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Mathematical Analysis of a Model of Interacting Economics with Absorptive Capacities

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Preface

The paper presents a mathematical analysis of an endogenous growth model for two economies with absorptive capacities. The model was proposed by Gernot Hutschenreiter (the Austrian Institute of Economic Research, WIFO), Yuri Kaniovski (IIASA, TED Project), and Arkadii Kryazhimskii (IIASA, DYN Project), and generalizes the one studied in Hutschenreiter et. al., 1996 [2]. The paper has been prepared in the course of the YSSP-1996 at IIASA as a part of a joint research in DYN and TED Projects. The research was partly supported by the Russian Fund of Fundamental Investigations (RFFI), project N 96-01-00890.

Contents

Mathematical Analysis of a Model of Interacting Economics with Absorptive Capacities

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1 Introduction

We study a coupled endogenous model of economic growth. It describes two interacting economies each with an absorptive capacity. Our analysis refers to the endogenous growth theory of Grossman and Helpman [1] who introduced a model for isolated (autarkic) development. The model associates a process of growth of the knowledge stock in an autarkic country A with the two-dimensional differential equation

$$\begin{aligned} \dot{n}^A &= \left[c^A n^A - \frac{\alpha}{v^A}\right]_+, \\ \dot{v}^A &= \rho v^A - \frac{1-\alpha}{n^A}. \end{aligned} \tag{1.1}$$

Here $(n^A, v^A) = (n^A(t), v^A(t)) \in \mathbb{R}^2$ describes the economic state of country A at time t; $n^A(t)$ characterizes the *measure of intermediate goods* invented before time t; $v^A(t)$ reflects the *value of the representative firm* or the cost for developing a unit of scientific production (blueprint). Notation $[p]_+$ stands for the maximum of p and 0. Parameters c^A, ρ, α are arbitrary positive reals. They have the following economic sense: $c^A = L^A/a$ where L^A is the endogenous *labor supply*, $1/a$ stands for the *amount of labor* needed to develop a unit of new product; ρ is the *discount rate*; α is related to the *elasticity of the substitution*; it is assumed that $\alpha \in (0,1)$ (for details, see [1]).

The analysis in [1] is primarily concerned with a *steady solution* to (1.1) characterized by the fact that the *aggregate equity value* $V^A = 1/n^A v^A$ remains constant; equivalently

$$n^A(t)v^A(t) \equiv \text{const.} \tag{1.2}$$

For a steady solution to (1.1) one has

$$n^A(t) = n^A(0)e^{g^A t} \tag{1.3}$$

where the coefficient

$$g^A = (1-\alpha)c^A - \alpha\rho \tag{1.4}$$

represents the *steady state growth rate* of the autarkic economy A. A remarkable feature of the steady solution $(n^A(t), v^A(t))$ is that it meets the so-called *perfect-foresight* condition; the latter claims that the stock market sets the value of the firm at time t equal to the present value of its steam of profits subsequent to t:

$$v^A(t) = \int_t^\infty e^{-\rho(t-s)} \frac{1-\alpha}{n^A(s)} ds. \tag{1.5}$$

The notion of *absorptive capacity* was introduced in Hutschenreiter et. al. [2] to cover the situation of coupled economies in countries A and B, where A, a technological leader with greater labor supply L^A, is an autarky and B, a technological follower with smaller labor supply L^B, is able to absorb knowledge originated in A and use it in its own manufacturing. Country A is assumed to have reached the steady state growth rate,

$$\dot{n}^A = g^A n^A. \tag{1.6}$$

The economic growth in country B is due to utilization of knowledge developed domestically and that absorbed from the knowledge stock of country A:

$$\begin{aligned} \dot{n}^B &= \left[c^B(n^B + \gamma_B(n^B)n^A) - \frac{\alpha}{v^B}\right]_+, \\ \dot{v}^B &= \rho v^B - \frac{1-\alpha}{n^B}. \end{aligned} \tag{1.7}$$

Here $c^B = L^B/a$ and function $\gamma_B(n^B)$ reflects the absorptive capacity of B, i.e. the share of the knowledge developed in A which is absorbed by B. The absorptive capacity $\gamma_B(n^B)$ grows as n^B increases.

It was stated in [2] that, under appropriate relations between the parameters, there exist solutions to (1.6)–(1.7) such that

$$n^B(t) = n^B(0)e^{g^A t}(1 + O(t)) \tag{1.8}$$

where $O(t) \to 0$ as $t \to +\infty$. In other words, the growth rate of the follower reaches asymptotically that of the leader. Besides, the aggregate equity value in B in the long run approaches that in country A,

$$\lim_{t\to+\infty} \frac{1}{n^B(t)v^B(t)} = \lim_{t\to+\infty} \frac{1}{n^A(t)v^A(t)} = c^A + \rho, \tag{1.9}$$

and the perfect-foresight condition holds true for both economies. Rigorous proves are given in Kryazhimskii [3].

In the present paper we explore a symmetric case when both economies have absorptive capacities (this setting was suggested by G. Hutschenreiter). We focus primarily on a formal part of the theory postponing the accurate justification of the model and the complete conceptual analysis of the results to the future.

Consider a growth model for countries A and B under the assumption that each of them is able to absorb the technological advances developed by the other one. Thus, we arrive at the following system of differential equations:

$$\begin{aligned} \dot{n}^A &= \left[c^A(n^A + \gamma_A(n^A)n^B) - \frac{\alpha_1}{v^A}\right]_+, \\ \dot{v}^A &= \rho_1 v^A - \frac{1-\alpha_1}{n^A}, \\ \dot{n}^B &= \left[c^B(n^B + \gamma_B(n^B)n^A) - \frac{\alpha_2}{v^B}\right]_+, \\ \dot{v}^B &= \rho_2 v^B - \frac{1-\alpha_2}{n^B}. \end{aligned} \tag{1.10}$$

Here $\gamma_i(n)$, $i = A, B$, are nonnegative monotonically increasing functions representing absorptive capacities of countries A and B. We assume that

$$\lim_{n\to+\infty} \gamma_i(n) = \Gamma_i \le 1, \qquad i = A, B,$$

Our principal mathematical problem is to find sufficient conditions for the existence of solutions $n^A(t)$, $v^A(t)$, $n^B(t)$, $v^B(t)$ to system (1.10) such that

$$\lim_{t\to+\infty} \frac{n^A(t)}{n^B(t)} = r \tag{1.11}$$

with $r \neq 0, \infty$. We shall refer to such solutions as *balanced solutions*. Along a balanced solution the growth rates of countries A and B are asymptotically equalized. An additional restriction on balanced solutions will be the asymptotic counterpart of the steady growth condition (1.2):

$$\lim_{t\to+\infty} n^A(t)v^A(t) = \sigma_1, \quad \lim_{t\to+\infty} n^B(t)v^B(t) = \sigma_2. \tag{1.12}$$

Let us give a brief description of our mathematical approach to identifying balanced solutions including those satisfying the asymptotic steady growth condition (1.12). First, we select regions where n^A and n^B increase to infinity along the solutions to (1.10). This allows us to reduce the number of variables by taking n^A for the independent variable (a new "time" in (1.10) instead of t). Thus, we arrive at an equivalent three-dimensional system. Next, we pass to the inverse $t = 1/n^A$ reducing therefore the original problem to the asymptotic analysis, with $t \to +0$, of a three-dimensional differential equation of the form

$$\dot{\vec{x}} = \frac{F(t,\vec{x})}{t}. \tag{1.13}$$

This equation has a singularity in the right-hand side — the state velocities are not bounded in a vicinity of the plane $t = 0$. We need to learn how a solution to (1.13) can reach the plane $t = 0$ starting at a point (t_0, x_0) in the half-space $t > 0$. We use a modified technique of resolution of singularities and end up with a complete theory of low codimension singularities for differential equation (1.13). The theory is presented in section 2. In subsection 2.1 one-dimensional equations are explored. We give a detailed characterization of solutions having finite limits as $t \to +0$ in terms of Taylor's expansions of $F(t, \vec{x})$. In subsection 2.2 the technique is generalized for the case of an arbitrary dimension.

Applications to the interactive growth model (1.10) are given in section 3. Our principal results are as follows.

1. For an open subset of parameters of system (1.10) there exists a solution to (1.10) satisfying the relations:

$$\lim_{t\to+\infty} \frac{n^B(t)}{n^A(t)} = x_0, \quad \lim_{t\to+\infty} n^A(t)v^A(t) = y_0, \quad \lim_{t\to+\infty} n^B(t)v^B(t) = x_0 z_0. \tag{1.14}$$

   Here $(x_0, y_0, z_0) \in \mathbb{R}^3_+$ is uniquely specified as a solution of

$$\frac{1-\alpha_2}{x(\rho_2 - \alpha_1\rho_1 + c^A(1-\alpha_1)(1+\Gamma_1 x))} = \frac{1}{\rho_2 x + c^B(x+\Gamma_2)},$$
$$y = \frac{1}{\rho_1 + c^A(1+\Gamma_A x)},$$
$$z = \frac{1}{\rho_2 x + c^B(x+\Gamma_B)}.$$

2. This solution has the asymptotics

$$n^i(t) = n^i(0)e^{\mu^i t}(1 + O(t)), \quad i = A, B$$

where

$$\mu^A = (1-\alpha_1)c^A(1 + \Gamma_A x_0) - \alpha_1\rho_1 = (1-\alpha_2)c^B(1+\Gamma_B/x_0) - \alpha_2\rho_2 = \mu^B.$$

The solution is therefore balanced, i.e. meets (1.11)–(1.12) with $r \neq 0, \infty$. Asymptotically it leads to the same growth rate $\mu_A = \mu_B$ in countries A and B. We emphasize that this common growth rate (arising in interacting economies) exceeds each of the autarkic growth rates:

$$\mu^A > g^A = (1-\alpha_1)c^A - \alpha_1\rho_1, \quad \mu^B > g^B = (1-\alpha_2)c^B - \alpha_2\rho_2.$$

3. Along the above balanced solution the perfect–foresight condition is valid for each country,

$$v^A(t) = \int_t^\infty e^{\rho_1(s-t)}\frac{1-\alpha_1}{n^A(s)}ds, \quad v^b(t) = \int_t^\infty e^{\rho_2(s-t)}\frac{1-\alpha_2}{n^B(s)}ds.$$

We conclude with a sensitivity analysis of limit growth rates μ^A, μ^B with respect to variations of parameters.

2 Ordinary differential equations with unbounded right–hand sides

2.1 One–dimensional equations

In this section we develop a technique for qualitative analysis of differential equations with unbounded right–hand sides. In the next section, the technique will be applied to the interactive economic growth model (1.10). We treat an ordinary differential equation with a singularity in the right–hand side:

$$\frac{dx}{dt} = \frac{F(t,x)}{t}. \tag{2.1}$$

Here $(t,x) \in \mathbb{R}^2$, function $F(t,x)$ is smooth enough (say, C^0 in both variables and Lipschitz in x). What we wish to learn is in what way a solution to (2.1) can reach the border line $t = 0$ if it starts at some point (t_0, x_0) in the half–plane $t > 0$ (we assume the reverse time current).

The simplest case is the following. Assume that $F(0, x_0) \neq 0$ for some $x_0 \in \mathbb{R}^1$, e.g. $F(0, x_0) > 0$. Denote

$$D_\epsilon = \{(t,x)\,|\,0 < t < \epsilon,\ |x - x_0| < \epsilon\}.$$

Given a pair $(t_1, x_1) \in D_\epsilon$, let $\bar{x}(t)$ be a solution to (2.1) such that $\bar{x}(t_1) = x_1$.

Lemma 2.1. *Let* $F(t,x) \ge C > 0$ *if* $(t,x) \in D_\epsilon$, *then there exists* $\tilde{t} \in (e^{-2\epsilon/C}t_1, t_1)$ *such that* $\bar{x}(\tilde{t}) = x_0 - \epsilon$.

Remark 2.1. Since $F(t,x)$ is a continuous function at least, the set of those ϵ that meet the conditions of Lemma 2.1 is not empty.

Remark 2.2. Lemma 2.1 claims that the trajectory $\bar{x}(t)$ inevitably hits the lower boundary of the open square D_ϵ after staying in D_ϵ (in the reverse time) no longer than $t_1(1 - e^{-2\epsilon/C})$ (see Fig. 1).

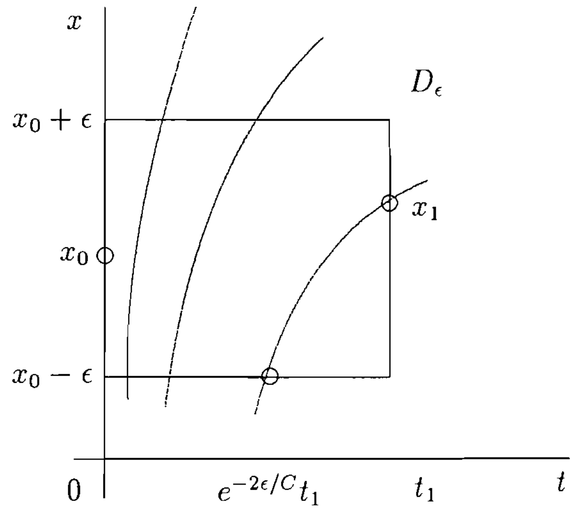

Fig. 1.

Proof of Lemma 2.1. We have $dx/dt > 0$ provided that $(t,x) \in D_\epsilon$, i.e. $\bar{x}(t)$ is a strictly increasing function. Let $t' < t_1$ be any instant such that $\bar{x}(t) \in D_\epsilon$ for $t \in (t', t_1)$. It follows from (2.1) that

$$\frac{d\bar{x}(t)}{F(t,\bar{x}(t))} = \frac{dt}{t},$$

or, equivalently,

$$\int_{t'}^{t_1} \frac{d\bar{x}(t)}{F(t,\bar{x}(t))} = \int_{t'}^{t_1} \frac{dt}{t}.$$

Hence,

$$\ln t_1 - \ln t' \le \frac{1}{C}\int_{t'}^{t_1} d\bar{x}(t) = \frac{1}{C}(\bar{x}(t_1) - \bar{x}(t')).$$

Since $\bar{x}(t_1) \le x_0 + \epsilon$ and $\bar{x}(t') \ge x_0 - \epsilon$, it follows that $\ln t_1 - \ln t' \le 2\epsilon/C$, or $t' \ge t_1 e^{-2\epsilon/C}$, which implies the statement to be proved. Q.E.D.

Much more complicated behavior of solutions to (2.1) can be observed in the vicinity of a point $(0, x_0)$ where $F(0, x_0) = 0$. Everywhere below we deal with simple zeros of function $F(0,x)$, i.e. when $F(0,x_0) = 0$, $\frac{\partial F}{\partial x}(0,x_0) \ne 0$. In particular, the latter inequality implies function $F(0,x)$ to change sign any time as x runs through an arbitrary interval including x_0. Thus, we exclude the zeroes of $F(0,\cdot)$ which are touchpoints.

We need the following auxiliary statement.

Lemma 2.2. *Given two points* x_1 *and* x_2 *and a time instant* $\bar{t}$ *such that*

$$\min_{0\le t\le \bar t} F(t,x_2) > 0, \qquad \max_{0\le t\le \bar t} F(t,x_1) < 0.$$

Let $\bar x(t)$ be a solution to (2.1) such that $\bar x(\bar t) = \bar x \in (x_1, x_2)$. Then $\bar x(t)$, $t \in (0,\bar t)$, has no intersections with the lines $x = x_1$ and $x = x_2$, i.e. for any $t \in (0,\bar t)$ we have $\bar x(t) \in (x_1, x_2)$.

Proof. Denote $\{A_+ = \{t \in (0,\bar t) : \bar x(t) = x_2\}$. Assume that A_+ is not empty and set $t^* = \sup\{t : t \in A_+\}$. It is easily seen that $t^* < \bar t$. Due to the inequality $F(t^*, x(t^*)) > 0$, the function $\bar x(t)$ increases at t^*, so there exists $t^{**} \in (t^*, \bar t)$ such as $\bar x(t^{**}) > x_2$. This contradicts the definition of t^*. The case $x = x_1$ can be treated in the same manner. Q.E.D.

Proposition 2.1. *Assume that there exists $\epsilon_0 > 0$ such that $F(0,x) > 0$ if $x \in (x_0, x_0+\epsilon_0)$ and $F(0,x) < 0$ if $x \in (x_0-\epsilon_0, x_0)$. Then there exists D_ϵ such that for any point $(\bar t, \bar x) \in D_\epsilon$ the solution $\bar x(t)$ to (2.1) which is passing through $(\bar t, \bar x)$ tends to x_0 as t tends to $+0$, i.e. $\lim_{t\to+0} \bar x(t) = x_0$, (see Fig. 2.)*

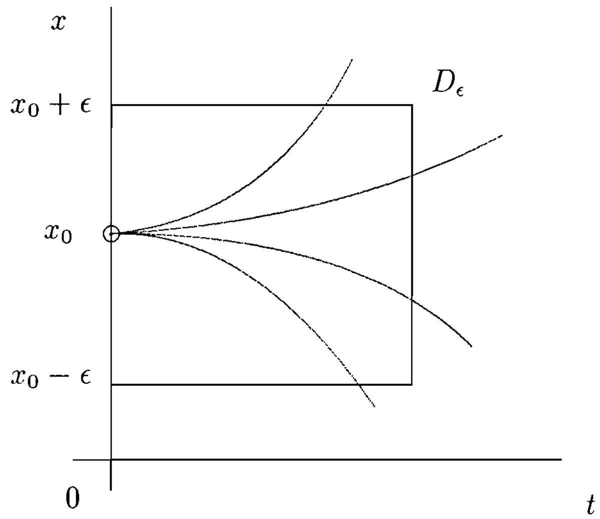

Fig. 2.

Proof. Let us choose $\epsilon < \epsilon_0$ so small that $F(t, x_0+\epsilon) > 0$, $F(t, x_0-\epsilon) < 0$ if $0 < t < \epsilon$. Let $(\bar t, \bar x)$ be an arbitrary point in D_ϵ and $\bar x(t)$ be the trajectory of (2.1) such as $\bar x(\bar t) = \bar x$. Denote by $\mathfrak{A}$ the set of α–limit points of $\bar x(t)$. By the definition of α–limit set, $x \in \mathfrak{A}$ if and only if $\liminf_{t\to+0} |\bar x(t) - x| = 0$. We assert that $\mathfrak{A} = \{x_0\}$.

As a consequence of Lemma 2.2, $\mathfrak{A}$ is not empty and belongs to the interval $x \in (x_0-\epsilon, x_0+\epsilon)$. Suppose to the contrary that $\mathfrak{A}$ contains a point $x_1 \ne x_0$, say $x_1 > x_0$. Then $F(t,x) > 0$ in a neighborhood of $(0, x_1)$. Due to the definition of $\mathfrak{A}$, for any $\delta > 0$ there exists $0 < \tilde t < \delta$ such that $|\bar x(\tilde t) - x_1| < \delta$. If $\bar x(\tilde t) < x_1$, it follows from Lemma 2.2 that the closure of the set $\bigcup_{t\in(0,\tilde t)} \bar x(t)$ belongs to the rectangle

$$\{(t,x)\,|\,0 \le t \le \delta, \quad x_0 - \epsilon \le x \le \bar x(\tilde t)\}$$

and does not contain x_1. Hence, $\bar x(t) \ge x_1$ for any $0 < t \le \tilde t$. If $\mathfrak{A}$ contains also some point $x_2 > x_1$, then a similar argument leads to the conclusion that $\bar x(t) \ge x_2$ on some interval $t \in (0, \delta_1)$ and $x_1 \notin \mathfrak{A}$ therefore. It means that $\mathfrak{A}$ consists of a single point, i.e. $\mathfrak{A} = \{x_1\}$, which contradicts Lemma 2.1. Q.E.D.

Proposition 2.2. *Assume that $F(0,x) < 0$ if $x \in (x_0, x_0 + \epsilon_0)$ and $F(0,x) > 0$ if $x \in (x_0 - \epsilon_0, x_0)$ for some $\epsilon_0 > 0$. Then there exists a pair $(\bar{x}, \bar{t})$ with the following property: the trajectory $\bar{x}(t)$ of (2.1), $\bar{x}(\bar{t}) = \bar{x}$, tends to x_0 as t tends to $+0$.*

Proof. Let us fix x_1 and x_2, $x_1 < x_2$ such that

1) $F(0,x_1) > 0, \quad F(0,x_2) < 0$;

2) there are no zeroes of $F(0,x)$ on (x_1, x_2) except for x_0.

Let us also fix a small $\bar{t} > 0$ such that functions $F(\cdot, x_i)$ do not change sign on $(0, \bar{t})$, $i = 1, 2$. Consider a subset A_+ of the interval (x_1, x_2) defined as follows. We say that $\bar{x} \in A_+$ if and only if the solution $\bar{x}(t)$ to (2.1) with $\bar{x}(\bar{t}) = \bar{x}$ intersects the upper boundary $x = x_2$ at some $t \in (0, \bar{t})$. If we replace x_2 by x_1 in this definition, we define another subset of (x_1, x_2) which we call A_-. Since solutions to (2.1) are continuous functions of initial data and all trajectories in the corresponding neighborhood intersect the straightlines $x = x_i$ $(i = 1, 2)$ at non-zero angles with non-zero velocities, it follows that both A_+ and A_- are open. It can be easily verified that both A_+ and A_- are one-connected sets, hence their supplement in (x_1, x_2) is some non-empty segment $[x_-, x_+] \subset (x_1, x_2)$. We assert that an arbitrary solution $\bar{x}(t)$ to (2.1) with $\bar{x}(\bar{t}) \in [x_-, x_+]$ tends to x_0 as t tends to $+0$.

Being intersected with a line $x = \text{const}$ where $F(t,x)$ does not change sign, a solution to (2.1) cannot return to this line in the reverse time. This idea has been already used in the proofs of Lemma 2.2 and Proposition 2.1. Repeating step by step the proof of Proposition 2.1 one can easily show that the α-limit set of $\bar{x}(t)$ is non-empty, that it cannot contain two distinct points and, consequently, that it coincides with x_0 (see Fig. 3). Q.E.D.

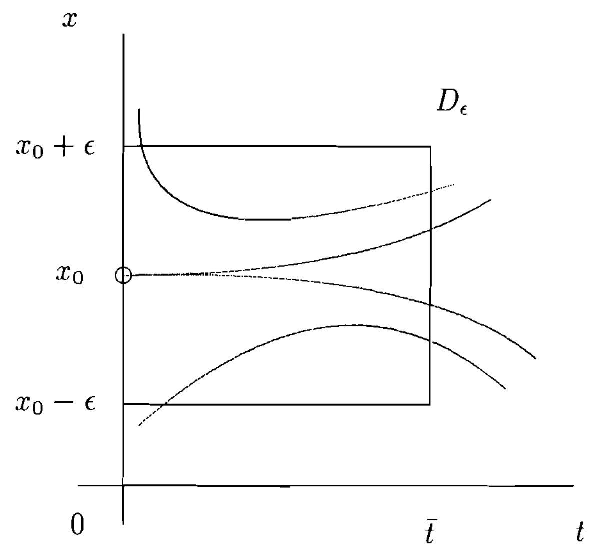

Fig. 3.

To clarify the behavior of solutions to (2.1) in a vicinity of a point $(0, x_0)$ in which $F(0, x_0) = 0$, one needs further assumptions regarding $F(t,x)$. Assume that $F(t,x)$ can be represented in the following form:

$$F(t,x) = f(x) + t\,g(x) + t^2\,h(t,x), \tag{2.2}$$

where $h(t,x)$ is a C^0-function in (t,x) and a C^1-function in x, $g(x)$ is C^1, and $f(x)$ is a C^2-function. Representation (2.2) can be easily obtained from Taylor's formula if we apply it to an arbitrary C^2-function of (t,x). (The restrictions on smooth properties of $F(t,x)$ can be slightly weakened indeed, but it is not principal.)

Now (2.1) is equivalent to the following system

$$\frac{dx}{dt} = \frac{f(x)}{t} + g(x) + th(t,x). \tag{2.3}$$

Let us consider the point x_0 such that

$$f(x_0) = 0, \quad \frac{\partial f}{\partial x}(x_0) = \alpha, \quad g(x_0) = \beta.$$

A standard method of exploring systems with singularities is so-called "blowing-up procedure" or "resolution of singularities" — transfer to a special coordinate system where a one-to-one correspondence is lost. These new coordinates map the point in which the system degenerates on an entire manifold in the new space. Let us introduce instead of (t, x) the coordinates (t, y) as follows:

$$x = x_0 + ty,$$

hence, $y = (x - x_0)/t$. The point $t = 0$, $x = x_0$ is associated to the straight-line $t = 0$, $y \in \mathbb{R}^1$ on the plane (t, y). System (2.3) converts now to the following one

$$\frac{dy}{dt} = -\frac{y}{t} + \frac{1}{t}\left(\frac{f(x_0 + ty)}{t} + g(x_0 + ty) + th(t, x_0 + ty)\right),$$

or, which, in its turn, is equivalent to the system

$$\frac{dy}{dt} = \frac{(\alpha - 1)y + \beta}{t} + r(t,y), \tag{2.4}$$

for $r(t,y) \stackrel{\text{def}}{=} h(t, x_0+ty) + \frac{g(x_0 + ty) - \beta}{t} + \frac{f(x_0 + ty) - \alpha t y}{t^2}$ being a C^0-function in (t, y) and C^1 in y on the half-space $t \geq 0$. (The derivatives at the line $t = 0$ are understood as limits as $t \to +0$ of corresponding derivatives for $t > 0$.) We shall assume that $r(t, y)$ is uniformly bounded, i.e.

$$|r(t,y)| \leq r_1 \quad \text{for all} \quad (t,y) \quad \text{in the half-plane} \quad t > 0,$$

because only this case is important for all that follows. We explore here all non-degenerate cases i.e. all possible relationships between (α, β) except for the cases $(\alpha = 1, \beta = 0)$ and $\alpha = 0$.

Case 1. $\alpha = 1$, $\beta \neq 0$. A general solution to the equation $dy/dt = \beta/t$ can be written as $y = \beta \ln t + C$. If we consider C as a function of t, i.e. $y = \beta \ln t + C(t)$, and substitute the latter relation for y in the equation $\dot{y} = \beta/t + r(t, y)$, we arrive at the following equation for $C(t)$:

$$\frac{dC}{dt} = r(t, \beta \ln t + C).$$

Function $r(\cdot, \cdot)$ being uniformly bounded, we get

$$|C(t) - C(t_0)| = \int_{t_0}^{t} |r(t, \beta \ln t + C(t))|\, dt \leq r_1(t - t_0),$$
$$y(t) = \beta \ln t + C(t_0) + \xi(t_0, t)(t - t_0),$$

where $t_0 > 0$ is some fixed initial time moment, $C(t_0) = y_0 - \beta \ln t_0$, and the absolute value of function $\xi(t_0, t)$ does not exceed r_1. The corresponding solutions to (2.3) reads

$$x(t) = x_0 + \beta t \ln t + tC(t),$$

where $C(t)$ is bounded. Thus, any trajectory $x(t)$ approaches x_0 as t tends to $+0$, at the same time its derivative tends to $-\infty$ if $\beta > 0$. The behavior of this family of solutions (for $\beta > 0$) is given in Fig. 4.

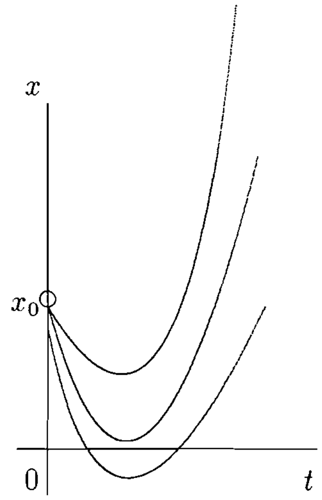

Fig. 4.

Case 2. $\alpha \neq 1$. In the vicinity of points $(0, y_0)$, $y_0 \neq -\beta/(\alpha-1)$, the behavior of solutions to (2.4) is described by Lemma 2.2. Consider the point $y_0 = -\beta/(\alpha-1)$. A general solution to the homogeneous system

$$\frac{dy}{dt} = \frac{(\alpha-1)y + \beta}{t}$$

can be written as

$$y = Ct^{\alpha-1} - \frac{\beta}{\alpha-1}. \tag{2.5}$$

Supposing $C = C(t)$ and substituting (2.5) into (2.4), we get

$$\frac{dC}{dt} = \frac{r\left(t, Ct^{\alpha-1} - \dfrac{\beta}{\alpha-1}\right)}{t^{\alpha-1}}.$$

Integration of the latter relation yields

$$\int_{t_0}^{t} dC = \int_{t_0}^{t} \frac{r\,dt}{t^{\alpha-1}}.$$

Case 2.(i) : $\alpha > 1$, $\alpha \neq 2$. We have

$$|C(t) - C(t_0)| \leq r_1 \left| \frac{t^{2-\alpha}}{2-\alpha} - \frac{t_0^{2-\alpha}}{2-\alpha} \right|,$$

hence,

$$|C(t) - C(t_0)| \le \frac{\text{const}}{t^{\alpha-2}},$$
$$y(t) = C_0 t^{\alpha-1} - \frac{\beta}{\alpha-1} + O(t),$$
$$x(t) = x_0 - \frac{\beta t}{\alpha-1} + C_0 t^{\alpha} + O(t^2).$$

In particular, all $x(t)$ are tangent to the line $x = x_0 - \beta t/(\alpha-1)$ at the point $t = 0$, $x = x_0$ (see Fig. 5).

Case 2.(ii) : $\alpha = 2$. Now

$$|C(t) - C(t_0)| \le r_1 |\ln t - \ln t_0|,$$
$$y(t) = -\frac{\beta}{\alpha-1} + O(t \ln t),$$
$$x(t) = x_0 - \frac{\beta t}{\alpha-1} + O(t^2 \ln t).$$

The phase portrait of $x(t)$ is the same as in Fig. 5.

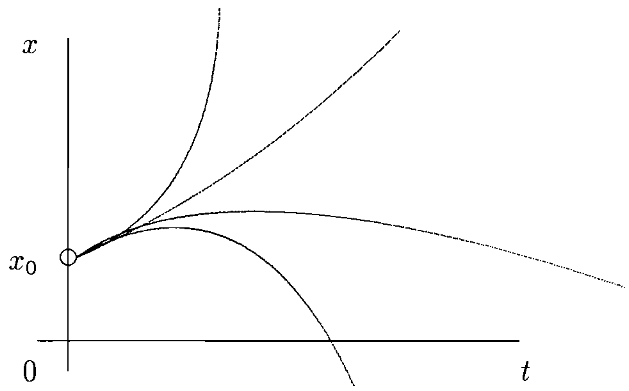

Fig. 5.

Case 2.(iii) : $0 < \alpha < 1$. Consider the integral equation:

$$C(t_1) - C(t_0) = \int_{t_0}^{t_1} t^{1-\alpha} r \, dt.$$

Since function $r(\cdot,\cdot)$ is uniformly bounded in D_ϵ, then, if the corresponding solution does not leave D_ϵ, the integral $\int_{\varepsilon_1}^{\varepsilon_2} t^{1-\alpha} r \, dt$ can be uniformly upper estimated by const $\cdot |\varepsilon_2 - \varepsilon_1|$. This implies both convergence of the integral $\int_0^{t_1} t^{1-\alpha} r \, dt$ and existence of the limit of $C(t_0)$ as $t_0 \to +0$.

Consider the mapping $\mathbb{R}^1 \to \mathbb{R}^1$ which relates $C(t_1)$ with $C(0)$ for some small t_1,

$$C(t_1) \longrightarrow C(0) = C(t_1) - \int_0^{t_1} t^{1-\alpha} r \, dt.$$

As far as the corresponding solution does not leave D_ϵ the derivative

$$\frac{\partial r(t, C(t_1))}{\partial C(t_1)} = \frac{\partial r}{\partial y} \cdot \frac{\partial y(t)}{\partial y(t_1)} \cdot \frac{\partial y(t_1)}{\partial C(t_1)} = O(1) t_1^{\alpha-1}$$

can be uniformly upper estimated. This implies that

$$\frac{\partial C(0)}{\partial C(t_1)} = 1 + O(t_1)$$

and by Implicit Function Theorem the mapping is reversible. Now we have $C(t) = C_0 + t^{2-\alpha}\xi(C_0, t)$, where $\xi(C_0, t)$ and its derivative in t are bounded as $t \to +0$,

$$y(t) = \frac{C_0}{t^{1-\alpha}} - \frac{\beta}{\alpha - 1} + \xi(C_0, t)t,$$

$$x(t) = x_0 + C_0 t^\alpha - \frac{\beta}{\alpha - 1}t + \xi(C_0, t)t^2. \tag{2.6}$$

All solutions (except for the single one which corresponds to the value $C_0 = 0$) have a vertical tangent line at the point $(0, x_0)$ (see Fig. 6).

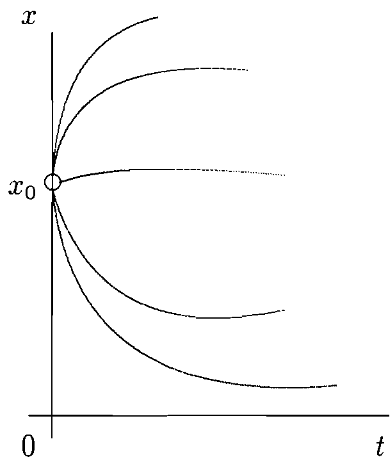

Fig. 6.

Case 2.(iv) : $\alpha < 0$. The representation (2.6) is still valid. There is a single solution with a finite limit (the one which corresponds to the value $C_0 = 0$) and all others leave the neighborhood of the point $(0, x_0)$, see Fig. 7.

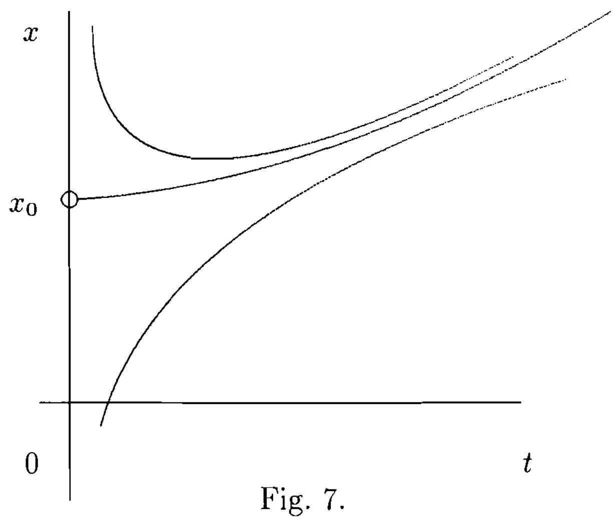

Fig. 7.

Let us put together the results we have obtained in the following proposition.

Proposition 2.3. *Let $F(0, x_0) = 0$ and $\frac{\partial F}{\partial x}(0, x_0) = \alpha$, $g(x_0) = \beta$. There exists $\epsilon > 0$ that solutions to (2.3) with initial data in D_ϵ have the following property as $t \to +0$:*

1) $\alpha = 1$, $\beta \neq 0$, *then being tangent to the vertical axis at* $(0, x_0)$ *the solutions approach* $(0, x_0)$*;*

2) $\alpha > 1$, *then the solutions tend to* $(0, x_0)$ *being tangent to the line* $x = -\beta t/(\alpha - 1)$ *at the limit point;*

3) $0 < \alpha < 1$, *then the solutions tend to* $(0, x_0)$ *and all of them except for a single one have the vertical tangent line at the limit point;*

4) $\alpha < 0$, *then there exists a single solution approaching* $(0, x_0)$, *all others leave the neighborhood* D_ϵ *in the reverse time.*

2.2 Multi–dimensional equations

Up to now, we have dealt with one–dimensional equations with unbounded right–hand sides. Let us switch now to the multidimensional case. Consider the system

$$\dot{x} = \frac{1}{t} F(t, x). \tag{2.7}$$

Here $x \in \mathbb{R}^n$, $t > 0$, $F \in C^1$. Let $F(0, x_0) \neq 0$ at a point $x_0 \in \mathbb{R}^n$. Denote $D_\epsilon = \{(t, x) \,|\, 0 < t < \epsilon,\ \|x - x_0\| < \epsilon\}$ where $\|\cdot\|$ means the standard Euclidean norm in $\mathbb{R}^n$, $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$.

Lemma 2.3. *If* $F(0, x_0) \neq 0$ *then there exists* $\epsilon > 0$ *such that any solution* $\bar{x}(t)$ *to (2.7),* $(\bar{t}, \bar{x}(\bar{t})) \in D_\epsilon$, *leaves* D_ϵ *in the reverse time during a period not greater than* $\bar{t}(1 - e^{-2\epsilon/F^*})$ *where* $F^* = \sup_{i=1,\ldots,n} \inf_{(t,x)\in D_\epsilon} |F_i(t, x)|$. *In other words, if* $\bar{x}(t) \in D_\epsilon$ *at* $t \in (t_1, \bar{t})$ *then* $t_1 \geq \bar{t} e^{-2\epsilon/F^*}$.

Proof. Straightforward integration of (2.7) yields

$$x(\bar{t}) - x(t_1) = \int_{t_1}^{\bar{t}} F(t, x(t)) \frac{dt}{t}.$$

Since $F(t, x) \neq 0$ in D_ϵ, one can take ϵ small enough to obtain the inequality $|F_i(t, x)| \geq F^*$ in D_ϵ at least for some i, $1 \leq i \leq n$. It follows that

$$\|x(\bar{t}) - x(t_1)\| \geq |x_i(\bar{t}) - x_i(t_1)| \geq F^*(\ln \bar{t} - \ln t_1).$$

As a result, $t_1 \geq \bar{t} e^{-2\epsilon/F^*}$. Q.E.D.

Everywhere below in this section we assume x_0 to be an isolated zero of the following system of equations

$$F_i(0, x) = 0, \qquad i = 1, \ldots, n, \tag{2.8}$$

i.e. we assume that there exists an open neighborhood $\mathfrak{U} \in \mathbb{R}^n$ of x_0 such that for any $y \in \mathfrak{U}$, $y \neq x_0$, we have $F_i(y) \neq 0$ for some i. The simplest sufficient condition for x_0 to be such isolated zero is non–degeneracy of the Jacobian of the mapping $F(0, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto F(0, x)$, at the point x_0,

$$\det \begin{vmatrix} \dfrac{\partial F_1}{\partial x_1}(0, x_0) & \ldots & \dfrac{\partial F_1}{\partial x_n}(0, x_0) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_n}{\partial x_1}(0, x_0) & \ldots & \dfrac{\partial F_n}{\partial x_n}(0, x_0) \end{vmatrix} \neq 0.$$

Indeed, this implies that the mapping F is a local diffeomorphism, hence, it locally takes the value $F_1 = 0, \ldots, F_n = 0$ only at the point x_0. But sometimes, instead of checking the correspondent Jacobian, it is convenient to prove the uniqueness of the solution to (2.8) directly.

The simplest analogue of Proposition 2.1 is the following statement.

Proposition 2.4. *Assume that $F(0, x_0) = 0$. Denote by $\mathcal{J}$ the Jacobian of the mapping $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x \mapsto F(0, x)$. Let the quadratic form $\mathfrak{c}(x) = x^T \mathcal{J} x$ be strictly positive definite, i.e. $\mathfrak{c}(x) > 0$ for any $x \neq 0$. Then there exists $\epsilon > 0$ such that any solution to (2.7) with initial data in D_ϵ tends to x_0 as t tends to $+0$.*

Proof. Consider the function $N(x) = \|x - x_0\|^2$ and differentiate it along solutions to (2.7). We have

$$\Xi(t, x) = \frac{d}{dt}\Big|_{(2.7)} N(x) = \frac{2}{t}\sum_{i=1}^{n}(x_i - x_{i0})F_i(t, x).$$

If $F(t, x)$ is smooth enough, then

$$F_i(t, x) = F_i(t, x_0) + \sum_{j=1}^{n}\frac{\partial F_i}{\partial x_j}(t, x_0)(x_j - x_{j0}) + o_i^1(t, x - x_0),$$

where

$$\lim_{x \to x_0}\frac{o_i^1(t, x - x_0)}{\|x - x_0\|} = 0, \quad i = 1, \ldots, n,$$

and the limits are uniform with respect to all $t \in [0, \epsilon_0]$. It follows that

$$\frac{t}{2}\Xi(t, x) = \sum_{i=1}^{n}(x_i - x_{i0})F_i(t, x_0) + \mathfrak{d}(t, x) + o^2(t, x - x_0), \tag{2.9}$$

where

$$\mathfrak{d}(t, x) = \sum_{i,j=1}^{n}\frac{\partial F_i}{\partial x_j}(t, x_0)(x_i - x_{i0})(x_j - x_{j0})$$

and

$$\lim_{x \to x_0}\frac{o^2(t, x - x_0)}{\|x - x_0\|^2} = 0.$$

By assumption, the quadratic form $\mathfrak{d}(0, x) = \mathfrak{c}(x)$ is strictly positive definite, hence there exists $A_1 > 0$ such that

$$\mathfrak{d}(0, x) \geq A_1\|x - x_0\|^2.$$

In view of continuity arguments, if $\epsilon_0 > 0$ is small enough then there exists $A_2 > 0$ such that

$$\mathfrak{d}(t, x) \geq A_2\|x - x_0\|^2$$

for all $0 \leq t \leq \epsilon_0$. It follows that there exists $r_0 > 0$ and a constant $A_3 > 0$ such that

$$\mathfrak{d}(t, x) + o^2(t, x - x_0) \geq A_3\|x - x_0\|^2$$

for all $\|x - x_0\| \leq r_0$ and $0 \leq t \leq \epsilon_0$. Fix r_0, let x be an arbitrary point of the sphere $\|x - x_0\| = r_0$ and consider $\Xi(t, x)$ as $t \to +0$.

By assumption, $F_i(t, x_0) \to 0$, hence, if $r_0\|F(t, x_0)\| < A_3 r_0^2$, the derivative $\Xi(t, x)$ is positive. It follows that for some $\epsilon = \epsilon(r_0) \in (0, \epsilon_0)$ all solutions to (2.7) with initial points inside the cylinder

$$C_{r_0,\epsilon} = \{t, x \,|\, \|x - x_0\| \leq r_0, \ 0 \leq t \leq \epsilon\} \tag{2.10}$$

does not leave $C_{r_0,\epsilon}$ in the reverse time. Thus, their α–limit sets are not empty. Let $\tilde{x}(t)$ be such solution and assume that a point $x_1 \neq x_0$ belongs to its α–limit set. Setting $r_0 = \|x_1 - x_0\|$ and arguing as above one concludes that there exists $\epsilon > 0$ such that $\tilde{x}(t)$ lies strictly inside $C_{r_0,\epsilon}$ for all $t \in (0, \epsilon)$ (the velocity of decreasing of the function $N(x)$ along $\tilde{x}(t)$ is separated from zero). This contradicts the definition of the α–limit set. Q.E.D.

To check the strictly positive defineteness of the quadratic form $\mathfrak{c}(x)$ one can use, for example, the standard Silverstre criterion for the matrix $\frac{1}{2}(\mathcal{J} + \mathcal{J}^T)$.

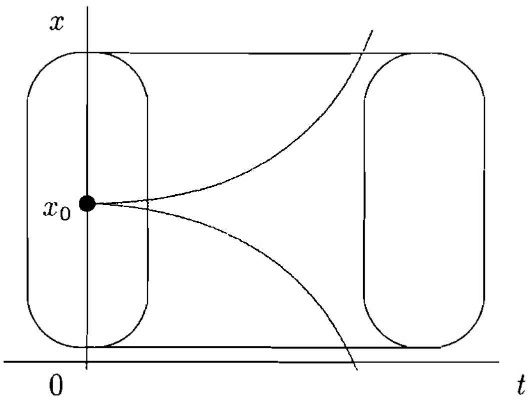

Fig. 8.

The next statement describes the situation which is directly opposite to that in Proposition 2.4.

Proposition 2.5. *Assume that $F(0, x_0) = 0$ and the quadratic form $\mathfrak{c}(x)$ defined in Proposition 2.4 is strictly negative definite. Then there exists a solution to (2.7) approaching x_0 as t tends to $+0$.*

Proof. Consider the set $C_{r_0,\epsilon}$ defined in (2.10). In the same way as in the proof of Proposition 2.4 one can show that for any fixed small enough r_0 there exists $\epsilon = \epsilon(r_0) > 0$ such that $\Xi(t, x) < 0$ for any $t \in (0, \epsilon)$ at any x: $\|x - x_0\| = r_0$ (the function $\Xi(t, x)$ is defined in (2.9)). This implies that solutions to (2.7) intersect the boundary of $C_{r_0,\epsilon}$ transversally (under non–zero angle and with non–zero velocity).

Let (t_1, x_1) be a point on $\partial C_{r_0,\epsilon} = \{t, x \,|\, \|x - x_0\| = r_0, \ t \in (0, \epsilon)\}$ and denote by $\hat{x}(t)$ the solution to (2.7) starting at (t_1, x_1). We conclude that $\hat{x}(t)$ intersects the "cover" of the cylinder $C_{r_0,\epsilon}$ — the plane $t = \epsilon$ — at some point x_2 inside the ball $B_{r_0,\epsilon} \stackrel{\text{def}}{=} \{(t, x) \,|\, t = \epsilon, \ \|x - x_0\| < r_0\}$. Consider the mapping $\mathfrak{p} : \partial C_{r_0,\epsilon} \longrightarrow B_{r_0,\epsilon}$, $(t_1, x_1) \mapsto (\epsilon, x_2)$. It is easy to see that $\mathfrak{p}$ is a diffeomorphism, hence, its image is an open subset of $B_{r_0,\epsilon}$ whose homotopic type is the same as that of $n - 1$–dimensional sphere. Thus, the mapping $\mathfrak{p}$ is not an epimorphism and the supplement $B_{r_0,\epsilon} \backslash \operatorname{Im} \mathfrak{p}$ is a closed non–empty set.

Let $\tilde{x} \in B_{r_0,\epsilon} \backslash \operatorname{Im} \mathfrak{p}$ and denote by $\tilde{x}(t)$ the solution to (2.7) passing through $(\epsilon, \tilde{x})$. The solution $\tilde{x}(t)$ has a non–empty α–limit set as $t \to +0$. Assume that its α–limit set contains some $x^* \neq x_0$. Setting $r^* = \|x^* - x_0\|$ and proceeding as above we conclude that $\tilde{x}(t)$ is separated from the ball $\|x - x_0\| \leq \|x^* - x_0\|$ for $t \in (\epsilon^*(r^*))$. This contradicts the definition of the α–limit set. Hence, $x^* = x_0$ which implies the statement to be proved. Q.E.D.

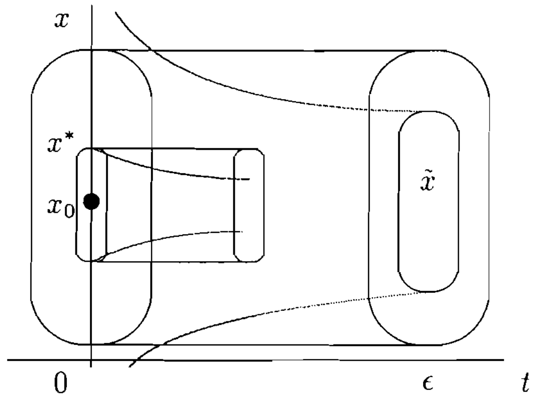

Fig. 9

A simplest generalization of Proposition 2.5 to the case of non-definite in signs quadratic forms is the following.

Proposition 2.6. *Let* $\mathfrak{c}(\xi) = (\mathcal{J}\xi, \xi)$ *(*$\mathcal{J} = \frac{DF}{Dx}(0, x_0)$, $\det \mathcal{J} \neq 0$*) be a non-degenerate quadratic form such that* $\max \mathfrak{c}(\xi) \leq \mu\|\xi\|^2$ *for some* $\mu < 1$. *Then there exists a solution* $\tilde{x}(t)$ *to (2.7) for which* $\lim\limits_{t \to +0} \tilde{x}(t) = x_0$.

Proof. Consider the cone $C_R = \{t, x \,|\, g_R(t,x) = 0\}$ for $g_R(t,x) = R^2t^2 - \|x - x_0\|^2$. Differentiate g_R along a solution to (2.7) at points of C_R. We have

$$\Xi(t,x) \stackrel{\text{def}}{=} \frac{1}{2}\frac{d}{dt}\bigg|_{(2.7)} g_R(t,x) = R^2t - (x - x_0, F(t,x)/t)$$
$$= t - \frac{1}{t}\left(x - x_0, F(t,x_0) + \frac{\partial F}{\partial x}(t,x_0)(x - x_0) + O(\|x - x_0\|^2)\right).$$

At points of C_R we have:

$$(x - x_0, F(t,x_0)) = (x - x_0, t\frac{\partial F}{\partial t}(0,x_0)) + O(t^2),$$
$$\frac{\partial F}{\partial x}(t,x_0)(x - x_0) = \mathcal{J}(x - x_0) + \left(\frac{\partial F}{\partial x}(t,x_0) - \mathcal{J}\right)(x - x_0),$$

where $\left\|\left(\frac{\partial F}{\partial x}(t,x_0) - \mathcal{J}\right)(x - x_0)\right\| \leq O(t)\|x - x_0\| = O(t^2)$. Thus, up to higher order terms,

$$\Xi(t,x) = R^2t - \frac{1}{t}\left((x - x_0, \mathcal{J}(x - x_0)) + (x - x_0, t\frac{\partial F}{\partial t}(0,x_0))\right) + O(t^2).$$

Since $\mathfrak{c}(x - x_0) \leq \mu\|x - x_0\|^2 = \mu R^2t^2$, then $\Xi(t,x) \geq (1-\mu)R^2t - R\|\frac{\partial F}{\partial t}(0,x_0)\| \cdot t + O(t^2) > 0$ for large enough $R > 0$ and small $t > 0$. We conclude that solutions to (2.7) are strictly transversal to the boundary of cone C_R. Consider the following mapping along solutions to (2.7)

$$\mathfrak{p} : C_R^\epsilon \longrightarrow B$$

where $C_R^\epsilon = \{t, x \,|\, 0 < t < \epsilon,\ \|x - x_0\| = Rt\}$ and $B = \{t, x, \,|\, t = \epsilon,\ \|x - x_0\| < R\epsilon\}$ (the mapping $\mathfrak{p}$ associates a point $(\epsilon, x) \in B$ with a point on C_R^ϵ through which the corresponding solution to (2.7) is passing). As above, $\mathfrak{p}$ is a homeomorphism and is not an epimorphism. It implies the existence of a solution to (2.7) which does not leave the interior of C_R in the reverse time current. Q.E.D.

To outline one possible generalization of Proposition 2.3, let us prove the following statement (Proposition 2.7 below). Assume that $F(t, x)$ can be represented in the form:

$$F(t, x) = f(x) + t\,g(x) + t^2 h(t, x),$$

where f, g, h are C^2 vector–functions such as

$$f(x_0) = 0, \quad \frac{Df}{Dx}(0, x_0) = A, \quad g(x_0) = a$$

(here A is $(n \times n)$–matrix, $a \in \mathbb{R}^n$.) Set $y = (x - x_0)/t$, then

$$\begin{aligned} \dot{y} &= -\frac{1}{t} y + \frac{1}{t^2}\Big(f(x_0 + ty) + tg(x_0 + ty) + t^2 h(t, x_0 + ty)\Big) \\ &= \frac{1}{t}((A - E)y + a + tH(t, y)), \end{aligned}$$

where

$$H(t, y) = h(t, x_0 + ty) + \frac{g(x_0 + ty) - a}{t} + \frac{f(x_0 + ty) - A\,ty}{t^2}$$

is C^2.

Proposition 2.7. *Assume that eigenvalues $\lambda_1, \ldots, \lambda_n$ of matrix $A - E$ in (2.11) are positive or have positive real parts. Then there exists a neighborhood D_ϵ of the point $(0, x_0)$ such that solutions to (2.7) with initial data in D_ϵ have the following properties:*

1) the solutions approaches $(0, x_0)$ as t tends to $+0$;

2) if Jordan form of the matrix $A - E$ is diagonal then the solutions are tangent at $(0, x_0)$ to the vector $-(A - E)^{-1} a$.

Consider first the linear system

$$\dot{y} = \frac{1}{t}((A - E)y + a), \tag{2.11}$$

for E being the unit $(n \times n)$–matrix. A general solution to (2.11) can be written as follows

$$y(t) = e^{(A-E)\tau}\left[C_0 + \int_0^\tau e^{-(A-E)s} a\, ds\right], \tag{2.12}$$

where $\tau = \ln t$.

We need the following auxiliary result.

Lemma 2.4. *Let $(n \times n)$–matrix M belong to one of the following three types:*

i) the Jordan representation of the matrix M reads

$$\begin{pmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{pmatrix}$$

where $\lambda_1, \ldots, \lambda_n$ *are real numbers (not necessarily different);*
ii) $n = 2$ *and the Jordan form of* M *is as follows:*

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

where $\phi \in [0, 2\pi)$*;*
iii) the Jordan form of M *is*

$$\begin{pmatrix} \lambda & 1 & \ldots & 0 & 0 \\ 0 & \lambda & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda & 1 \\ 0 & 0 & \ldots & 0 & \lambda \end{pmatrix}.$$

Then the matrix e^{Mt} *can be represented in the forms*
i)

$$\begin{pmatrix} e^{\lambda_1 t} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{\lambda_n t} \end{pmatrix};$$

ii)

$$e^{t\cos\phi} \begin{pmatrix} \cos(t\sin\phi) & -\sin(t\sin\phi) \\ \sin(t\sin\phi) & \cos(t\sin\phi) \end{pmatrix};$$

iii)

$$e^{\lambda t} \begin{pmatrix} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \ldots & \frac{1}{(n-1)!} \\ 0 & 1 & 1 & \frac{1}{2!} & \ldots & \frac{1}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Proof. Case *i)* is evident. Let us consider case *ii)*. We have

$$e^{Mt} = E + Mt + M^2\frac{t^2}{2!} + \ldots + M^n\frac{t^n}{n!} + \ldots.$$

Since $M^n = \begin{pmatrix} \cos n\phi & -\sin n\phi \\ \sin n\phi & \cos n\phi \end{pmatrix}$ and $\cos n\phi = \mathrm{Re}\, e^{in\phi}$, $\sin n\phi = \mathrm{Im}\, e^{in\phi}$, we conclude that the element in the upper left corner of e^{Mt} is equal to

$$\begin{aligned} [e^{Mt}]_{11} &= \mathrm{Re}\left(1 + e^{i\phi}t + e^{i2\phi}\frac{t^2}{2!} + \ldots + e^{in\phi}\frac{t^n}{n!} + \ldots\right) \\ &= \mathrm{Re}\, e^{e^{i\phi}t} \quad = \quad e^{t\cos\phi}\cos(t\sin\phi). \end{aligned}$$

Other elements of the matrix e^{Mt} can be calculated in the same way.
Consider case *iii)*. To calculate e^{Mt}, denote by I the matrix

$$I = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Then

$$M^k = (\lambda E + I)^k = \sum_{\alpha=0}^{k} C_k^\alpha \lambda^{k-\alpha} I^\alpha$$

where I^α is the $(n \times n)$–matrix which has units on the $(\alpha+1)$–th upper–diagonal and zeroes on all others. Now

$$\begin{aligned} e^{Mt} &= \sum_{k=0}^{\infty} \frac{M^k t^k}{k!} = \sum_{k=0}^{\infty}\sum_{\alpha=0}^{k} C_k^\alpha \frac{\lambda^{k-\alpha} t^{k-\alpha}}{k!} I^\alpha \\ &= \sum_{k=0}^{\infty}\sum_{\alpha=0}^{k} \frac{\lambda^{k-\alpha} t^{k-\alpha}}{\alpha!(k-\alpha)!} I^\alpha. \end{aligned}$$

This is an upper–triangular matrix for which the $(l+1)$–th diagonal contains the elements

$$\sum_{k=l}^{\infty} \frac{(\lambda t)^{k-l}}{l!(k-l)!} = \frac{1}{l!}\sum_{k=l}^{\infty} \frac{(\lambda t)^{k-l}}{(k-l)!} = \frac{1}{l!} e^{\lambda t}.$$

Thus,

$$e^{Mt} = e^{\lambda t} \begin{pmatrix} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \dots & \frac{1}{(n-1)!} \\ 0 & 1 & 1 & \frac{1}{2!} & \dots & \frac{1}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Q.E.D.

Corollary 2.1. *Under conditions of Lemma 2.4 solutions to the equation*

$$\dot{y} = My + m$$

can be represented in the Jordan basis in one of the following forms:

i)

$$y(t) = \begin{pmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} C_0^1 \\ \vdots \\ C_0^n \end{pmatrix} - \begin{pmatrix} m_1/\lambda_1 \\ \vdots \\ m_n/\lambda_n \end{pmatrix};$$

ii)

$$\begin{aligned} y(t) = e^{t\cos\phi} &\begin{pmatrix} \cos(t\sin\phi) & -\sin(t\sin\phi) \\ \sin(t\sin\phi) & \cos(t\sin\phi) \end{pmatrix} \cdot \begin{pmatrix} C_0^1 \\ C_0^2 \end{pmatrix} + \\ + &\begin{pmatrix} (-m_1\cos\phi - m_2\sin\phi)\cos(t\sin\phi) + (m_1\sin\phi - m_2\cos\phi)\sin(t\sin\phi) \\ (-m_2\cos\phi + m_1\sin\phi)\cos(t\sin\phi) + (m_2\sin\phi + m_1\cos\phi)\sin(t\sin\phi) \end{pmatrix}; \end{aligned}$$

iii)

$$y(t) = e^{\lambda t} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} C_0^1 \\ C_0^2 \\ \vdots \\ C_0^{n-1} \\ C_0^n \end{pmatrix} +$$

$$-\frac{1}{\lambda}\cdot\begin{pmatrix} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \dots & \frac{1}{(n-1)!} \\ 0 & 1 & 1 & \frac{1}{2!} & \dots & \frac{1}{(n-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}^{-1}\cdot m.$$

The proof is simple and consists in direct integrating the relation

$$y(t) = e^{Mt}y(0) + e^{Mt}\int_0^t e^{-Ms}m\,ds.$$

Proof of Proposition 2.7. Without loss of generality one can assume that matrix $A - E$ in (2.11) is already reduced to its Jordan form. Applying Corollary 2.1 to each of the Jordan boxes one can check that corresponding coordinates of solutions to the linear system (2.11) can be written as

$$t^{\mathrm{Re}\ \lambda_i}c(t) + d(t),$$

where $c(t)$ and $d(t)$ are C^∞ everywhere except for $t = 0$ and are uniformly bounded as $t \to +0$. The most "irregular" case takes place for complex-conjugated eigenvalues. In this case $d(t)$ consists of the following terms:

$$d_1\cos(\ln t\cdot\mathrm{Im}\ \lambda_1) + d_2\sin(\ln t\cdot\mathrm{Im}\ \lambda_1).$$

Returning back to original system of the coordinates x we obtain the assertion of the proposition for the non-disturbed linear equation. Solutions to the non-linear equation can be obtained by setting $C_0 = C_0(t)$. As before, we can prove that function $C_0(t)$ does not change the asymptotics which is, hence, the same as for solutions of the linear system. Q.E.D.

3 A model of technological leading–following

Consider the following differential equation

$$\begin{aligned} \dot{n}^A &= \left[c^A(n^A + \gamma_1(n^A)n^B) - \frac{\alpha_1}{v^A}\right]_+, \\ \dot{v}^A &= \rho_1 v^A - \frac{1-\alpha_1}{n^A}, \\ \dot{n}^B &= \left[c^B(n^B + \gamma_2(n^B)n^A) - \frac{\alpha_2}{v^B}\right]_+, \\ \dot{v}^B &= \rho_2 v^B - \frac{1-\alpha_2}{n^B}. \end{aligned} \tag{3.1}$$

Note that $(n^A, v^A, n^B, v^B) \in \mathbb{R}^4$ stand for state variables and c^A, c^B, ρ_1, ρ_2, α_1, α_2 are some positive constants, $\alpha_i \in (0,1)$, $i = 1,2$. Functions $\gamma_i(\cdot)$ are positive, increasing and continuously differentiable, $\lim\limits_{n\to+\infty}\gamma_i(n) = \Gamma_i > 0$, $i = 1,2$. We consider solutions to (3.1) such that all n^A, v^A, n^B, v^B are positive simultaneously. The goal is to determine "asymptotically balanced" solutions to (3.1) that means exactly that there exists

$$\lim_{t\to+\infty}\frac{n^A(t)}{n^B(t)}=\gamma\in(0,\infty).$$

The first possible case which allows a complete analysis is the following.

3.1 The upper asymptotics ($n^A \to +\infty$, $n^B \to +\infty$, $v^A \to +\infty$, $v^B \to +\infty$).

Thus, the initial values $n^A(0)$, $v^A(0)$, $n^B(0)$, $v^B(0)$ being large enough:

$$n^A(0) > r_0, \quad v^A(0) > r_0, \quad n^B(0) > r_0, \quad v^B(0) > r_0 \tag{3.2}$$

for some $r_0 > 0$, we compare the growth of two components of the solution: $n^A(t)$ and $n^B(t)$ as $t \to +\infty$.

Lemma 3.1. *There exists such r_0 that solutions to (3.1) with initial conditions (3.2) are monotonously increasing in each coordinate.*

Indeed, we prove another statement which implies Lemma 3.1 as a consequence. Namely, we show that Lemma 3.1 remains in place even if we refuse the preliminary restriction $\dot{n}^A \geq 0$, $\dot{n}^B \geq 0$.

Lemma 3.2. *Consider the system*

$$\begin{aligned}
\dot{n}^A &= c^A(n^A+\gamma_1(n^A)n^B)-\frac{\alpha_1}{v^A},\\
\dot{v}^A &= \rho_1 v^A-\frac{1-\alpha_1}{n^A},\\
\dot{n}^B &= c^B(n^B+\gamma_2(n^B)n^A)-\frac{\alpha_2}{v^B},\\
\dot{v}^B &= \rho_2 v^B-\frac{1-\alpha_2}{n^B}.
\end{aligned}\tag{3.3}$$

There exists an open subset of initial points for which solutions to (3.3) are monotonously increasing in each coordinate.

Proof. Indeed, r_0 can be chosen in such a way that $\dot{n}^A > 0$, $\dot{n}^B > 0$ at the initial moment $t = 0$. Hence, $n^A(t)$, $v^A(t)$ increase at least for all small enough $t > 0$. The differences $c^A n^A - \alpha_1/v^A$ and $\rho_1 v^A - (1-\alpha_1)/n^A$ begin to grow therefore. But since the velocities $\dot{n}^A(t)$ and $\dot{v}^A(t)$ grow, the values $n^A(t)$ and $v^A(t)$ in their turn remain such that the right hand sides are positive, $\dot{n}^A > 0$, $\dot{v}^A > 0$. The pair $(n^B(t), v^B(t))$ can be considered in the same way. The statement of Lemma 3.2 is proved. In particular, it implies that systems (3.1) and (3.2) are asymptotically equivalent if the initial data are large enough.

It follows from Lemmas 3.1 and 3.2 that the function $n^A(t)$ is strictly increasing in an appropriate region. Hence, we can take the value of $n^A(t)$ as the new independent variable (instead of t). We obtain

$$\begin{aligned}
\frac{dv^A}{dn^A} &= \frac{\rho_1 v^A-(1-\alpha_1)/n^A}{c^A(n^A+\gamma_1(n^A)n^B)-\alpha_1/v^A},\\
\frac{dn^B}{dn^A} &= \frac{c^B(n^B+\gamma_2(n^B)n^A)-\alpha_2/v^B}{c^A(n^A+\gamma_1(n^A)n^B)-\alpha_1/v^A},\\
\frac{dv^B}{dn^A} &= \frac{\rho_2 v^B-(1-\alpha_2)/n^B}{c^A(n^A+\gamma_1(n^A)n^B)-\alpha_1/v^A}.
\end{aligned}\tag{3.4}$$

Since we are looking for solutions for which $\lim_{n^A \to +\infty} n^B/n^A = \gamma \in (0, \infty)$, we exclude for a while the functions $v^A(t)$, $v^B(t)$ from the analysis. It is important only that

$$\lim_{n^A \to +\infty} v^A(n^A) = \lim_{n^A \to +\infty} v^B(n^A) = +\infty.$$

Setting $n^A = \tau$, $n^B = n$, $c^B/c^A = c$, $-\alpha_2/(c^B v^B) = f_1(\tau)$, $-\alpha_1/(c^A v^A) = f_2(\tau)$, we arrive at the equation

$$\frac{dn}{d\tau} = c\frac{n + \gamma_2(n)\tau + f_1(\tau)}{\tau + \gamma_1(\tau)n + f_2(\tau)}.$$

It is convenient to express the last equation in terms of the variables $x = n/\tau$, $s = 1/\tau$. Straightforward calculations lead to the system

$$\frac{dx}{ds} = \frac{1}{s}\left[x - c\frac{x + \gamma_2(x/s) + s\,f_1(1/s)}{1 + \gamma_1(1/s)x + s\,f_2(1/s)}\right]. \tag{3.5}$$

Denote

$$F(s, x) = x - c\frac{x + \gamma_2(x/s) + s\,f_1(1/s)}{1 + \gamma_1(1/s)x + s\,f_2(1/s)}.$$

Equation (3.5) is a particular case of what has been studied in section 2 for equation (2.1). It is easy to see that the balanced solutions to (3.3) correspond to the values x_0 such as $F(0, x_0) = 0$. This leads to the following quadratic equation:

$$x(1 + \Gamma_1 x) - c(x + \Gamma_2) = 0.$$

Hence,

$$x = \frac{c - 1 \pm \sqrt{(c-1)^2 + 4c\Gamma_1\Gamma_2}}{2\Gamma_1}. \tag{3.6}$$

Denote by x_+ the root which corresponds to the sign "+" in (3.6) and denote by x_- the other root. It is worth noting that x_+ and x_- have always opposite signs, $x_+ > 0$, $x_- < 0$. Hence, the function $F(0, x)$ is increasing at x_+ and Proposition 2.1 proves the following statement.

Proposition 3.1. *There exists an open set of initial values $n^A(0)$, $v^A(0)$, $n^B(0)$, $v^B(0)$ such that the corresponding solutions to system (3.1) have the following properties:*
1) $n^A(t) \to +\infty, \quad n^B(t) \to +\infty, \quad v^A(t) \to +\infty, \quad v^B(t) \to +\infty;$
2) $\lim_{t \to +\infty} \frac{n^B(t)}{n^A(t)} = x_+$, *where* $x_+ = \frac{c - 1 + \sqrt{(c-1)^2 + 4c\Gamma_1\Gamma_2}}{2\Gamma_1}$.

The qualitative behavior of solutions to (3.4) is depicted in Fig. 10. To analyze the picture in Fig. 10 in the vicinity of the point $(0, x_+)$ in more details one needs to study the function $F(t, x)$. Impose the following additional restriction.

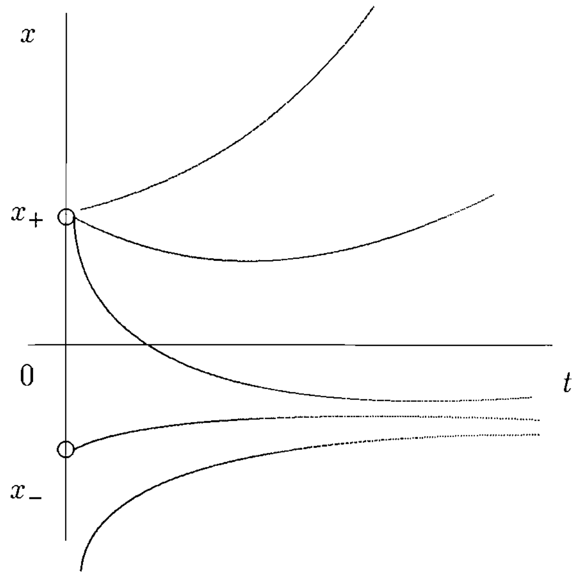

Fig. 10.

Assumption 3.1. *Functions $\gamma_1(n)$ and $\gamma_2(n)$ admit the following representation:*

$$\gamma_i(n) = \Gamma_i - \frac{\delta_i}{n} + \frac{\epsilon_i(n)}{n^2}, \quad i = 1, 2,$$

where δ_1, δ_2 stand for positive constants and functions $\epsilon_1(n)$, $\epsilon_2(n)$ are uniformly bounded as $n \to +\infty$.

From now on we shall be assuming that Assumption 3.1 holds true. We need the following auxiliary result.

Lemma 3.3. *Consider the solution to (3.4) with initial conditions $v^A(n_0^A) = v_0^A$, $n^B(n_0^A) = n_0^B$, $v^B(n_0^A) = v_0^B$, where $(1/n_0^A, n_0^A/n_0^B) \in \mathbb{R}^2$ is close enough to $(0, x_+) \in \mathbb{R}^2$ and the assertions of Proposition 3.1 hold true for it. Assume also that*

$$\frac{\rho_1}{c^A(1+\Gamma_1 x_+)} > 1, \quad \frac{\rho_2}{c^B(1+\Gamma_2 x_+)} > 1.$$

Then there exist $v_1 \geq 0$, $v_2 \geq 0$ such that $v^A \geq v_1 n^A$, $v^B \geq v_2 n^B$.

Proof. Since $n^B(n^A)/n^A$ tends to x_+ as $n^A \to +\infty$, by Lemma 3.1, we have

$$\frac{dv^A}{dn^A} = \left(\frac{\rho_1}{c^A(1+\Gamma_1 x_+)} + f(v^A, n^A)\right)\frac{v^A}{n^A},$$

where $f(v^A, n^A) \to 0$ as $n^A \to +\infty$. Hence,

$$\frac{dv^A}{dn^A} \geq \frac{v^A}{n^A},$$

for all large n^A or, equivalently,

$$\frac{d\ln v^A}{d\ln n^A} \geq 1.$$

Consequently, $\ln v^A(n^A) - \ln v^A(n_0^A) \geq \ln n^A - \ln n_0^A$, which implies that $v^A \geq \text{const} \cdot n^A$. The estimate for v^B can be obtained in the same manner. Q.E.D.

If conditions of Lemma 3.1 hold true, then the following representation for the function $F(s,x)$ takes place:

$$F(s,x) = x - c\frac{x+\Gamma_2 - \delta_2 s/x}{1+(\Gamma_1 - \delta_1 s)x} + s^2 h(s,x),$$

where $h(s,x)$ is uniformly bounded for $(s,x) \in D_\epsilon$. By Taylor's expansion, up to higher order terms, we get

$$F(s,x) = \frac{-c\Gamma_2 + (1-c)x + \Gamma_1 x^2}{1+\Gamma_1 x} + s\left(\frac{\delta_2 c}{x(1+\Gamma_1 x)} - \frac{c(\Gamma_2 + x)\delta_1 x}{(1+\Gamma_1 x)^2}\right) + s^2\tilde{h}(s,x).$$

As was shown in section 2, the behavior of the system

$$\frac{dx}{dt} = \frac{f(x)}{t} + g(x) + th(t,x)$$

in a vicinity of a root x_0 of the function $f(x)$ depends on two parameters, namely $\frac{\partial f}{\partial x}(x_0) = \alpha$ and $g(x_0) = \beta$. Straightforward calculations give

$$\begin{aligned}
\alpha &= \frac{1-c+2\Gamma_1 x_+}{1+\Gamma_1 x_+} = 2 - \frac{1+c}{1+\Gamma_1 x_+}, \\
\beta &= \frac{\delta_2 c}{x_+(1+\Gamma_1 x_+)} - \frac{c(\Gamma_2 + x_+)\delta_1 x_+}{(1+\Gamma_1 x_+)^2}.
\end{aligned}$$

It is easy to see that $\alpha \in (0,2)$ and all possible relationships between α and β depend on the values c, Γ_1, Γ_2, δ_1, δ_2. This allows to determine how solutions to system (3.5) reach the limit point. It can be done using the theory developed in section 2 (Proposition 2.3).

3.2 Intermediate growth ($n^A/n^B \to \kappa$, $n^A v^A \to \sigma_1$, $n^B v^B \to \sigma_2$).

Our goal is to find such relations between the parameters of system (3.1) that allow for the asymptotics given above where σ_1 and σ_2 are some positive constants. Our plan is as follows. Though, a priori, we cannot be sure now that the variable n^A is monotone increasing, we still transfer to n^A as to an independent variable instead of t. Then using the technique developed in section 2 we determine some region in the space of variables n^A, n^B, v^A, v^B where solutions to (3.1) have the desired asymptotics. In particular, we shall be able to show that n^A increases in this region.

First of all let us show that this solution satisfies "perfect-foresight" condition (1.5). The following statement contains a simple necessary and sufficient condition for the perfect-foresight property.

Lemma 3.4. *For any non-decreasing positive function $n(t)$ a solution to the equation*

$$\dot{v} = \rho v - \frac{1-\alpha}{n(t)} \tag{3.7}$$

satisfies for the relation

$$v(t) = \int_t^\infty \frac{1-\alpha}{n(t)} e^{\rho(t-s)} ds \tag{3.8}$$

if and only if

$$\limsup_{t\to+\infty} |v(t)| < \infty.$$

Proof. The general solution to (3.7) can be written as

$$v(t) = e^{\rho t}(v(0) - \int_0^t \frac{1-\alpha}{n(s)} e^{-\rho s} ds).$$

Since $n(t)$ is not decreasing, then the following integral is converging

$$\int_0^\infty \frac{1-\alpha}{n(s)} e^{-\rho s} ds \overset{\text{def}}{=} C_0.$$

Now

$$v(t) = e^{\rho t}(v(0) - C_0 + \int_t^\infty \frac{1-\alpha}{n(t)} e^{-\rho s} ds).$$

The relation (3.8) is equivalent to $v(0) = C_0$. By Lopital rool, the limit of the function

$$\frac{\int_t^\infty \frac{1-\alpha}{n(s)} e^{-\rho s} ds}{e^{-\rho t}}$$

as $t \to +\infty$ can be upper estimated by

$$\limsup_{t\to+\infty} \frac{-\frac{1-\alpha}{n(t)} e^{-\rho t}}{-\rho e^{-\rho t}} \le \frac{1-\alpha}{n(0)\rho}.$$

It follows that the condition $v(0) = C_0$ is equivalent to the inequivality $\limsup_{t\to+\infty} |v(t)| < \infty$. Q.E.D.

Corollary 3.1. *The perfect-foresight condition holds true along any solution to (3.1) with the asymptotics* $n^i(t) \to +\infty$, $v^i(t)n^i(t) \to \sigma_i \in (0, \infty)$ *as* $t \to \infty$ $(i = A, B)$.

Consider equation (3.4) and set

$$n^A = \tau, \quad \frac{n^B}{n^A} = x, \quad n^A v^A = y, \quad n^A v^B = z.$$

Straightforward calculations lead to the system

$$\begin{aligned}
\frac{dx}{d\tau} &= \frac{1}{\tau}\left(-x + \frac{c^B(x + \gamma_2(x\tau)) - \alpha_2/z}{c^A(1 + \gamma_1(\tau)x) - \alpha_1/y}\right), \\
\frac{dy}{d\tau} &= \frac{1}{\tau}\left(y + \frac{\rho_1 y - (1-\alpha_1)}{c^A(1 + \gamma_1(\tau)x) - \alpha_1/y}\right), \\
\frac{dz}{d\tau} &= \frac{1}{\tau}\left(z + \frac{\rho_2 z - (1-\alpha_2)/x}{c^A(1 + \gamma_1(\tau)x) - \alpha_1/y}\right).
\end{aligned}$$

Setting $s = 1/\tau$ we arrive at the system

$$\begin{aligned}
\frac{dx}{ds} &= \frac{1}{s}\left(x - \frac{c^B(x+\gamma_2(x/s)) - \alpha_2/z}{c^A(1+\gamma_1(1/s)x) - \alpha_1/y}\right), \\
\frac{dy}{ds} &= \frac{1}{s}\left(-y + \frac{-\rho_1 y + (1-\alpha_1)}{c^A(1+\gamma_1(1/s)x) - \alpha_1/y}\right), \\
\frac{dz}{ds} &= \frac{1}{s}\left(-z + \frac{-\rho_2 z + (1-\alpha_2)/x}{c^A(1+\gamma_1(1/s)x) - \alpha_1/y}\right).
\end{aligned} \tag{3.9}$$

As follows from Lemma 2.3, $(x_0, y_0, z_0, s = 0)$ is a stable point for system (3.9) only if the following equations hold:

$$\begin{aligned}
x - \frac{c^B(x+\Gamma_2) - \alpha_2/z}{c^A(1+\Gamma_1 x) - \alpha_1/y} &= 0, \\
-y + \frac{-\rho_1 y + (1-\alpha_1)}{c^A(1+\Gamma_1 x) - \alpha_1/y} &= 0, \\
-z + \frac{-\rho_2 z + (1-\alpha_2)/x}{c^A(1+\Gamma_1 x) - \alpha_1/y} &= 0,
\end{aligned} \tag{3.10}$$

or, equivalently,

$$\begin{gathered}
xz(c^A y(1+\Gamma_1 x) - \alpha_1) - c^B yz(x+\Gamma_2) + \alpha_2 y = 0, \\
-c^A y(1+\Gamma_1 x) - \rho_1 y + 1 = 0, \\
-xz(c^A y(1+\Gamma_1 x) - \alpha_1) - \rho_2 xyz + (1-\alpha_2)y = 0.
\end{gathered} \tag{3.11}$$

The second equation in (3.11) yields

$$y = \frac{1}{\rho_1 + c^A(1+\Gamma_1 x)}. \tag{3.12}$$

If we substitute the expression for y from (3.12) to the third equation in (3.11), we get

$$\begin{aligned}
z &= \frac{(1-\alpha_2)y}{x(c^A y(1+\Gamma_1 x) - \alpha_1) + \rho_2 xy} \\
&= \frac{1-\alpha_2}{x(\rho_2 - \alpha_1\rho_1 + c^A(1-\alpha_1)(1+\Gamma_1 x))}.
\end{aligned}$$

On the other hand, if we add the first equation of (3.11) to the third one, we have

$$z = \frac{1}{\rho_2 x + c^B(x+\Gamma_2)}.$$

It allows to exclude z and set the following equation for determining x:

$$\frac{1-\alpha_2}{x(\rho_2 - \alpha_1\rho_1 + c^A(1-\alpha_1)(1+\Gamma_1 x))} = \frac{1}{\rho_2 x + c^B(x+\Gamma_2)},$$

or, equivalently,

$$
\begin{aligned}
c^A\Gamma_1(1-\alpha_1)x^2 + (-\alpha_1\rho_1 + \alpha_2\rho_2 + c^A(1-\alpha_1) \\
-c^B(1-\alpha_2))x - (1-\alpha_2)c^B\Gamma_2 = 0.
\end{aligned} \tag{3.13}
$$

The discriminant of the quadratic equation (3.13) is always positive and only one root of (3.13) is positive. Denote the root by x_0. It uniquely determines the coordinates y_0, z_0, hence we have a single point (x_0, y_0, z_0) on the plane $s = 0$ where system (3.9) can possess asymptotically stable solutions.

To apply Proposition 2.6 to (3.9) we have to impose some further restrictions on $\gamma_i(n)$, $i = 1, 2$. Let Assumption 3.1 holds true. Then (3.9) can be written as follows

$$
\begin{aligned}
\frac{dx}{ds} &= \frac{1}{s}\Bigg(F_1 + s\left(\frac{c^B\delta_2/x}{c^A(1+\Gamma_1 x) - \alpha_1/y} - \frac{c^B(x+\Gamma_2) - \alpha_2/z}{(c^A(1+\Gamma_1 x) - \alpha_1/y)^2}c^A\delta_1 x\right) \\
&\qquad + s^2 f_1(s,x,y,z)\Bigg), \\
\frac{dy}{ds} &= \frac{1}{s}\left(F_2 + s\frac{-\rho_1 y + (1-\alpha_1)}{(c^A(1+\Gamma_1 x) - \alpha_1/y)^2}c^A\delta_1 x + s^2 f_2(s,x,y,z)\right), \\
\frac{dz}{ds} &= \frac{1}{s}\left(F_3 + s\frac{-\rho_2 z + (1-\alpha_2)/x}{(c^A(1+\Gamma_1 x) - \alpha_1/y)^2}c^A\delta_1 x + s^2 f_3(s,x,y,z)\right).
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(x,y,z) &= x - \frac{c^B(x+\Gamma_2) - \alpha_2/z}{c^A(1+\Gamma_1 x) - \alpha_1/y}, \\
F_2(x,y) &= -y + \frac{-\rho_1 y + (1-\alpha_1)}{c^A(1+\Gamma_1 x) - \alpha_1/y}, \\
F_3(x,y,z) &= -z + \frac{-\rho_2 z + (1-\alpha_2)/x}{c^A(1+\Gamma_1 x) - \alpha_1/y}.
\end{aligned}
$$

To apply Proposition 2.6 one has to check that the maximal positive eigenvalue of the quadratic form $\mathfrak{c}(\xi) = \xi^T \mathcal{J} \xi$, $\xi \in \mathbb{R}^3$,

$$
\mathcal{J} = \begin{pmatrix}
\frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\
\frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\
\frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z}
\end{pmatrix}
$$

is less than the unit at the point (x_0, y_0, z_0). Let us calculate $\mathcal{J}$ in an explicit way. Denote

$$
\mu = c^A(1+\Gamma_1 x_0) - \alpha_1/y_0 = (1-\alpha_1)c^A(1+\Gamma_1 x_0) - \alpha_1\rho_1.
$$

Straightforward calculations yield

$$
\begin{aligned}
&\mu\frac{\partial F_1}{\partial x} = \mu + c^A\Gamma_1 x_0 - c^B, &&\mu\frac{\partial F_1}{\partial y} = \frac{\alpha_1 x_0}{y_0^2}, &&\mu\frac{\partial F_1}{\partial z} = -\frac{\alpha_2}{z_0^2}, \\
&\mu\frac{\partial F_2}{\partial x} = -c^A\Gamma_1 y_0, &&\mu\frac{\partial F_2}{\partial y} = -c^A(1+\Gamma_1 x_0) - \rho_1, &&\mu\frac{\partial F_2}{\partial z} = 0, \\
&\mu\frac{\partial F_3}{\partial x} = -c^A\Gamma_1 z_0 - \frac{1-\alpha_2}{x_0^2}, &&\mu\frac{\partial F_3}{\partial y} = -\frac{\alpha_1 z_0}{y_0^2}, &&\mu\frac{\partial F_3}{\partial z} = -\mu - \rho_2.
\end{aligned}
$$

Hence,

$$\mathcal{J} = \frac{1}{\mu}\begin{pmatrix} \mu + c^A\Gamma_1 x_0 - c^B & \alpha_1 x_0/y_0^2 & -\alpha_2/z_0^2 \\ -c^A\Gamma_1 y_0 & -c^A(1+\Gamma_1 x_0) - \rho_1 & 0 \\ -c^A\Gamma_1 z_0 - (1-\alpha_2)/x_0^2 & -\alpha_1 z_0/y_0^2 & -\mu - \rho_2 \end{pmatrix}.$$

Consider $\mathcal{J}$ under the following assumptions on the parameters of (3.9):

$$\alpha_1 = \alpha_2 = 0, \qquad \rho_1 = \rho_2 = \rho, \qquad \Gamma_1 = 0.$$

It follows from (3.13) that

$$x_0 = \frac{c^B\Gamma_2}{c^A - c^B}, \quad y_0 = \frac{1}{\rho + c^A}, \quad z_0 = \frac{c^A - c^B}{(\rho + c^A)c^B\Gamma_2}.$$

For matrix $\mathcal{J}$ we have

$$\mathcal{J} = \frac{1}{c^A}\begin{pmatrix} c^A - c^B & 0 & 0 \\ 0 & -(c^A+\rho) & 0 \\ -\frac{(c^A-c^B)^2}{(c^B)^2\Gamma_2^2} & 0 & -(c^A+\rho) \end{pmatrix}.$$

To simplify the analysis let us put $c^A - c^B = \epsilon$ for some small positive ϵ, then up to higher order terms, matrix $\mathcal{J}$ is diagonal with the eigenvalue ϵ/c^A, $-(c^A-\rho)/c^A$, $-(c^A+\rho)/c^A$. Thus, for some set of parameters, the matrix meets the conditions of Proposition 2.6. Since the eigenvalues of the corresponding quadratic form are continuous in the parameters, we obtain the following result.

Proposition 3.2. *There exists an open set of parameters α_1, α_2, ρ_1, ρ_2, c^A, c^B, Γ_1, Γ_2 such that for an open subset of initial points $n^A(0)$, $n^B(0)$, $v^A(0)$, $v^B(0)$ the corresponding solutions to (3.1) have intermediate balanced asymptotics:*

$$\begin{aligned} \lim_{t\to\infty} \frac{n^B(t)}{n^A(t)} &= x_0, \\ \lim_{t\to\infty} v^A(t)n^A(t) &= y_0, \\ \lim_{t\to\infty} v^B(t)n^A(t) &= z_0, \end{aligned} \tag{3.14}$$

where (x_0, y_0, z_0) are uniquely specified by (3.13).

4 Conclusions

The mathematical analysis performed in the previous section leads to observations plainly interpreted in economic terms. Recall that it has been proved that if ρ_1 and ρ_2 are large enough, there exists a family of solutions to system (3.3) having the asymptotics (3.14). Hence, even when the ratio costs of developing new technologies are high enough, there exist balanced regimes of growth such that the measures of products tend to infinity, n^A, $n^B \to \infty$ as $t \to +\infty$, while the values of the representative firms asymptotically vanish, v^A, $v^B \to 0$ as $t \to +\infty$.

The second and the third equalities in (3.14) show that for these balanced regimes the asymptotic counterpart of Grossman&Helpman steady growth condition (1.2) holds. The fact that $v^A(t), v^B(t) \to +0$ as $t \to +\infty$ easily implies that for both economies the perfect-foresight condition is satisfied, i.e. $v^A(t) = \int_t^\infty \frac{1-\alpha_1}{n^A(s)} e^{\rho_1(t-s)} ds$, $v^B(t) = \int_t^\infty \frac{1-\alpha_2}{n^B(s)} e^{\rho_2(t-s)} ds$. The proof reflects that given in [2].

It is easy to see that along solutions to (3.3) having the asymptotics (3.14) we have

$$\begin{aligned} \dot{n}^A &= \left(c^A(1 + x_0\Gamma_1) - \frac{\alpha_1}{y_0}\right) n^A + O(t, n^A), \\ \dot{n}^B &= \left(c^B(1 + \frac{\Gamma_2}{x_0}) - \frac{\alpha_2}{x_0 z_0}\right) n^B + O(t, n^B). \end{aligned}$$

where the terms $O(t, n^A)$, $O(t, n^B)$ are uniformly bounded as $(t, n^A) \in \mathbb{R}_+^2$, $(t, n^B) \in \mathbb{R}_+^2$. Hence,

$$\begin{aligned} n^A(t) &= n^A(0)e^{\mu^A t} + \tilde{n}^A(t), \\ n^B(t) &= n^B(0)e^{\mu^B t} + \tilde{n}^B(t), \end{aligned}$$

where $\mu^A = c^A(1 + x_0\Gamma_1) - \alpha_1/y_0$, $\mu^B = c^B(1 + \Gamma_2/x_0) - \alpha_2/x_0 z_0$, and

$$\lim_{t \to +\infty} e^{-\mu^i t}\tilde{n}^i(t) = 0, \qquad i = A, B.$$

Consequently, it holds that

$$\dot{n}^A(t)/n^A(t) \to \mu^A, \quad \dot{n}^B(t)/n^B(t) \to \mu^B \quad \text{as } t \to +\infty;$$

in other words, μ^A and μ^B represent the limit growth rates for countries A and B, respectively. The first equation in system (3.10) for $x = x_0$, $y = y_0$ implies that $\mu^A = \mu^B$. Using (3.12), one gets

$$\mu^A = (1 - \alpha_1)c^A(1 + \Gamma_1 x_0) - \alpha_1\rho_1 = (1 - \alpha_2)c^B(1 + \Gamma_2/x_0) - \alpha_2\rho_2 = \mu^B$$

where x_0 solves the quadratic equation (3.13). Note that if we formally set $\Gamma_1 = \Gamma_2 = 0$, then μ^A and μ^B turn into the autarkic steady state growth rates g^A and g^B respectively. Thus, we arrive at the following principal conclusion: growth rate $\mu^A = \mu^B$ in the economies interacting through the absorptive capacities (Γ_1 and Γ_2 are positive) strictly dominates each of the autarkic growth rates, $\mu^A > g^A$, $\mu^B > g^B$. Straightforward calculations give

$$\frac{\partial \mu^A}{\partial \Gamma_1} = \frac{(1 - \alpha_1)c^A(1 - \alpha_2)c^B\Gamma_2}{2c^A\Gamma_1(1 - \alpha_1)x_0 + (\alpha_1\rho_1 + \alpha_2\rho_2 + c^A(1 - \alpha_1) - c^B(1 - \alpha_2))}. \tag{4.1}$$

The denominator in the latter ratio equals the derivative at x_0 of the quadratic polynomial on the left-hand side of (3.13). Since x_0 is the largest root of the polynomial, the derivative is positive. Since the numerator in (4.1) is positive, $\mu^A(\Gamma_1)$ is strictly increasing. Similarly, we obtain that $\mu^B(\Gamma_2)$ is also strictly increasing. In other words, for each country it is beneficial to develop its own absorptive capacity; the greater is the limit value of the absorptive capacity (Γ_1 for country A and Γ_2 for country B) the greater is the asymptotic growth rate (provided all other parameters are fixed).

We summarize as follows.

Proposition 4.1. *Let the economies of countries A and B interact through absorptive capacities whose limit values are $\Gamma_1 > 0$ and $\Gamma_2 > 0$, respectively. Let parameters α_1, α_2,*

ρ_1, ρ_2, c^A, c^B, Γ_1, Γ_2 of the economies lie in an appropriate region specified in Proposition 3.2. Then there exists a balanced regime of growth $n^A(t)$, $n^B(t)$, $v^A(t)$, $v^B(t)$ for A and B such that:

1. *For both economies the asymptotic counterpart of Grossman&Helpman steady growth criterion holds true:*

$$\lim_{t\to+\infty} v^A(t)n^A(t) = y_0 > 0,$$
$$\lim_{t\to+\infty} v^B(t)n^B(t) = z_0x_0 > 0.$$

2. *For both economies the perfect-foresight condition is satisfied,*

$$v^A(t) = \int_t^\infty e^{\rho_1(s-t)}\frac{1-\alpha_1}{n^A(s)}ds, \quad v^B(t) = \int_t^\infty e^{\rho_2(s-t)}\frac{1-\alpha_2}{n^B(s)}ds.$$

3. *The asymptotic growth rates μ_A, μ_B in products $n^A(t)$, $n^B(t)$ of countries A and B are equal and given by*

$$\mu^A = (1-\alpha_1)c^A(1+\Gamma_1 x_0) - \alpha_1\rho_1 = (1-\alpha_2)c^B(1+\Gamma_2/x_0) - \alpha_2\rho_2 = \mu^B.$$

4. *The growth rate μ_A is increasing in Γ_1 and the growth rate μ_B is increasing in Γ_2.*

5. *The common growth rate in interacting economies strictly dominates the autarkic growth rates in both countries, $\mu^A > g^A$, $\mu^B > g^B$.*

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