$f'(\underline{\beta},\sigma|\underline{\underline{X}},\underline{y}) \propto f^o(\underline{\beta},\sigma)\ L(\underline{\underline{X}},\underline{y}|\underline{\beta},\sigma) \tag{A6}$$

$$\propto f^o(\underline{\beta},\sigma) \prod_{i=1}^{n} f_e(y_i - \underline{\beta}\underline{x}_i|\underline{\beta},\sigma) \tag{A7}$$

$$\propto \frac{1}{\sigma^{n+1}} \exp\left\{-\frac{1}{2\sigma^2}\left[\nu S^2 + (\underline{\beta} - \hat{\underline{\beta}})^t \underline{\underline{X}}^t\underline{\underline{X}}(\underline{\beta} - \hat{\underline{\beta}})\right]\right\} \quad , \tag{A8}$$

in which ν is the degrees-of-freedom,

$$\nu = n - k \quad , \tag{A9}$$

$$\hat{\underline{\beta}} = (\underline{\underline{X}}^t\underline{\underline{X}})^{-1}\ \underline{\underline{X}}^t\underline{Y} \quad , \tag{A10}$$

and

$$S^2 = \frac{(\underline{Y} - \underline{\underline{X}}\hat{\underline{\beta}})^t\ (\underline{Y} - \underline{\underline{X}}\hat{\underline{\beta}})}{\nu} \quad . \tag{A11}$$

The centerline passing through any line $x = x_o$ is a weighted sum of the random variable $\underline{\beta}$,

$$\bar{y} = \beta_1 + \beta_2 x_o + \beta_3 x_o^2 + \cdots \quad ; \tag{A12}$$

and given the posterior distribution of equation A8, Zellner (1971) shows this weighted sum to be distributed as a univariate Student t,

$$f(\bar{y}|\underline{\underline{X}},\underline{Y},\underline{x}_o) \propto \left[\nu + \frac{(\bar{y} - \hat{y})^2}{s^2 c}\right]^{-(\nu+1)/2} \quad , \tag{A13}$$

in which

$$\hat{y} = \underline{x}_o \hat{\underline{\beta}} \tag{A14}$$

and

$$c = \underline{x}_o (\underline{\underline{X}}^t \underline{\underline{X}})^{-1} \underline{x}_o \quad . \tag{A15}$$

The procedure for using the pdf of centerline location along any line $x = x_o$ in conjunction with the pdf of width to predict body location is described in Section III.

Three simple trend models were fitted to the data (linear, quadratic, and cubic) with equal *a priori* probabilities (i.e., 1/3). These probabilities were updated using Bayes' Theorem to arrive at posterior model probabilities, then used in forming the weighted or composite model for predictions:

$$p'(k) = \frac{p^{o}(k)\ L(\underline{\underline{X}},\underline{y}|k)}{\sum_{k=1}^{3} p^{o}(k)\ L(\underline{\underline{X}},\underline{y}|k)} \tag{A16}$$

in which

$$p^{o}(k = 2) = p^{o}(k = 3) = p^{o}(k = 3) = \frac{1}{3} \tag{A17}$$

and

$$L(\underline{\underline{X}},\underline{y}|k) = \prod_{i=1}^{n} f_t\left[\frac{y_i - \underline{x}_i\hat{\beta}}{\sqrt{h_{ii}}}\right] , \tag{A18}$$

where the conditional distribution of the observations is Student t with ν degrees-of-freedom, and h_{ii} is the (i,i)th element of the inverse of H,

$$H = \frac{1}{s^2} (\underline{\underline{I}} - \underline{\underline{X}}(2\underline{\underline{X}}^t\underline{\underline{X}})^{-1}\ \underline{\underline{X}}^t) \quad . \tag{A19}$$

The weighted sum of the model predictions is formed as in Section III, to generate final predictions.

The probability distribution of body width inferred from each of the models is computed using the marginal posterior distribution of σ and the relationship,

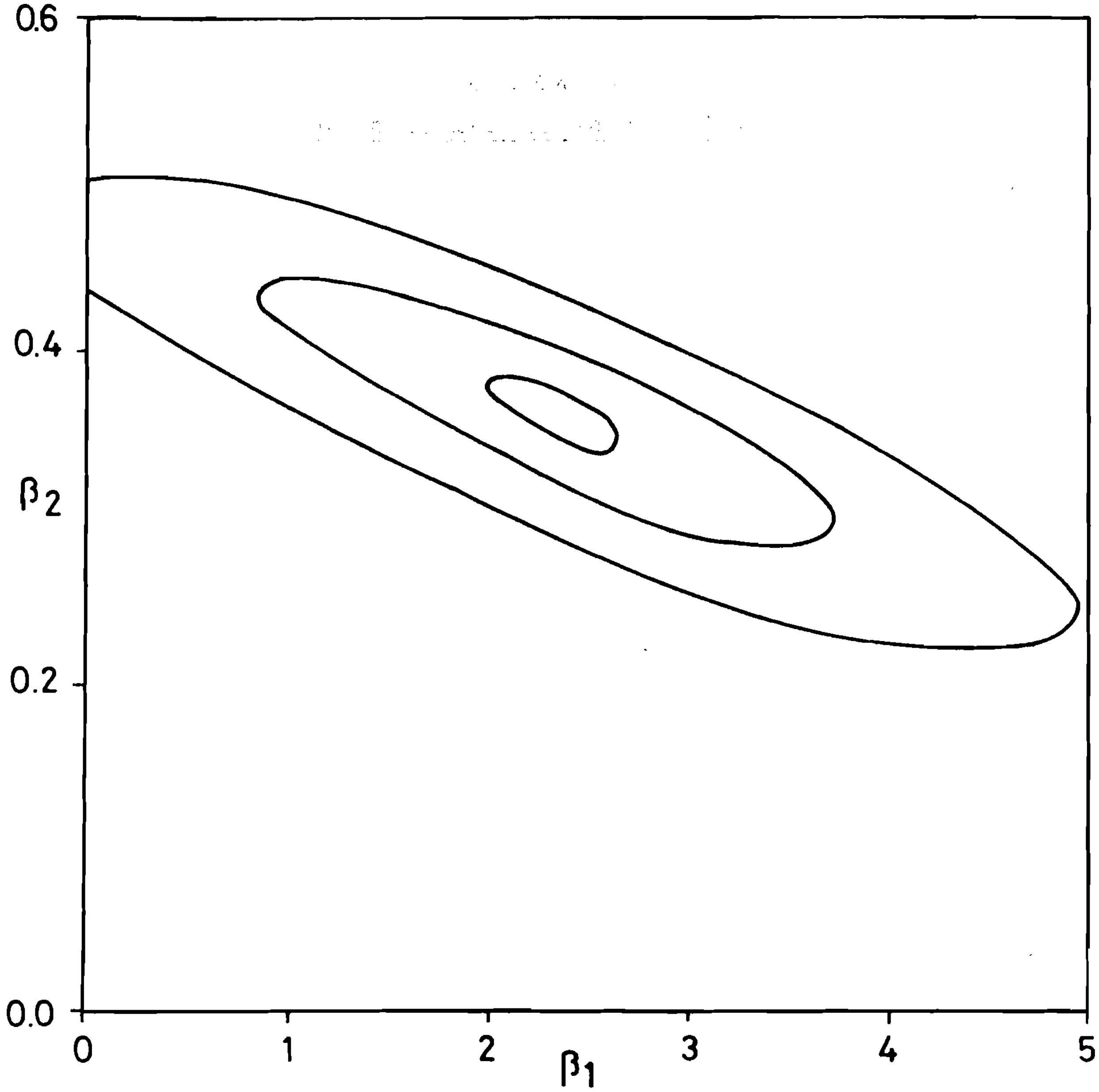
$$f(w|\underline{\underline{X}},\underline{y}) = \int_{\sigma=0}^{\infty} f(w|\underline{\underline{X}},\underline{y},\sigma)\ f(\sigma|\underline{\underline{X}},\underline{y})\ d\sigma \tag{A20}$$

$$= \int_{\sigma=0}^{\infty} \frac{\sqrt{2}\ w^2}{\sigma^3\ \sqrt{\pi}} \exp(-(w^2/2\sigma^2)) \left[\frac{2(\nu s^2/s)^{\nu/2}}{\Gamma(\nu/2)}\right] \frac{1}{\sigma^{\nu+1}} \exp(-(\nu s^2/2\sigma^2))\ d\sigma \tag{A21}$$

$$= \frac{(\nu s^2/2)^{\nu/2}}{\Gamma(\nu/2)} \ \frac{\sqrt{2}\ w^2}{\sqrt{\pi}} \left[\frac{\Gamma[(\nu+3)/2]}{[\frac{1}{2}(\nu s^2 + w^2)]^{\frac{\nu+3}{2}}}\right] . \tag{A22}$$

APPENDIX B

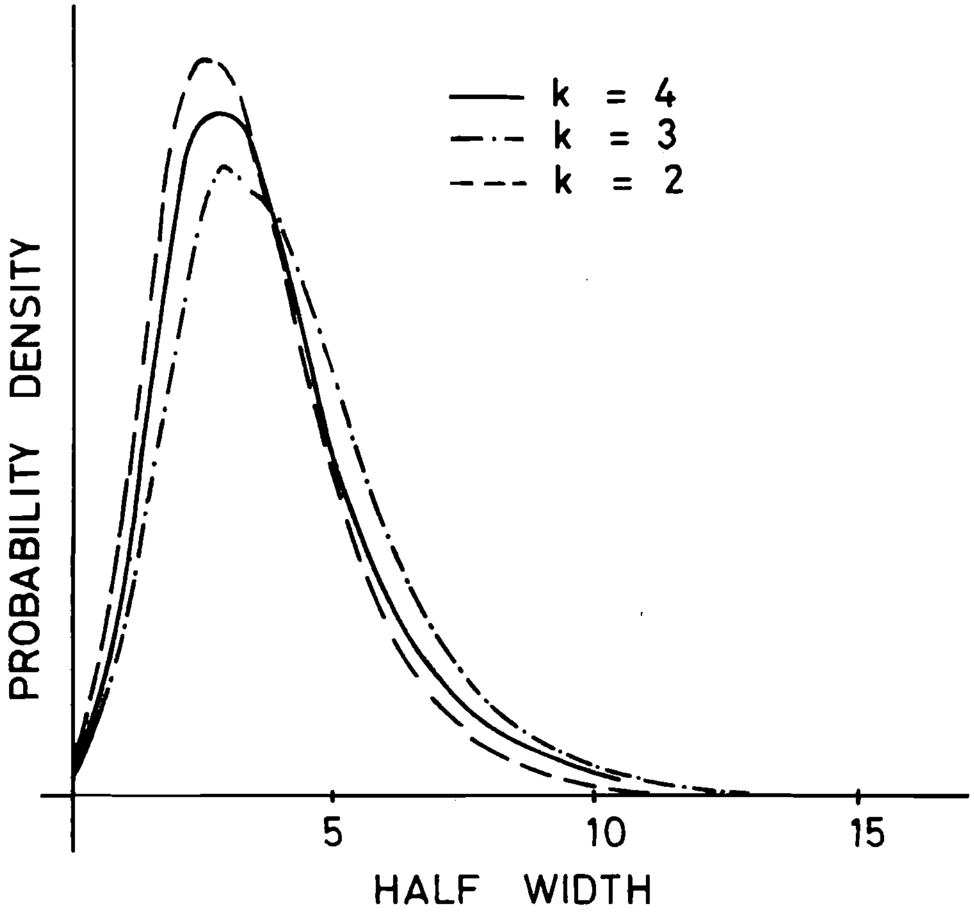
A Second Numerical Example



POSTERIOR DISTRIBUTION OF $\underline{\beta}$

LINEAR MODEL

FIGURE B.1



POSTERIOR HALF WIDTH DISTRIBUTIONS

FIGURE B.2

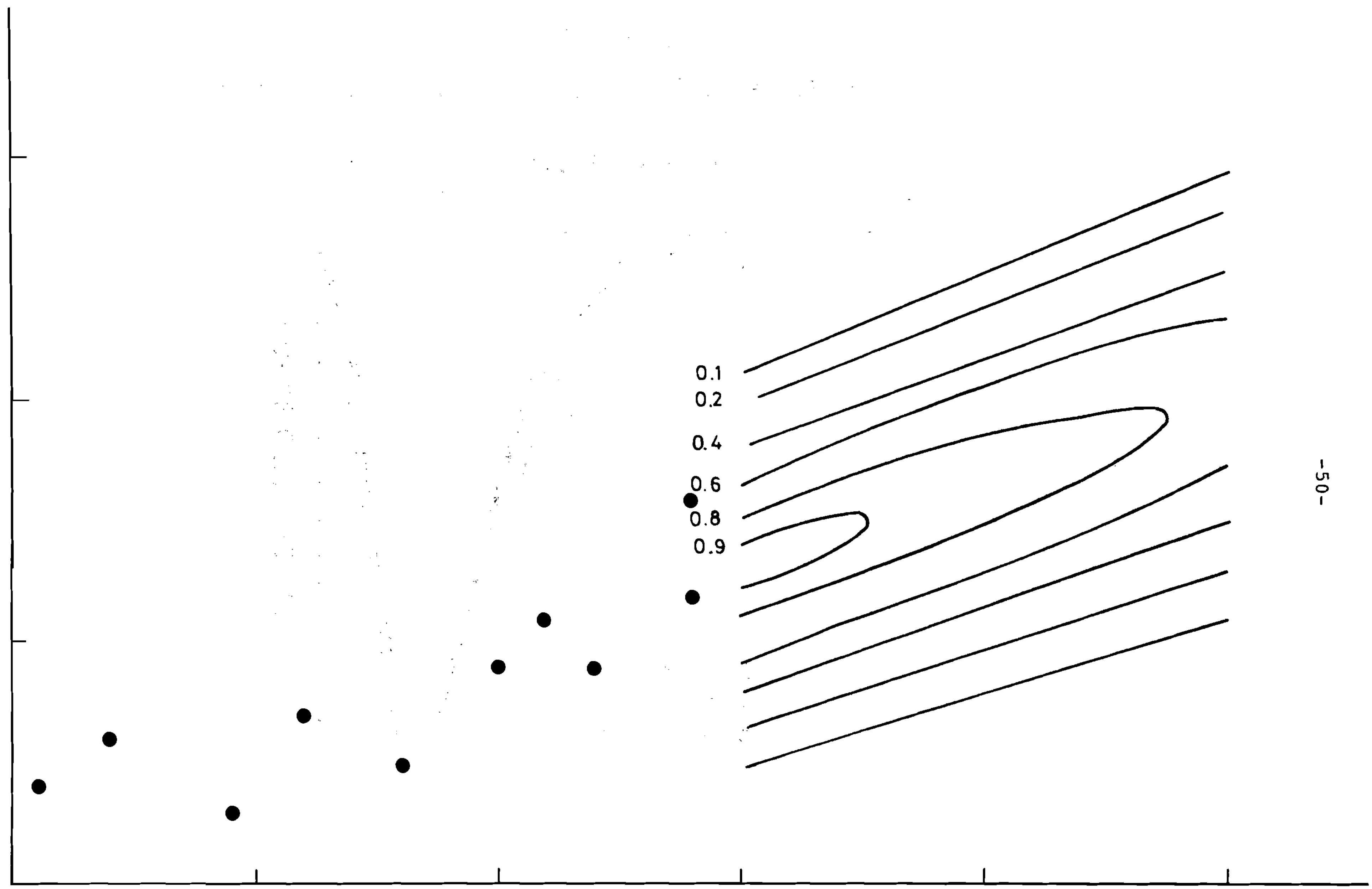


FIGURE B.3 : LINEAR MODEL

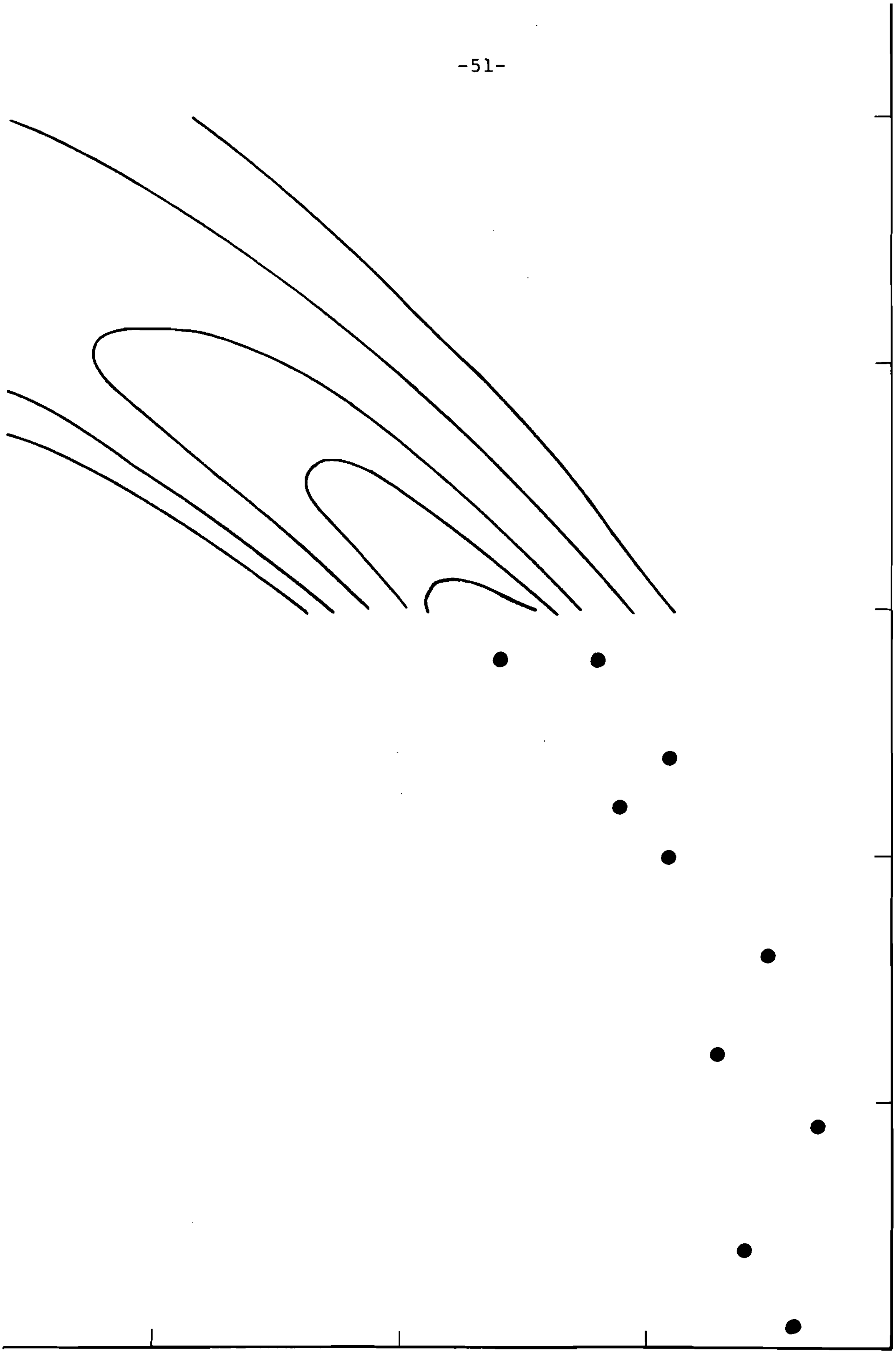


FIGURE B.4: QUADRATIC MODEL

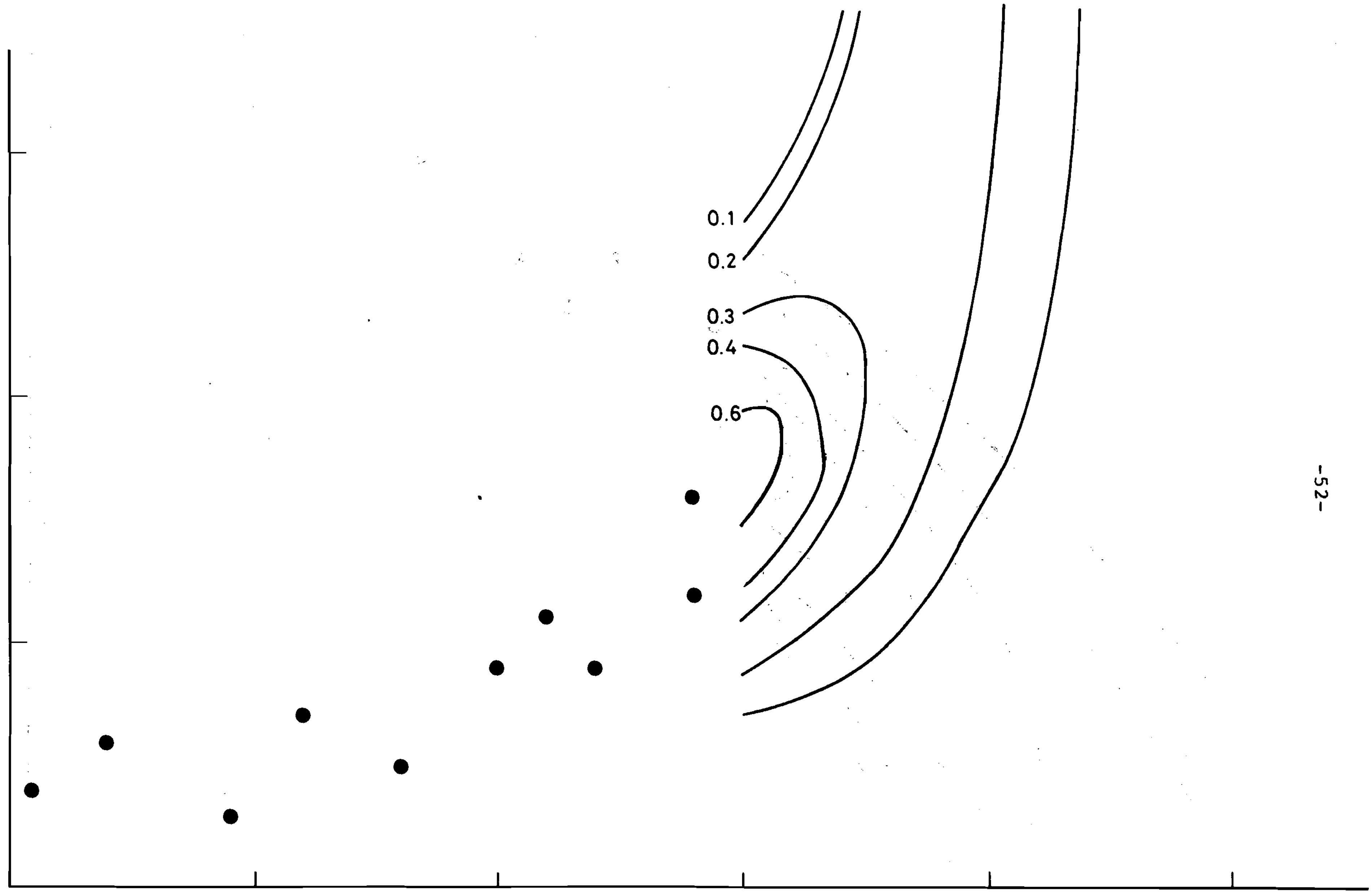


FIGURE B.5: CUBIC MODEL

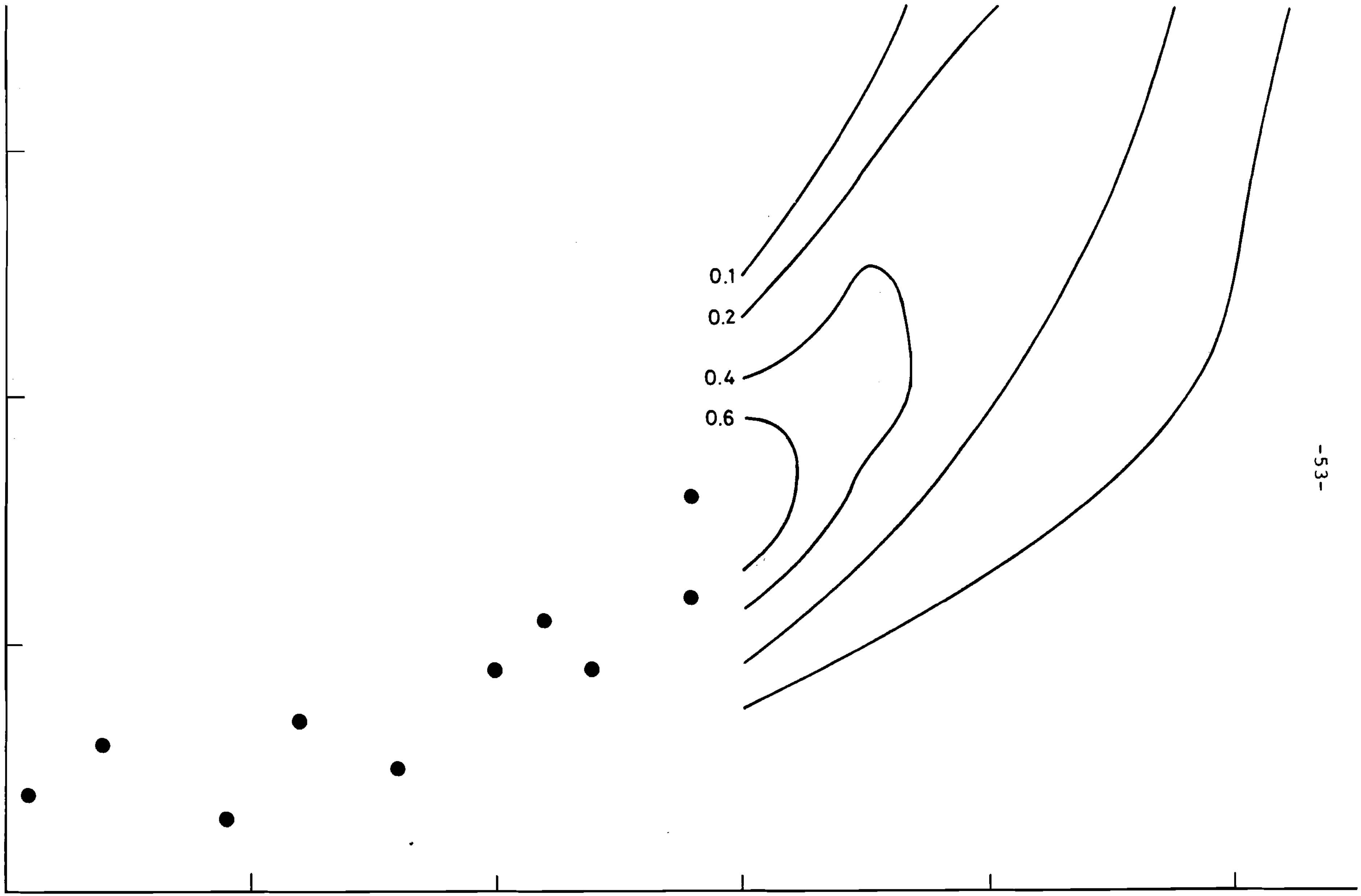


FIGURE B.6: COMPOSITE

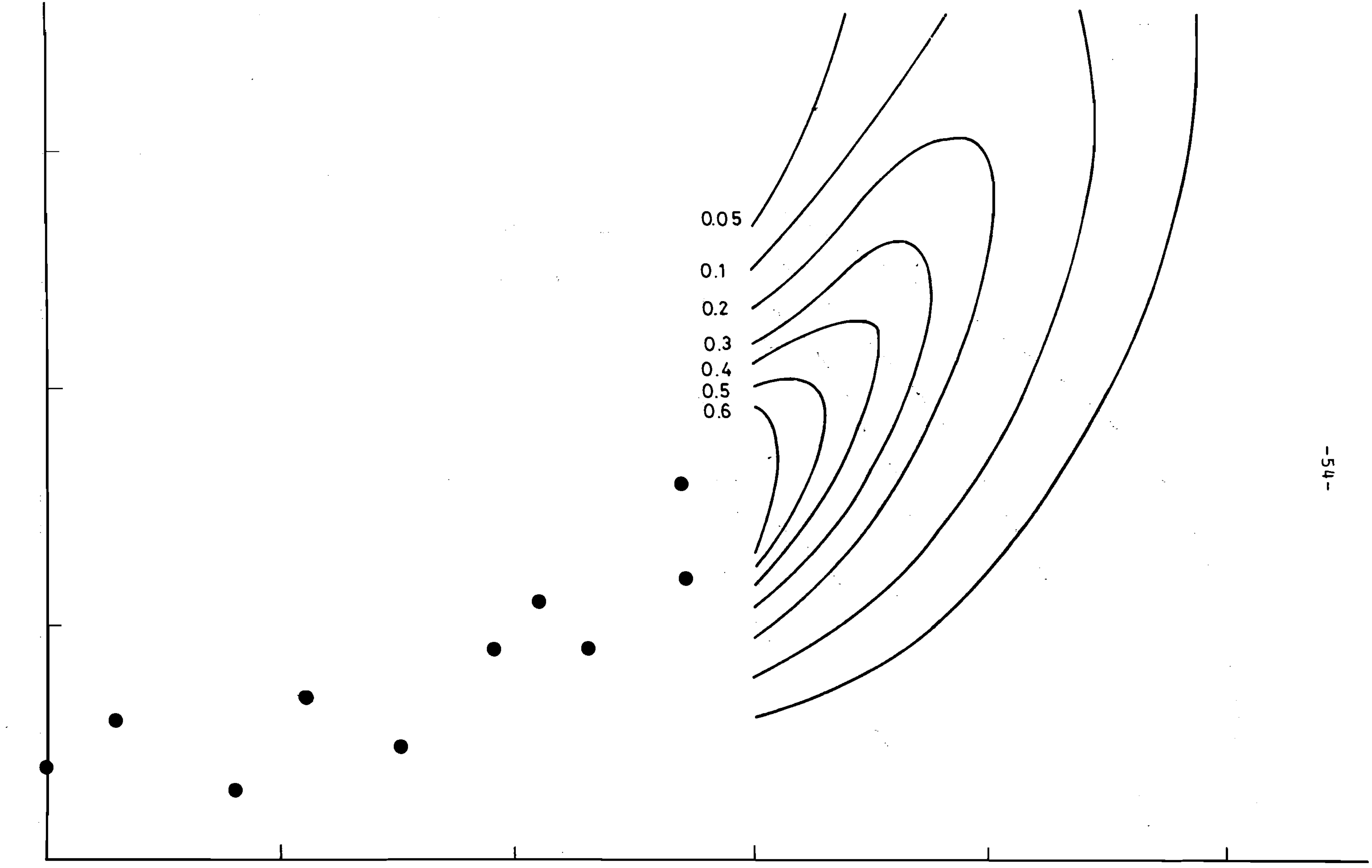


FIGURE B.7: COMPOSITE WITH LENGTH CORRECTION

APPENDIX C
Informed Priors

The use of non-informative or diffuse priors in Bayesian analysis has been a source of controversy, and indeed a point of major criticism of Bayesian methods by members of the frequentist school. This controversy is discussed in several places (e.g., Jeffreys, 1966; Savage, et al., 1963; Zellner, 1971), and so won't be summarized here.

When prior information on feelings does exist, some prior pdf on $\underline{\beta}$ and σ which accounts for this information should be used (i.e., rather than "uninformed" priors). Since the procedure for updating a prior distribution on $\underline{\beta}$ and σ by sample data rapidly becomes intractable unless the shape of the prior distribution is judiciously chosen, one is well advised to select this distribution in coordination with the likelihood function. One such distribution is the conjugate of the likelihood function which has the property of closure under Bayesian updating. That is, a conjugate distribution is one which when updated by the likelihood function yields a distribution of the same family, but with different parameters. Zellner shows that for normal multiple regression the conjugate distribution is

$$f(\underline{\beta},\sigma|\underline{\underline{X}},\underline{y}) \propto \frac{1}{\sigma^{n+1}} \exp\left[-\frac{1}{2\sigma^2}(\underline{y} - \underline{\underline{X}}\underline{\beta})^t(\underline{y} - \underline{\underline{X}}\underline{\beta})\right] , \tag{C1}$$

or the same as the posterior distribution generated using the "uninformed" prior.

Assessment of subjective probabilities in terms of this distribution is clearly complicated, but as a first approximation, marginal distributions of $\underline{\beta}$ and σ might be assessed independently. The marginal distribution of $\underline{\beta}$ is multivariate normal, which for multivariate assessments is easier than most; and the marginal distribution of σ is inverted-gamma, which being univariate is at least straightforward. It is also conceivable that specialized methods of assessment, by sketching ranges and most-probable axes on a map, say, could be developed.

APPENDIX D

Symbol List

C	constant = $\underline{x}_o(\underline{\underline{X}}^t\underline{\underline{X}})^{-1}\ \underline{x}_o$
c	cost of drilling
e	error term
$f^o(\cdot)$	prior probability density function
$f'(\cdot\|\cdot)$	posterior probability density function
g(d)	relationship of production to distance
k	order of polynomial
$L(\cdot\|\cdot)$	likelihood function
M_i	model number i
$p^o(\cdot)$	prior probability
$p'(\cdot\|\cdot)$	posterior probability
pdf	"probability density function"
$Pr[x_o,y_o]$	probability body located at point (x_o,y_o)
S^2	sum of squared errors in regression
$(\underline{\underline{X}},\underline{y})$	data set: known body locations
y'	random location of body centerline given x
$\hat{y}$	expected location of centerline given x
$\underline{z}$	data set = $(\underline{\underline{X}},\underline{y})$
$\underline{\beta} = \{\beta_1,\ldots,\beta_k\}$	regression coefficients
$\hat{\underline{\beta}}$	most likely value of regression coefficients
$\Gamma[\cdot]$	gamma function
σ^2	variance of error about center, and parameter of width distribution

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