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Reconstruction of Boundary Sources through Sensor Observations

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1. Introduction. Problem Formulation.

The paper deals with a problem of reconstruction of extremal boundary disturbances in a parabolic system. The reconstruction is perfomed on the basis of inaccurate observations of linear signals on system's states. Extremality of disturbances is understood with respect to a given linear functional. The paper ajoins theory of inverse problems for distributed systems (see, e.g., [Lavrentyev et al.,1980; Banks and Kunisch, 1982; Kurzhanski and Khapalov, 1989; Barbu, 1991; Osipov et al., 1991; Maksimov, 1993; Ainseba, 1994]) and theory of ill-posed problems (see, e.g., [Tikhonov and Arsenin, 1979], [Vasiliev, 1981]).

A variant of a substantial problem formulation is as follows. A flow of heat enters a solid body occupying space Ω through boundary domains $\gamma_1, \ldots, \gamma_n$. At time t the instant velocity of the heat flow coming through γ_k is represented as $v_k(t)\bar{\gamma}_k$ where $\bar{\gamma}_k$ is the square of γ_k. Values $v_k(t)$ are not available for direct observation. Instead, the average temperature of the body in spatial domains $\Omega_1, \ldots, \Omega_m \subset \Omega$ (as a variant, the temperature at points $\eta_1, \ldots, \eta_m \in \Omega$) is observed. It is required to reconstruct the minimum value of a linear functional $I = I(v(\cdot))$ over all $v(\cdot) = (v_1(\cdot), \ldots, v_n(\cdot))$ compatible with the observed data. A reconstruction algorithm should be robust to observation errors.

In our analysis we generalize the approach of [Kryazhimskii, and Osipov, 1993] based on the technique of adjoint equations (see [Lions, 1971] and [Marchuk, 1982]), and reduce the reconstruction problem to an extremal problem on the solutions of an appropriate finite-dimensional integral equation. To solve the latter problem, the method of convex optimization proposed in [Kryazhimskii, and Osipov, 1987] (see also [Kryazhimskii, 1994] and [Ermoliev et al., 1995]) is utilized. We design two iterative reconstruction algorithms and estimate rates of convergence.

Specify the problem. Let Ω be an open bounded domain in $\mathbf{R}^n$ with sufficiently smooth boundary Γ, Ax be a second order partial differential operator in Ω with smooth real coefficients,

$$(Ax)(\eta) = \sum_{i,j=1}^{n} \frac{\partial}{\partial \eta_i}(a_{ij}(\eta)\frac{\partial x(\eta)}{\partial \eta_j}) + \sum_{j=1}^{n} b_j(\eta)\frac{\partial x(\eta)}{\partial \eta_j} + c_0(\eta),$$

and the principal part uniformly strongly elliptic in Ω,

$$\sum_{i,j=1}^{n} a_{ij}(\eta)\eta_i\eta_j \geq \gamma \sum_{j=1}^{n} \eta_j^2, \quad a_{ij} \equiv a_{ji}, \quad \gamma > 0.$$

We consider the parabolic boundary control problem

$$\dot{x}_t(t,\eta) = (Ax(t))(\eta) \quad \text{in} \quad T \times \Omega, \quad T = [0, \vartheta], \tag{1.1}$$

$$x(0,\eta)=x_0(\eta),\quad \eta\in\Omega,$$
$$x(t,\sigma)=g(t,\sigma)\quad \text{in}\quad T\times\Gamma,$$

where

$$g(t,\sigma)=\sum_{k=1}^{n}\omega_k(\sigma)v_k(t)+f(t,\sigma),\qquad v=\{v_1,\ldots,v_n\}\in\mathbf{R}^n,$$

$\omega_k\in\mathbf{L}_2(\Gamma)$, $f(\cdot)\in\mathbf{L}_2(T;\Gamma)$, $x_0\in H=\mathbf{L}_2(\Omega)$. The n-dimensional control $v(\cdot)=(v_k(\cdot))_{k=1}^n\in\mathbf{L}_2(T;\mathbf{R}^n)$ treated as a disturbance effects the system through the Dirichlet boundary conditions. The disturbance is uncontrolled and not available for observation. It is assumed (see [Lasiecka, 1980]) that the closed and densely defined operator $A_1 : H\supset D(A)\to H$ determined by $A_1y=Ay$, $y\in D(A)=\{h\in H : Ah\in H,\ \ h\big|_\Gamma=0\}$ represents the infinitisimal generator of a strongly continuous semigroup of linear continuous operators $Q(t)$, $t\in\mathbf{R}$.

An (abstract) solution $x(\cdot):T\mapsto\mathbf{L}_2(\Omega)$ of problem (1.1) is defined by

$$x(t)=Q(t)x_0+\int_0^t AQ(t-\tau)\{\sum_{k=1}^{n}v_k(\tau)D\omega_k+Df(\tau)\}\,d\tau \tag{1.2}$$

(see [Lasiecka, 1980]). Here $w_v(t)=\sum_{k=1}^n v_k(t)D\omega_k+Df(t)$, $D\in\mathbf{L}(\mathbf{L}_2(\Gamma);H)$ is the Dirichlet map, i. e. $D\omega=y_\omega$ is the generalized solution of the elliptic equation

$$\begin{aligned} Ay_w&=0\quad \text{in}\quad \Omega\\ y_w\big|_\Gamma&=w\quad \text{in}\quad \Gamma;\end{aligned}$$

equivalently, y_w is the unique element of H such that

$$\int_\Omega y_wA\psi\,d\eta=\int_\Gamma w\frac{\partial\psi}{\partial\sigma}\,d\sigma\qquad \text{for all } \psi\in D(A).$$

It is known (see [Lasiecka, 1978, 1980]) that under the above conditions, for every $x_0\in H$ and $v(\cdot)\in H_1=\mathbf{L}_2(T;\mathbf{R}^n)$ there exists a unique (abstract) solution $x(\cdot)$ of problem (1.1); we denote it by $x(\cdot;x_0,v(\cdot))$.

Introduce the set of all admissible disturbances:

$$V_*=\{v(\cdot)\in W^{1,2}(T;\mathbf{R}^n) : v(0)=0,\ |\dot v(\cdot)|_{H_1}\le M\}; \tag{1.3}$$

here and in what follows

$$W^{1,2}(T;\mathbf{R}^n)=\{v(\cdot)\in H_1 : \dot v(\cdot)\in H_1\}.$$

Let $P:H\to\mathbf{R}^m$ be a linear continuous observation operator:

$$Px=\{\langle p_1,x\rangle_H,\ldots,\langle p_m,x\rangle_H\},\ p_i\in H,\ i\in[1:m].$$

We assume that, given a solution $x(\cdot)=x(\cdot;x_0,v(\cdot))$ corresponding to an (unknown) disturbance $v(\cdot)\in V_*$, the m-dimensional vector

$$z(t)=Px(t) \tag{1.4}$$

is observed at each $t\in T$. The observation result $\xi(t)$ is, in general, inaccurate:

$$|\xi(t) - z(t)|_{\mathbf{R}^m} \le h, \qquad \xi(\cdot) = \xi^h(\cdot). \tag{1.5}$$

We suppose that $\xi^h(\cdot) \in H_2 = \mathbf{L}_2(T; \mathbf{R}^m)$. Denote by V_z the set of all admissible disturbances $v(\cdot)$ compatible with $z(\cdot)$:

$$V_z = \{v(\cdot) \in V_* : \; Px(t; v(\cdot)) = z(t) \quad \text{for all } t \in T\}.$$

Let

$$I(v(\cdot)) = g_1' v(\vartheta) + \int_0^\vartheta g_2'(t) v(t)\, dt \tag{1.6}$$

be a linear functional on space $W^{1,2}(T; \mathbf{R}^n)$ ($g_1 \in \mathbf{R}^n$, $g_2 \in \mathbf{L}_2(T; \mathbf{R}^n)$; prime stands for transposition),

$$I_z^0 = \min \{ \; I(v(\cdot)) : v(\cdot) \in V_z \; \}, \tag{1.7}$$

and

$$V_0(z) = \arg\min \{ \; I(v(\cdot)) : v(\cdot) \in V_z \; \};$$

note that

$$V_0(z) \ne \emptyset \tag{1.8}$$

(see below Theorem 2.1).

Problem 1. Given a family of observation results $\xi^h = \xi^h(\cdot) \in H_2$, $h > 0$, satisfying (1.5), build a family of admissible disturbances $v^h(\cdot) = v(\cdot; \xi^h)$, $h > 0$, such that

$$I(v^h(\cdot)) \to I(v_0(\cdot)) = I_z^0, \quad v^h(\cdot) \to v_0(\cdot) \in V_0(z) \quad \text{in} \quad C(T; \mathbf{R}^n) \quad \text{as} \quad h \to 0. \tag{1.9}$$

2. Problem reduction.

We reduce Problem 1 to another one more convenient for investigation. Further on, the following condition is assumed.

Condition 2.1. a) $\omega_j \in H^{2\beta-1}(\Gamma)$, $\beta = [0.25n + 1.75 + 0.5\varepsilon] + 1$ ($\varepsilon \in (0, 1/2)$), $j \in [1:n]$, b) $x_0 \in H^{2\beta-1/2}(\Omega)$, c) $f(\cdot) \in H^{2\beta,\beta}(T; \Gamma)$.

Symbol $[a]$ denotes the integer part of a, $x^0(\cdot) = x(\cdot; x_0, 0)$, $x_j(\cdot) = x(\cdot; D\omega_j, 0)$. The algorithms described below solve Problem 1 under assumptions weaker than Condition 2.1. However, the estimates of the convergence rate (see (1.9)) provided in Sec.5 require functions $x(\cdot)$, $x^0(\cdot)$, $x_j(\cdot)$ to be sufficiently smooth (see below Lemma 2.1). The needed smoothness is ensured under Condition 2.1.

Lemma 2.1 *Let $u_j(\cdot) \in W^{1,2}(T; \mathbf{R})$, $u_j(0) = 0$, $j \in [1:n]$. Then $D\omega_j \in C(\Omega)$, $x_j(\cdot)$, $\dot{x}_j(\cdot)$, $x^0(\cdot)$, $\dot{x}^0(\cdot) \in C(T; C(\Omega))$, $x(\cdot) \in C(T; C(\Omega)$ and*

$$x(t) = x^0(t) + \int_0^t \sum_{j=1}^n \Big\{ x_j(t-\tau) - D\omega_j \Big\} \dot{u}_j(\tau)\, d\tau, \qquad t \in T. \tag{2.1}$$

Proof. If Condition 2.1 a) is true, then

$$D\omega_j \in C(\Omega), \quad x_j(\cdot), \dot{x}_j(\cdot) \in C(T; C(\Omega)), \quad \ddot{x}_j(\cdot) \in \mathbf{L}_2(T; C(\Omega)). \tag{2.2}$$

Indeed, as follows from [Lasiecka, 1980] (see also Theorem 7.4 in [Lions and Magenes, 1972]), we have $D \in \mathbf{L}_2(H^{2\nu}(\Gamma); H^{2\nu+1/2}(\Omega))$ for all $\nu \in \mathbf{R}$. Let $\nu = \beta - 0.5$. Theorem 6.1 from [Lasiecka, 1980] implies

$$x_j(\cdot) \in \mathbf{L}_2(T; H^{2\beta+1/2-\varepsilon}(\Omega)), \quad \dot{x}_j(\cdot) \in \mathbf{L}_2(T; H^{2(\beta-1)+1/2-\varepsilon}(\Omega)),$$
$$\ddot{x}_j(\cdot) \in \mathbf{L}_2(T; H^{2(\beta-2)+1/2-\varepsilon}(\Omega)).$$

Therefore (see Theorem 9.8 in [Lions and Magenes, 1972]), changing, if necessary, $x_j(t,\eta)$, $((t,\eta) \in T \times \Omega)$ on a set of zero measure we obtain that for an integer β satisfying $2(\beta-2) + 0.5 - \varepsilon > 0.5n$ the following is true:

$$H^{2\beta+1/2-\varepsilon}(\Omega) \subset H^{2(\beta-1)+1/2-\varepsilon}(\Omega) \subset H^{2(\beta-2)+1/2-\varepsilon}(\Omega) \subset C(\Omega),$$

$$D\omega_j \in C(\Omega), \quad \ddot{x}_j(\cdot) \in L_2(T; C(\Omega)).$$

The validity of (2.2) is established. Analogously, we deduce that if $f(\cdot) \in H^{2\beta,\beta}(T;\Gamma)$, $x_0 \in H^{2\beta-1}(\Omega)$, then

$$x_j^0(\cdot) \in C(T; C(\Omega)), \quad \dot{x}_j^0(\cdot) \in C(T; C(\Omega)), \quad (\, \ddot{x}_j^0(\cdot) \in \mathbf{L}_2(= x(T; C(\Omega))\,).$$

To verify equality (2.1) it is sufficient to integrate by parts the term

$$\int_0^t AQ(t-\tau) \sum_{k=1}^n v_k(\tau) D\omega_k \, d\tau.$$

(see [Lasiecka, 1980]). Lemma is proved.

Introduce the linear operator $F : H_1 \to H_2$,

$$(Fu(\cdot))(\sigma) = \int_0^\sigma \varphi(t,\sigma) u(t)\, dt, \quad \sigma \in T, \tag{2.3}$$

where $\varphi(t,\sigma) = \varphi(\sigma - t)$ is the $m \times n$-dimensional matrice whose k-th row has the form

$$\varphi_k(t,\sigma) = \{\{\varphi_k(t,\sigma)\}_1, \{\varphi_k(t,\sigma)\}_2, \ldots, \{\varphi_k(t,\sigma)\}_n\},$$

$$\{\varphi_k(t,\sigma)\}_j = \{\varphi_k(\sigma - t)\}_j = \begin{cases} \langle x_j(\sigma - t) - D\omega_j,\, p_k\rangle_H, & \sigma \ge t, \\ 0, & \sigma < t, \end{cases} \qquad j \in [1:n].$$

Theorem 2.1 *We have $v(\cdot) \in V_z$ if and only if*

$$\dot{v}(\cdot) \in U_*, \qquad (F\dot{v}(\cdot))(t) = b_z(t) \qquad \textit{for all } t \in T.$$

Here

$$b_z(t) = \{\, b_z(t)_1, \ldots, b_z(t)_m \,\} \in \mathbf{R}^m,$$

$$b_z(t)_k = z_k(t) - \langle x^0(t),\, p_k\rangle_H \in \mathbf{R}, \qquad k \in [1:m],$$

$$U_* = \{u(\cdot) \in H_1 : \ |u(\cdot)|_{H_1} \le M\}, \tag{2.4}$$

and $z(t) = \{z_1(t), \ldots, z_m(t)\}$ is defined by to (1.4).

The theorem follows from Lemma 2.1. Note that Theorem 2.1 implies (1.8). For any $v(\cdot) \in V_*$ such that $\dot{v}(t) = u(t)$ for a.a. $t \in T$ the following equality

$$I(v(\cdot)) = J(u(\cdot)) \equiv \int_0^{\vartheta} g'(t)u(t)\,dt \tag{2.5}$$

is valid, provided $g(\vartheta) = g_1$, $\dot{g}(t) = -g_2(t)$ for a.a. $t \in T$. Thus, Problem 1 is replaced by the following one. Let

$$J_z^0 = \min \{\ J(u(\cdot)) : u(\cdot) \in U_z\ \}, \tag{2.6}$$

$$U_z = \{u(\cdot) \in U_* : Fu(\cdot) = b_z\}, \qquad U_0(z) = \arg\min \{\ J(u(\cdot)) : u(\cdot) \in U_z\ \}.$$

Problem 2. Given a family of observation results $\xi^h = \xi^h(\cdot) \in H_1$, $h > 0$, satisfying (1.5), build a family $u^h(\cdot) = u(\cdot;\xi^h)$, $h > 0$, from U_* such that

$$J(u^h(\cdot)) \to J(u_0(\cdot)) = J_z^0, \quad u^h(\cdot) \to u_0(\cdot) \in U_0(z) \quad \text{weakly in} \quad H_1 \quad \text{as} \quad h \to 0. \tag{2.7}$$

It is easily seen that if family $u^h(\cdot)$ solves Problem 2, then family $v^h(\cdot)$

$$v^h(\tau) = \int_0^{\tau} u^h(\nu)\,d\nu, \quad \tau \in T,$$

solves Problem 1, and

$$I(v^h(\cdot)) = J(u^h(\cdot)). \tag{2.8}$$

Futher on we focus on Problem 2.

3. Solution algorithm. 1.

We use the approach of [Kryazhimskii and Osipov, 1987; Kryazhimskii, 1994]. By Theorem 2.1 and equality (2.8) we have

$$J_z^0 = \min \{\ J(v) : v \in U_*,\ Fv = b_z\ \}. \tag{3.1}$$

Denote

$$J_z^0[\gamma] = \min \{\ J(v) : v \in U_*,\ |Fv - b_z|^2_{H_2} \ \le \gamma\ \}. \tag{3.2}$$

It is clear that

$$J_z^0[\gamma] \le J_z^0, \qquad \lim_{\gamma \to +0} J_z^0[\gamma] = J_z^0. \tag{3.3}$$

Let $U_z^0[\gamma,\beta]$ ($\beta \ge 0,\ \gamma > 0$) be the set of all $u(\cdot) \in U_*$ such that

$$|Fu - b_z|^2_{H_2} \ \le \gamma, \qquad J(u) \le J_z^0 + \beta.$$

In what follows, we set $N = \{0, 1, 2, \ldots\}$, and $\langle\cdot,\cdot\rangle_{H_2}$ stands for the scalar product in H_2. Fix a family of mappings

$$V_\alpha : \{i, y, \xi(\cdot)\} \mapsto V_\alpha(i, y, \xi) : N \times H_1 \times H_2 \to U_*, \quad \alpha > 0.$$

For every $\delta, \alpha, h > 0$ define sequence $y^\alpha_{\delta,h}(i), \quad i \in N$, in H_2, as follows :

$$y^\alpha_{\delta,h}(i+1) = y^\alpha_{\delta,h}(i) + v^h_i \delta, \quad v^h_i = V_\alpha(i, y^\alpha_{\delta,h}(i), \xi^h), \quad y^\alpha_{\delta,h}(0) = 0, \qquad i \in N. \tag{3.4}$$

It is easily seen that by virtue of the convexity and closedness of set U_*, we have

$$y^\alpha_{\delta,h}(i)/(i\delta) \in U_*, \qquad i > 0.$$

Specify V_α. We put

$$V_\alpha(i, y, \xi) = \arg\min \{ \ 2\langle Fy - i\delta\, b^h_\xi, \ Fu\rangle_{H_2} + \alpha J(u) : \ u \in U_* \ \},$$

or, explicitly,

$$V_\alpha(i, y, \xi)(\nu) = u(\nu; \ i, y, \xi) \qquad (\nu \in T),$$

$$u(\nu; \ i, y, \xi^h) = \begin{cases} -M \dfrac{D_\alpha(\nu; \ i, y, \xi)}{|D_\alpha(\cdot; \ i, y, \xi)|_{H_1}}, & \text{if } |D_\alpha(\cdot\,; i, y, \xi)|_{H_1} \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

Here

$$D_\alpha(\nu; \ i, y, \xi) = 2 \int\limits_\nu^\vartheta \mu(\nu, \sigma; \ i, y, \xi) d\sigma + \alpha g(\nu), \qquad \nu \in T,$$

$$\mu(\nu, \sigma; \ i, y, \xi) = \varphi(\nu, \sigma)\nu_0(\sigma; \ i, y, \xi), \qquad \nu_0(\sigma; \ i, y, \xi) = \int\limits_0^\sigma \varphi(\varrho, \sigma) y(\varrho)\, d\varrho - i\delta\, b^h_\xi(\sigma),$$

$$b^h_\xi(\sigma) = \{b^h_\xi(\sigma)_1, \ldots, b^h_\xi(\sigma)_m\} \in \mathbf{R}^m, \tag{3.6}$$

$$b^h_\xi(\sigma)_k = \xi_k(\sigma) - \langle x_0(t), p_k\rangle_H \in \mathbf{R}, \qquad k \in [1:m].$$

Define constants K_J and K_F by $|J(v)| \ \leq K_J, \ \ |Fv - b_z|_{H_2} \ \leq K_F \ \ (v \in U_*, \ z(\cdot) \in \{z(\cdot) : z(t) = Px(t, v(t))$ for all $t \in T, \ v(\cdot) \in U_*\})$.

Theorem 3.1 *It holds that*

$$y^\alpha_{\delta,h}(i)/(i\delta) \in U^0_z\,[2K_J\alpha/(i\delta) + \varrho_i(\delta, h)/(i\delta), \ \varrho_i(\delta, h)/\alpha], \tag{3.7}$$

$$J^0_z\,[2K_J\alpha/(i\delta) + \varrho_i(\delta, h)/(i\delta)] - J^0_z \leq J(y^\alpha_{\delta,h}(i)/(i\delta)) - J^0_z \leq \varrho_i(\delta, h)/\alpha, \tag{3.8}$$

where

$$\varrho_i(\delta, h) = K^2_F\delta + 2K_F\sqrt{\vartheta}h\, i\delta, \qquad i \geq 1. \tag{3.9}$$

Proof. We follow the proof pattern of [Kryazhimskii, 1994]. Estimate recurrently the values

$$\Lambda_\alpha(i; \ y^\alpha_{\delta,h}(\cdot)) = |Fy^\alpha_{\delta,h}(i) - i\delta\, b_z|^2_{H_2} + \alpha\delta \sum_{j=1}^i J(v^h_{i-1}) - \alpha i\delta J^0_z, \qquad i \geq 1.$$

We have

$$\Lambda(1;\ y^{\alpha}_{\delta,h}(\cdot)) = 0. \tag{3.10}$$

Let

$$v_z \in U^0_z = \arg\min\ \{\ J(v) : v \in U_*,\ \ Fv = b_z\ \}.$$

By Theorem 2.1 $U^0_z = U_0(z)$. Therefore

$$\Lambda_\alpha(i+1;\ y^{\alpha}_{\delta,h}(\cdot)) = \Lambda_\alpha(i;\ y^{\alpha}_{\delta,h}(\cdot)) + 2\delta\langle\lambda_i,\ (Fv^h_i - b_z)\rangle_{H_2} +$$

$$+\,|Fv^h_i - b_z|^2_{H_2}\delta^2 + \alpha J(v^h_i) - \alpha J^0_z\delta \le$$

$$\le\ \Lambda_\alpha(i;\ y^{\alpha}_{\delta,h}(\cdot)) + [2\langle\lambda_i,\ Fv^h_i - b_z\rangle_{H_2} + \alpha J(v^h_i)]\delta + \tag{3.11}$$

$$+\,[2\langle\lambda_i,\ Fv_z - b_z\rangle_{H_2} + \alpha J(v_z)]\delta + K^2_F\delta^2, \qquad \lambda_i = Fy^{\alpha}_{\delta,h}(i) - i\delta\, b_z.$$

By virtue of (1.5)

$$|\xi^h(\cdot) - z(\cdot)|_{H_2} \le \sqrt{\vartheta}h.$$

Consequently,

$$|b_z(\cdot) - b^h_\xi(\cdot)|_{H_2} \le \sqrt{\vartheta}h, \tag{3.12}$$

$$\langle\lambda_i,\ Fv^h_i - b_z\rangle_{H_2} = \langle Fy^{\alpha}_{\delta,h}(i) - i\delta\, b^h_\xi + i\delta\,(b^h_\xi - b_z),\ Fv^h_i - b_z\rangle_{H_2}\ \le$$

$$\le \langle Fy^{\alpha}_{\delta,h}(i) - i\delta\, b^h_\xi,\ Fv^h_i - b_z\rangle_{H_2} + i\delta\,\sqrt{\vartheta}h\cdot K_F.$$

Similarly we obtain

$$\langle\lambda_i,\ Fv_z - b_z\rangle_{H_2}\ \le \langle Fy^{\alpha}_{\delta,h}(i) - i\delta\, b^h_\xi,\ Fv_z - b_z\rangle_{H_2} + i\delta\,\sqrt{\vartheta}h\cdot K_F. \tag{3.13}$$

Then the following equality is true

$$2\langle Fy - i\delta\, b^h_\xi,\ Fu\rangle_{H_2} + \alpha J(u) = 2\int\limits_0^{\vartheta} \nu_0(\sigma;\ i, y, \xi^h)\int\limits_0^{\sigma} \varphi(\nu,\sigma)u(\nu)\,d\nu\,d\sigma\ + \tag{3.14}$$

$$+\ \alpha\int\limits_0^{\vartheta} g'(\sigma)u(\sigma)\,d\sigma = \int\limits_0^{\vartheta} D'_\alpha(\nu;\ i, y, \xi^h)u(\nu)\,d\nu = \langle D_\alpha,\, u\rangle_{H_1}.$$

From (3.4), (3.5), (3.10)–(3.14) we derive that

$$\Lambda_\alpha(i+1;\ y^{\alpha}_{\delta,h}(\cdot)) \le \Lambda_\alpha(i;\ y^{\alpha}_{\delta,h}(\cdot)) + 4i\delta^2\sqrt{\vartheta}K_F h + K^2_F\delta^2 \le \varrho_i(\delta,h)(i+1)\delta. \tag{3.15}$$

Note that

$$\lambda_{\alpha/(i\delta)}(y^{\alpha}_{\delta,h}(i)/(i\delta)) \le \Lambda_\alpha(i;\ y^{\alpha}_{\delta,h}(\cdot))/(i\delta)^2 \le \varrho_i(\delta,h)/(i\delta). \tag{3.16}$$

Here

$$\lambda_a(y) =\ |Fy - b_z|^2_{H_2} +\ aJ(y) - aJ^0_z, \quad a > 0.$$

Besides, if $v \in U_*$, $\lambda_a(y) \le \varepsilon$, then the following relations hold:

$$y \in U_z^0\,[2K_J a + \varepsilon, \varepsilon/a], \tag{3.17}$$

$$J_z^0\,[2K_J a + \varepsilon] - J_z^0 \le J(y) - J_z^0 \le \varepsilon/a. \tag{3.18}$$

The estimates (3.7)–(3.9) follow from (3.15)–(3.18). The Theorem is proved.

Theorem 3.1 yields the following.

Theorem 3.2 *Let $\alpha(h) > 0$, $\delta(h) > 0$, and $i(h) \in N$ satisfy*

$$\alpha(h) \to 0, \quad \delta(h) \to 0, \quad \delta(h)i(h) \to \infty, \quad (hi(h)\delta(h) + \delta(h))/\alpha(h) \to 0 \text{ as } h \to 0.$$

Then the family

$$u^h(\cdot) = y^{\alpha(h)}_{\delta(h),h}(i(h))/(i(h)\delta(h)), \qquad h > 0,$$

solves Problem 2, i.e. convergences (2.7) take place.

4. Solution algorithm. 2.

Let us describe another solution algorithm. Introduce the mapping $p : (i, y, \xi) \mapsto p(i, y, \xi^h(\cdot)) : T \times H_1 \times H_2 \to H_1$:

$$p = p(\nu;\ i, y, \xi) = \int_0^\vartheta \mu(\nu, \sigma;\ i, y, \xi)\, d\sigma, \quad \nu \in T \tag{4.1}$$

(see (3.6)). For an arbitrary triple $(i, y, \xi) \in T \times H_2 \times H_2$, we represent element $g \in H_1$ (see (2.5)) in the form

$$g = \mu p + c_1, \tag{4.2}$$

$$\mu = \langle g, p\rangle_{H_1} |p|^{-2}_{H_1}, \qquad c_1 = g - \mu p, \tag{4.3}$$

and denote

$$\mu = \mu(i, y, \xi), \qquad p = p(i, y, \xi), \qquad c_1 = c_1(i, y, \xi).$$

Note that c_1 is orthogonal to p, i.e. $\langle c_1, p\rangle_{H_1} = 0$.

Let the mapping $V : (i, y, \xi) \mapsto V(i, y, \xi) \in N \times H_2 \times H_2 \to U_*$ be defined by

$$V(i, y, \xi)(\nu) = u(\nu;\ i, y, \xi) \quad (\nu \in T), \tag{4.4}$$

where

$$u(\nu) = u(\nu;\ i, y, \xi) = \begin{cases} -Mg(\nu)|g|^{-1}_{H_1}, & \text{if } p = 0 \\ \nu_1 p(\nu) + v_1(\nu), & \text{otherwise,} \end{cases} \tag{4.5}$$

and $p = p(\nu) = p(\nu;\ i, y, \xi)$, $\nu \in T$, is defined by (4.1). If $c_1 = c_1(i, y, \xi) \neq 0$, then we have

$$v_1 = -k_* c_1, \qquad k_* = ((M^2 - \nu_1^2 |p|^2_{H_1})/|c_1|^2_{H_1})^{1/2},$$

$$\nu_1 = \min\ \{\ \nu_0, g_2\}, \qquad \nu_0 = -\mathrm{sign}\{a_0 \frac{a_0\sqrt{a_1}}{\sqrt{a_0^2 + a_2\sqrt{a_2}}}\ \}, \tag{4.6}$$

$$a_0 = \mu |p|^2_{H_1}, \qquad a_1 = M^2 |c_1|_{H_1}, \qquad a_2 = |p|^2_{H_1} \cdot |c_1|_{H_1},$$

$$g_1 = -M|p|^{-1}_{H_1} = -\sqrt{a_1 a_2^{-1}}, \qquad g_2 = \min\ \{\ -g_1, \beta |p|^{-2}_{H1}\ \},$$

$$\beta = \beta(i, y, \xi) = \langle Fy - i\delta\, b^h_\xi,\ b^h_\xi \rangle_{H_2} + \varrho_1(h) =$$

$$= \int\limits_0^\vartheta \Big(\int\limits_0^\sigma \varphi(t,\sigma) y(t)\, dt - i\delta\, b^h_\xi(\sigma) \Big)'\ b^h_\xi(\sigma)\, d\sigma + \varrho_1(h),$$

$$\varrho_1(h) = 2K_F\sqrt{\vartheta}h + \vartheta h^2.$$

If $c_1 = c_1(i, y, \xi^h) = 0$, then

$$\nu_1 = \begin{cases} g_2, & a_0 > 0 \\ g_1, & a_0 < 0 \\ v_* \in [g_1, g_2], & a_0 = 0, \end{cases} \tag{4.7}$$

and $v_1 = v_1(\nu)$, $\quad \nu \in T$, is defined as an arbitrary element in space H_1 such that

$$\langle v_1, p \rangle_{H_1} = 0, \qquad \nu_1 |p|^2_{H_1} + |v_1|^2_{H_1}\ \leq M^2.$$

Below, in the proof of Theorem 4.1, we shall show that set $V(i, y, \xi)$ (4.4) is that of all solutions of the problem

$$\min\quad J(v), \tag{4.8}$$
$$v \in U_*,\ \langle Fy - i\delta\, b^h_\xi,\ Fv - b^h_\xi \rangle_{H_2}\ \leq \varrho_1(h).$$

Introduce the sequence $y_{\delta,h}(i) \in H_2$ $\ (\delta > 0, h \geq 0)$, $\ i \in N$, by

$$y_{\delta,h}(i+1) = y_{\delta,h}(i) + v_i\delta, \quad v_i = V(i, y_{\delta,h}(i), \xi^h).$$

Let

$$\varrho_*(\delta, h) = K_F^2\delta + 4\varrho_1(h).$$

Theorem 4.1 *It holds that*

$$y_{\delta,h}(i)/(i\delta) \in U^0_z\,[\varrho_*(\delta,h)/(i\delta),\ 0], \quad (i > 0),$$

$$J^0_z[\varrho_*(\delta,h)/(i\delta)] \leq J(y_{\delta,h}(i)/(i\delta)) \leq J^0_z.$$

Proof. It is sufficient to show that

$$|F(y_{\delta,h}(i)/(i\delta)) - b_z|^2_{H_2} \ \leq \varrho_*(\delta,h)/(i\delta), \quad J(y_{\delta,h}(i)/(i\delta)) \leq J^0_z.$$

It is easily seen that these inequalities are ensured by

$$|Fy_{\delta,h}(i\delta) - i\delta\, b_z|^2_{H_2} \ \leq \varrho_*(\delta,h)i\delta, \qquad \delta\sum_{j=1}^{i} J(v_{j-1}) \leq i\delta J^0_z. \tag{4.9}$$

Let us verify (4.9). First, we prove that $V(i,y,\xi^h)$ is the set of all solutions of problem (4.8). Using (4.1)–(4.3), transform problem (4.8) into

$$\min \ \mu\nu|p|^2_{H_1} + \ \langle c_1, v_1\rangle_{H1},$$

$$\{\nu, v_1\} \in \mathbf{R}^+ \times H_1, \quad \langle v_1, p\rangle_{H_1} = 0,$$

$$\nu^2|p|^2_{H_1} + |v_1|^2_{H_1} \ \leq M^2, \quad \nu \leq \beta|p|^{-2}_{H_1}, \qquad \mathbf{R}^+ = \{z \in \mathbf{R} : z \geq 0\}. \tag{4.10}$$

For a fixed ν the minimum in (4.10) is reached at

$$v_1 = -k_0 c_1, \qquad k_0 = ((M^2 - \nu^2|p|^2_{H_1})|c_1|^{-2}_{H_1})^{1/2}$$

if $c_1 \neq 0$. In the opposite case v_1 is an arbitrary element such that

$$\langle v_1, p\rangle_{H_1} = 0, \qquad \nu^2|p|^2_{H_1} + |v_1|^2_{H_1} \ \leq M^2.$$

In this case problem (4.10) is reduced to

$$\min \ \mu\nu|p|^2_{H_1} + ((M^2 - \nu^2|p|^2_{H_1})|c_1|^2_{H_1})^{1/2}, \tag{4.11}$$
$$\nu^2|p|^2_{H_1} \ \leq M^2, \ \nu \leq \beta|p|^{-2}_{H_1}$$

if $c_1 \neq 0$, and

$$\min \ \mu\nu|p|^2_{H_1}, \tag{4.12}$$
$$\nu^2|p|^2_{H_1} \ \leq M^2, \ \nu \leq \beta|p|^{-2}_{H_1}$$

if $c_1 = 0$. Rewrite problem (4.11) in the form

$$\min \{ \ a_0\nu - (a_1 - a_2\nu^2)^{1/2} : \nu \in [g_1, g_2] \ \}. \tag{4.13}$$

It is easily seen that the minimum in (4.13) is reached at $\nu = \nu_1$ (4.6). In turn, a solution of problem (4.12) is of the form (4.7). Thus, $V(i,y,\xi)$ is the set of all solutions of problem (4.8). Let us verify inequalities (4.9). Using (3.12), we obtain

$$|\langle Fy - b_z, \ Fv - b_z\rangle_{H_2} - \langle Fy - b^h_\xi, \ Fv - b^h_\xi\rangle_{H_2}| \ \leq$$

$$\leq \ |\langle Fy - b_z, \ b^h_\xi - b_z\rangle_{H_2}| + |\langle Fy - b_z, \ Fv - b^h_\xi\rangle_{H_2} - \langle Fy - b^h_\xi, \ Fv - b^h_\xi\rangle_{H_2}| \ \leq$$

$$\leq K_F\sqrt{\vartheta}h \ + \{ \, |Fv - b_z|_{H_2} \ + \ |b^h_\xi - b_z|_{H_2} \, \}\sqrt{\vartheta}h \leq \varrho_1(h).$$

Hence by the definition of $V(i, y, \xi^h)$ we have

$$J(v_i) \leq J_z^0. \tag{4.14}$$

Let $S(i) = Fy_{\delta,h}(i) - i\delta$. Then by virtue of the inclusion

$$v_i \in \{v \in U_* : \langle S(i),\ Fv - b_z\rangle_{H_2}\ \leq 2\varrho_1(h)\}$$

the following inequality holds:

$$|S(i+1)|^2_{H_2}\ =\ |S(i)|^2_{H_2} +\ 2\langle S(i),\ Fv_i - b_z\rangle_{H_2} \cdot \delta +$$

$$+ |Fv_i - b_z|^2_{H_2}\delta^2\ \leq\ |S(i)|^2_{H_2} + K_F^2\delta^2 + 4\varrho_1(h)\delta. \tag{4.15}$$

Besides,

$$|S(0)|_{H_2} = 0. \tag{4.16}$$

The inequality (4.9) follows from (4.14)–(4.16). The Theorem is proved.

Theorem 4.1 yields the following.

Theorem 4.2 *Let $\delta(h) > 0$ and $i(h) \in N$ satisfy*

$$\delta(h) \to 0, \quad \delta(h)i(h) \to \infty \text{ as } h \to 0.$$

Then the family

$$u^h(\cdot) = y_{\delta(h),h}(i(h))/(i(h)\delta(h)), \qquad h > 0,$$

solves Problem 2, i.e. convergences (2.7) take place.

5. Estimates of convergence rate.

In this Section we provide estimates of the convergence rate for the above described algorithms. Assume that the following condition is satisfied.

Condition 5.1 Functions $\varphi(\cdot)$ and $b_z(\cdot)$ are Lipshitz.

By Lemma 2.1 Condition 5.1 is ensured by Condition 2.1.

Fix a partition of interval T:

$$\{t_j\}_{j=0}^{E(\Delta)}, \qquad t_0 = 0, \qquad t_{j+1} = t_j + \Delta, \qquad t_{E(\Delta)} = \vartheta.$$

Introduce the operator $F_\Delta : H_1 \to H_2$ (a "Δ - approximation" to F):

$$(F_\Delta u(\cdot))(t) = \int_0^t \varphi_\Delta(t-\sigma)u(\sigma)\,d\sigma \qquad (t \in T), \tag{5.1}$$

where

$$\varphi_\Delta(t-\sigma) = \varphi(t_j - \sigma) \qquad (t \in [t_j, t_{j+1})), \qquad j \in [0:\ E(\Delta)-1], \qquad (\sigma \in T).$$

Denote by K a Lipschitz constant for functions $\varphi(t)$ and $b_z(t)$ on T. Let

$$b_z^\Delta(t) = b_z(t_j), \qquad t \in [t_j, t_{j+1}),$$

$$J_z^\Delta[\varepsilon] = \min \{ \; J(v) : v \in U_*, \; |F_\Delta v - b_z^\Delta|_{H_2} \; \le \varepsilon \; \}. \tag{5.2}$$

Note that the following inequalities are true:

$$|b_z - b_z^\Delta|_{H_2} \; \le K\sqrt{\vartheta}\Delta,$$

$$|F_\Delta u - Fu|_{H_2} \; \le \Delta K M \vartheta \qquad \forall u \in U_*. \tag{5.3}$$

By virtue of (5.3) we obtain

$$J_z^0 \ge J_z^0[\gamma^{1/2}] \ge J_z^\Delta[\gamma + \Delta k_*] \qquad (\gamma > 0), \qquad k_* = K(M\vartheta + \sqrt{\vartheta}),$$

hence

$$J_z^0 \ge J_z^0\,[(\varepsilon - k_*\Delta)^{1/2}] \ge J_z^\Delta[\varepsilon] \qquad (\varepsilon > k_*\Delta). \tag{5.4}$$

The next theorem provides an estimate for the difference between J_z^0 and $J_z^\Delta[\varepsilon]$ (recall that J_z^0, J_z^Δ, U_* and F_Δ are defined by (2.6), (5.2), (1.2) and (2.4)).

Theorem 5.1 *Let Condition 5.1 be fulfilled. Then for every $\varepsilon > \Delta k_*$ it holds that*

$$k_0 \varepsilon^{1/2}\,\Delta^{-1/4} \le J_z^\Delta[\varepsilon] - J_z^0 \le 0, \tag{5.5}$$

where k_0 does not depend on ε and Δ and can be computed explicitly.

Proof. The inequality

$$J_z^\Delta[\varepsilon] - J_z^0 \le 0$$

follows from (5.4). Let us verify that

$$J_z^\Delta[\varepsilon] - J_z^0 \ge k_0 \varepsilon^{1/2} \Delta^{-1/4}.$$

Note that

$$\langle l, \; F_\Delta v - b_z^\Delta \rangle_{H_2} = \int_0^\vartheta l'(t)\; \{ \int_0^t \varphi_\Delta(t-\sigma) v(\sigma)\, d\sigma - b_z^\Delta(t) \}\, dt =$$

$$= \int_0^\vartheta \psi_l'(\sigma) v(\sigma)\, d\sigma - \int_0^\vartheta l'(t) b_z^\Delta(t)\, dt,$$

where

$$\psi_l'(\sigma) = \int_\sigma^\vartheta l'(t)\varphi_\Delta(t-\sigma)\, dt \equiv \int_0^\vartheta l'(t)\varphi_\Delta(t-\sigma)\, dt,$$

$$l(\cdot) = \{\, l_1(\cdot), \ldots, l_m(\cdot)\} \in \mathbf{L}_2(T; \mathbf{R}^m);$$

l_r is the r-th component of vector $l \in \mathbf{R}^m$. Therefore inequality $|F_\Delta v - b_z^\Delta|_{H_2} \le \varepsilon$ can be rewritten into

$$\sup_{|l|_{H_2} \le 1} \{ \int_0^\vartheta \psi_l'(\sigma) v(\sigma)\, d\sigma - \int_0^\vartheta l'(\sigma) b_z^\Delta(\sigma)\, d\sigma \} \le \varepsilon. \tag{5.6}$$

Let $\Phi \subset H_1$ be the closure in H_1 of the linear hull of the set

$$\{\lambda(\sigma) \in \mathbf{R}^n,\ \sigma \in T : \exists t \in T \text{ I } k \in [1:m] \quad \lambda(\sigma) = \{\varphi_\Delta(t-\sigma)\}_k \text{ PRI P. W.}\, \sigma \in [0,t]\} \subset H_1;$$

here $\{\varphi_\Delta(t)\}_k$ is the k-th line of matrice $\varphi_\Delta(t)$, $k \in [1:m]$. Represent element $g \in H_1$ in the form

$$g = p_1 + p_2,$$

where $p_1 \in \Phi$, $\langle p_2, \lambda\rangle_{H_1} = 0 \quad (\lambda \in \Phi)$, i.e. $p_2 \perp \Phi$. Then

$$J(u) = \langle p_1, u_1\rangle_{H_1} + \langle p_2, u_2\rangle_{H_1} \quad (u \in U_*),$$

$$u = u_1 + u_2, \qquad u_1 \in \Phi, \qquad u_2 \perp \Phi,$$

$$p_1(\sigma) = \sum_{j=1}^{S} \alpha_j \varphi^*(t_j - \sigma), \quad \sigma \in T, \quad \varphi^* = \varphi_{k_j}(t_j - \sigma) \in \Phi, \quad 0 \le s \le E(\Delta) \times m.$$

Consequently

$$\langle p_1, u_1\rangle_{H_1} = \sum_{j=1}^{S} \alpha_j \{b_z(t_j)\}_{k_j} \equiv c \qquad (u_1 \in \Phi).$$

Therefore

$$J_z^0 = c + \min \{ \langle p_2, u_2\rangle_{H_1} : u = u_1 + u_2,\ u_1 \in \Phi,\ u_2 \perp \Phi,\ |u_1|_{H_1}^2 + |u_2|_{H_1}^2 \le M^2 \}.$$

It is easily seen that the above minimum is reached at the element

$$u^0 = u_1^0 + u_2^0,$$

$$u_1^0 = p_1 c |p_1|_{H_1}^{-2}, \quad u_2^0 = -p_2 (M^2 - c^2 |p_1|_{H_1}^{-2})^{1/2} |p_2|_{H_1}^{-1}.$$

Thus,

$$J_z^0 = c - (M^2 - c^2 |p_1|_{H_1}^{-2})^{1/2}. \tag{5.7}$$

Consider $J_z^\Delta[\varepsilon]$. We have

$$J_z^\Delta[\varepsilon] = \min \{ \langle p_1, u_1\rangle_{H_1} + \langle p_2, u_2\rangle_{H_1} : u = u_1 + u_2 \in U_*, \quad u_1 \in \Phi,\ u_2 \perp \Phi$$

$$\text{and inequality (5.6) is true } \}.$$

Let $d \in (0, \Delta]$ and vector functions $l_j(\cdot) \in H_2$, $j \in [1:s]$, be such that

$$l_j(t)_{k_j} = \begin{cases} d^{-1}, & t \in \Delta_j \equiv [t_j, t_j + d], \\ 0, & t \in T \setminus \Delta_j, \end{cases} \tag{5.8}$$

$$l_j(t)_r = 0 \quad \forall t \in T, \quad r \ne k_j, \quad r \in [1 : E(\Delta)].$$

Since $d \in (0, \Delta]$, the following inequalities hold:

$$\int_0^{\vartheta} l'_j(t)\varphi_\Delta(t-\sigma)_{k_j}\,dt = \varphi_\Delta(t_j-\sigma)_{k_j}, \tag{5.9}$$

$$\int_0^{\vartheta} l'_j(t) b_z^\Delta(t)\,dt = b_z^\Delta(t_j)_{k_j}.$$

Therefore, using (5.6), (5.8) and (5.9), we obtain

$$|\langle p_1, u\rangle_{H_1} - \sum_{j=1}^{S} \alpha_j \{b_z(t_j)\}_{k_j}| \;=\; |\langle p_1, u\rangle_{H_1} - c| \;\le \mu, \quad \mu = k_1 \varepsilon d^{-1/2}.$$

Hence

$$J_z^\Delta[\varepsilon] \ge c - \mu + J_*[\varepsilon], \tag{5.10}$$

where

$$J_*[\varepsilon] = \min\,\{\ \langle p_2, u_2\rangle_{H_1} : u = u_1 + u_2, \quad u_1 \in \Phi, \quad u_2 \perp \Phi, \tag{5.11}$$

$$|u|_{H_1} \;\le M, \quad |\langle p_1, u_1\rangle_{H_1} - c| \;\le \mu\ \}.$$

Fix a $u_1 = u_1^* \in \Phi$. We have

$$\arg\min\,\{\langle p_2, u_2\rangle_{H_1} : u = u_1^* + u_2, \quad u_2 \perp \Phi, \quad |u_1^* + u_2|_{H_1} \;\le M\ \} = \tag{5.12}$$
$$= -p_2 (M^2 - |u_1^*|_{H_1}^2)^{1/2}\, |p_2|_{H_1}^{-1},$$

and

$$\min\,\{\ \langle p_2, u_2\rangle_{H_1} : u = u_1^* + u_2, \quad u_2 \perp \Phi, \quad |u_1^* + u_2|_{H_1} \;\le M\ \} = \tag{5.13}$$

$$= -(M^2 - |u_1^*|_{H_1}^2)^{1/2} |p_2|_{H_1}.$$

Besides,

$$\arg\min\,\{\ |u_1|_{H_1} : c - \mu \le \langle p_1, u_1\rangle_{H_1} \le c + \mu\ \} = \tag{5.14}$$
$$= \min\,\{\ |c-\mu|, |c+\mu|\ \} p_1 \cdot |p_1|_{H_1}^{-2},$$
$$\mu = \mu(\varepsilon, d).$$

From (5.11)–(5.14) follows that

$$J_*[\varepsilon] = -(M^2 - \min\,\{\ |c-\mu|^2,\ |c+\mu|^2\ \} |p_1|_{H_1}^{-2})^{1/2}.$$

Hence by (5.7), (5.10) we deduce

$$J_z^\Delta[\varepsilon] - J_z^0 \ge \mu + (M^2 - c^2 |p_1|_{H_1}^{-2})^{1/2} -$$

$$- ((M^2 - c^2 |p_1|_{H_1}^{-2}) + (2|c|\mu - \mu^2)|p_1|_{H_1}^{-2})^{1/2} \ge k_2 \varepsilon^{1/2} d^{-1/4}.$$

Now inequality (5.5) follows with $d = \Delta$. The theorem is proved.

Theorem 3.1, 5.1 yield the following.

Theorem 5.2 *Let Condition 5.1 be fulfilled and δ-trajectory $y^\alpha_{\delta,h}(i)/(i\delta)$ be defined by (3.4), (3.5). Then*

$$k_0\{(2K_J\alpha/(i\delta)+\varrho_i(\delta,h)/(i\delta))^{1/2}\,\Delta^{-1/2}+k_*\Delta^{1/2}\}^{1/2}\leq \tag{5.15}$$

$$\leq J(y^\alpha_{\delta,h}(i)/(i\delta)-J^0_z\leq \varrho_i(\delta,h)/\alpha.$$

Proof. By virtue of (3.7), and taking into account the definition of set $U^0_z[\gamma,\beta]$, we conclude that

$$|F(y^\alpha_{\delta,h}(i)/(i\delta))-b_z|_{H_2}\leq\{2K_J\alpha/(i\delta)+\varrho_i(\delta,h)/(i\delta)\}^{1/2},$$

$$J(y^\alpha_{\delta,h}(i)/(i\delta))\leq J^0_z+\varrho_i(\delta,h)/\alpha.$$

Hence and from (5.3) we deduce

$$|F_\Delta(y^\alpha_{\delta,h}(i)/(i\delta))-b^\Delta_z|_{H_2}\leq\{2K_J\alpha/(i\delta)+\varrho_i(\delta,h)/(i\delta)\}^{1/2}+k_*\Delta.$$

Consequently (see the rule of definition $J^\Delta_z[\varepsilon]$ (5.2))

$$J(y^\alpha_{\delta,h}(i)/(i\delta))\leq J^\Delta_z\,[\{2K_J\alpha/(i\delta)+\varrho_i(\delta,h)/(i\delta)\}^{1/2}+k_*\Delta]. \tag{5.16}$$

The inequality (5.15) follows from (5.16) and Theorem 5.1. The theorem is proved.

The following theorem is stated similarly.

Theorem 5.3 *Let Condition 5.1 be fulfilled and δ-trajectory $y_{\delta,h}(i)$ be generated by mappings (4.4), (4.5). Then the inequality*

$$k_0\{(\varrho_*(\delta,h)/(i\delta))^{1/2}\,\Delta^{-1/2}+k_*\Delta^{1/2}\}^{1/2}\leq J(y_{\delta,h}(i)/(i\delta)-J^0_z\leq 0$$

is true.

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