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HYDRODYNAMICAL ASPECTS IN THE PROBLEM OF DETERMINING THE HEIGHT OF A DIKE ALONG RIVER REACHES SUBJECT TO FLOOD

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1. The problem of finding the required dike height along a channel or river reach is an important element of the river basin project, and in a simulation system, it is necessary to have a special block for this problem. It should contain the system of models and algorithms for design of the protection constructions. In this paper one of the simplest models of that type will be discussed.

Practically speaking, it is a very usual situation when an investment in a dike's construction is limited, and the given amount is not enough for the complete prevention of the flood under all meteorological conditions. Therefore, the problem of finding forms of the dike $D(x)$ which leads to minimum losses from the flood under the condition of a fixed investment for the dike construction $\hat{L}$ arises.

For an accurate statement of the problem it is necessary to make some assumptions:

1) Hypothesis of stationarity: river flow in a dike system is described by stationary Saint-Venant equations.

2) Hypothesis of complete certainty: stochastic features of the process could be left out of the account.

The first assumption presents the possibility of neglecting the initial step of the flood, and this is valid, for example, in the case of not very big floods or when upstream storage could have a dampening effect. The second assumption is quite usual for any engineering considerations which are oriented to a calculation of particular cases or to analysis of some particular cases.

2. Denote by S the losses from the flood. It will be some function of the amount of the water Q which overflows the dike along an interval $0 \leq x \leq X$

$$S = f(Q(x)) \quad . \tag{1}$$

Here $x \varepsilon [0,X]$ is the distance along the reach of the river or channel from its beginning. Let

$$\dot{Q} = q \quad . \qquad (2)$$

So $q(x)$ is the discharge of the overflow over a dike in the cross section of the river with coordinate x. This value is a function of the height of a dike $D(x)$ and the depth of water in the river $h(x)$, i.e. $q = q(D,h)$. This function is a complex and nonlinear one, having the form shown in Figure 1. We will return to discuss it later.

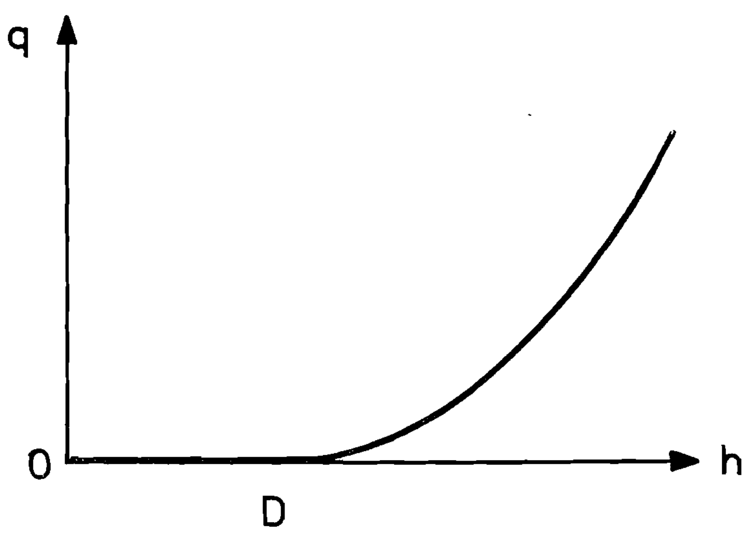

FIGURE 1.

Value h should satisfy the Saint-Venant equations and it depends on the height of a dike and other parameters of the problem. Later we will show how we could get an ordinary differential equation for $h(x)$ from those equations so that

$$\dot{h} = F(D,h,Q) \quad . \qquad (3)$$

Investment for dike construction also could be represented in a differential form

$$\dot{L} = \ell(D,x) \quad . \tag{4}$$

Variables h, Q and L should satisfy the following obvious conditions:

$$\begin{aligned} Q(0) &= 0 \quad , \\ Q(X) &= \text{free} \quad , \\ h(0) &= h_0 \quad , \\ h(X) &= \text{free} \quad , \\ L(0) &= 0 \quad , \\ L(X) &= \hat{L} \quad . \end{aligned} \tag{5}$$

Value h_0 is the depth of the water at the initial cross section of the river reach, a characteristic of the flow regime under consideration. $\hat{L}$ is the total investment for dike construction.

So the problem of finding the height of the dike is reduced to an optimal control problem where the height of the dike plays the role of control variable.

3. The following is an accurate statement of the problem under consideration: find control function D(x) and phase variables Q(x), h(x) and L(x) related by differential equations (2), (3) and (4) that satisfy the boundary conditions (5) such that functional S reaches its minimum value. This is the classical Meier problem and for its solution the classical method of variational calculation could be used.

Let us introduce Lagrange multipliers λ_Q, λ_h and λ_L and construct the Hamiltonian function

$$H = \lambda_Q q(D,h) + \lambda_h F(D,h,Q) + \lambda_L \ell(D) \quad . \tag{6}$$

Now we could write the equation for the impulses

$$\dot{\lambda}_Q = -\lambda_h \frac{\partial F}{\partial Q} \quad . \tag{7}$$

Since there are no restrictions on the function $Q(x)$ and the functional S is defined only by the value of $Q(x)$, the condition of transversality gives the following value of $\lambda_Q(x)$

$$\lambda_Q(x) = - \frac{dS}{dQ} \quad . \tag{8}$$

Then

$$\dot{\lambda}_L = - \frac{\partial H}{\partial L} = 0 \quad , \tag{9}$$

and therefore

$$\lambda_L = \text{const.}$$

Thus λ_L is a free parameter and could be chosen from condition (5).

Function λ_h satisfies the following equation

$$\lambda_h = -\lambda_Q \frac{\partial q}{\partial h} - \lambda_h \frac{\partial F}{\partial h} \quad , \tag{10}$$

and the condition

$$\lambda(X) = 0 \quad . \tag{11}$$

Since there are no restrictions on $D(x)$, we could use the Lagrange necessity condition

$$\frac{\partial H}{\partial D} = 0 \quad .$$

This leads to the following equation for D

$$\lambda_Q \frac{\partial q}{\partial D} + \lambda_h \frac{\partial F}{\partial D} + \lambda_L \frac{\partial \ell}{\partial D} = 0 \tag{12}$$

which defined the D-like function of phase variables, Lagrange multipliers and independent variable x:

$$D = D(\lambda_Q, \lambda_h, \lambda_L, Q, h, L, x) \quad . \tag{13}$$

Thus the problems of finding the shape of the dike, the regime of the flow and losses from the flood are reduced to the boundary value problem: differential equations (2), (3), (4), (7) and (10) with boundary conditions (5), (8) and (11).

To satisfy three conditions on the right end

$$L(X) = \hat{L} \quad , \qquad \lambda_h(X) = 0 \quad , \qquad \lambda_Q(X) = -\frac{dS}{dQ}$$

at our disposal there are three parameters $\lambda_h(0)$, $\lambda_Q(0)$ and constant λ_L. For the solution of this problem a standard Newton method program could be used.

4. Let us specify now the functions in differential equations of the problem. The steady state flow in the channel could be described by Saint-Venant equations

$$u \frac{du}{dx} = g(\Theta - \chi) - g \frac{dh}{dx} \tag{14}$$

$$u \frac{dh}{dx} + h \frac{du}{dx} = \frac{1}{B} q(x) \quad . \tag{15}$$

Here

u = the average velocity of the flow;

B = the width of the channel, which we assume constant;

g = the acceleration of gravity;

Θ = the slope of the channel bed;

χ = the "frictional slope" of the channel, nonlinear function of the velocity u;

$$\chi = \frac{u|u|}{\gamma R^n} \quad ; \tag{16}$$

R = the hydraulic radius equal to the ratio of the cross-section area of the water to the wetted perimeter;

γ and n = positive empirical parameters.

Equation (15) could be reduced to the following form

$$\frac{d}{dx}(uh) = \frac{1}{B} q(x) \quad ,$$

and hence

$$uh = (uh)_{x=0} + \frac{1}{B} Q(x) \quad . \tag{17}$$

The first term in the right-hand part of equation (17) is a constant, which should be given. Denote it by A. Thus

$$u = \frac{1}{h}\left(A + \frac{1}{B} Q\right) \tag{18}$$

and therefore

$$\frac{du}{dx} = \frac{\frac{qh}{B} - \left(A + \frac{Q}{B}\right)\frac{dh}{dx}}{h^2} \quad . \tag{19}$$

After substituting (18) and (19) into equation (14) and resolving for dh/dx we will have

$$\frac{dh}{dx} = \frac{g\Theta - g\dfrac{\left(A + \frac{Q}{B}\right)^2}{\gamma h^2 \left(\frac{Bh}{B + 2h}\right)^n} - q\dfrac{A + \frac{Q}{B}}{h^2 B}}{g - \dfrac{\left(A + \frac{Q}{B}\right)^2}{h^3}} \equiv F(D,h,Q) \quad . \tag{20}$$

5. Let us now consider the function q(h,D). Its qualitative behavior was shown in Figure 1. In engineering practice an empirical formula is used very often. With our notation it has the form

$$q(h,D) = \begin{cases} 0 & \text{if } h < D \\ m_\delta \sqrt{2g}(h - D)^{3/2} & \text{if } h \geq D \end{cases} . \qquad (21)$$

Here m_δ = the empirical coefficient. Equation (21) describes function q as differentiable with respect to its arguments.[1] We should also make a few remarks on the function $\ell(D)$. It is defined by the specifity of the project and the conditions of building the protective construction. Function $\ell(D)$ is always convex and could be approximated by parabola

$$\ell(D) = aD^b \quad ; \quad a > 0 \quad , b > 1 \quad . \qquad (22)$$

6. The presented problem could have different modifications. In particular, it could be formulated as a dual problem: find the minimum investment for dike construction under given losses from a flood, for example, a dike which guarantees zero losses from floods and has minimal investment for construction.

The main parameter of this problem, which defines its peculiarities, is the initial level of the flood, h(0). By changing this parameter and solving variational problems for each of its values, we will find function $S(h_0,\hat{L})$. The character of this function is shown in Figure 2.

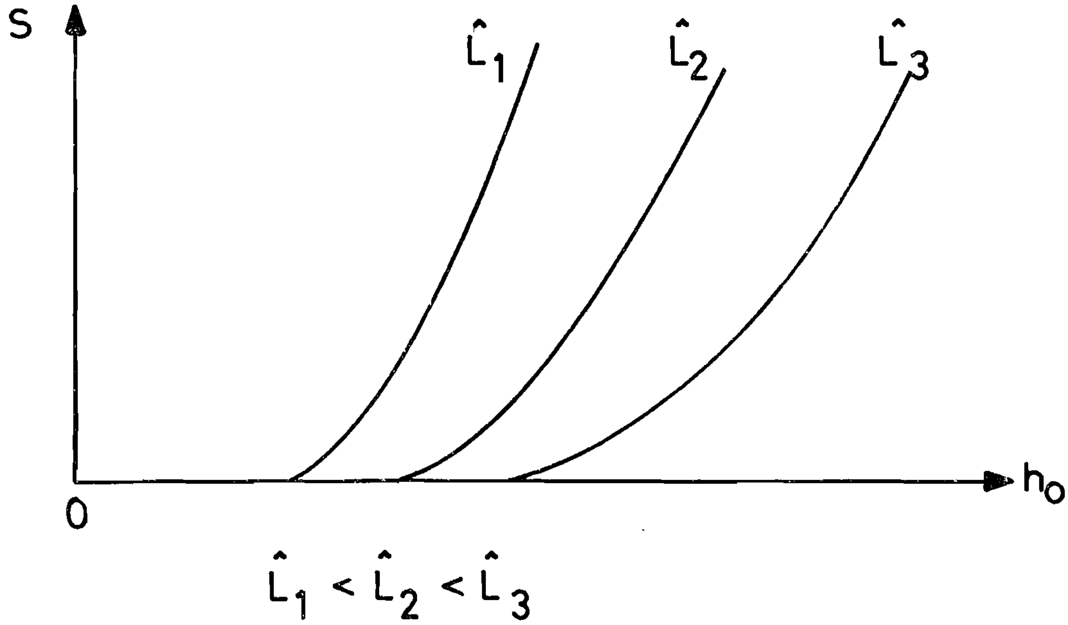

FIGURE 2.

[1]For a more detailed explanation of this formula see I.I. Agroskin, G.B. Dmitriev and F.I. Pikalov, "Energy", in Hydraulics, Moscow, Leningrad, 1964, pp. 256-257.

By solving dual problems we could find functions $\hat{L}(S,h_0)$. Such functions could serve as a base for making decisions on choosing the shape of the dike. The presented model does not need special software. Numerical solutions could be found by using the standard Newton method program.