Working Paper

Robust Control of Constrained Parabolic Systems with Neumann Boundary Conditions

Boris S. Mordukhovich and Kaixia Zhang

WP-96-88
July 1996

IIASA
International Institute for Applied Systems Analysis ◻ A-2361 Laxenburg ◻ Austria
Telephone: +43 2236 807 ◻ Fax: +43 2236 71313 ◻ E-Mail: info@iiasa.ac.at

Robust Control of Constrained Parabolic Systems with Neumann Boundary Conditions

Boris S. Mordukhovich and Kaixia Zhang

WP-96-88
July 1996

IIASA International Institute for Applied Systems Analysis ▫ A-2361 Laxenburg ▫ Austria
Telephone: +43 2236 807 ▫ Fax: +43 2236 71313 ▫ E-Mail: info@iiasa.ac.at

ROBUST CONTROL OF CONSTRAINED PARABOLIC SYSTEMS WITH NEUMANN BOUNDARY CONDITIONS [1]

BORIS S. MORDUKHOVICH and KAIXIA ZHANG

Department of Mathematics
Wayne State University
Detroit, MI 48202

E-mail: boris@math.wayne.edu   zhang@math.wayne.edu

Abstract. This paper presents recent results by the authors on minimax robust control design of parabolic systems with uncertain perturbations under pointwise state and control constraints. The design procedure involves multi-step approximations and essentially employs monotonicity properties of the parabolic dynamics as well as its asymptotics on the infinite horizon. The results obtained justify a suboptimal three-positional structure of feedback controllers in the Neumann boundary conditions and provide calculations of their optimal parameters to ensure the required state performance and stability under any admissible perturbations. The problem under consideration was originally motivated by control design in water resources but certaintly admits a much broader spectrum of applications.

Keywords: robust control, parabolic systems, uncertainty, minimax design, state feedback, state-control constraints, suboptimality, and stability.

1. INTRODUCTION

This paper is concerned with robust control design of constrained parabolic systems under uncertain disturbances (perturbations) and feedback controllers in the Neumann boundary conditions. Our interest to such problems was originally motivated by applications to automatic control of groundwater regimes in irrigation networks where the objective was to neutralize negative effects of uncertain weather conditions; see [9]. Here we consider a more

[1]This research was partly supported by the National Science Foundation under grants DMS-9206989 and DMS-9404128 and by the NATO contract CRG-950360.

general class of parabolic control systems that have a broad spectrum of practical applications.

Dynamical processes in such systems are described by linear second-order parabolic equations with boundary controllers and pointwise state and control constraints. One of the most remarkable features of these processes is their functioning in the presence of uncertain perturbations when only an admissible region is given and no probabilistic information is available. A natural approach to control design of uncertain systems is *minimax synthesis* (principle of guaranteed result) that provides the best system performance under worst perturbations and ensures an acceptable (at least stable) behavior under any admissible perturbations.

Such a minimax approach to feedback control design is related to theories of differential games and robust H_∞-control; see [2, 5, 6] and their references. However, we are not familiar with any results in these theories that could be directly applied to the parabolic systems considered below under *hard* (pointwise) control and state constraints.

In this paper we developed an effective multi-step approximation procedure to design *suboptimal* feedback controllers for constrained parabolic systems. This procedure is initiated in [9, 11] for the case of one-dimensional heat-diffusion equations and takes into account certain specific features of the parabolic dynamics with infinite horizon. Related results for more general parabolic equations with both Dirichlet and Neumann boundary conditions are presented in [11–13].

This paper contains new results for the case of Neumann boundary controllers. The results obtained include a justification of a suboptimal three-positional control structure with subsequent optimization of its parameters. The main goal is to ensure the desired state performance within required state constraints for all admissible perturbations and to minimize the given (energy type) cost functional in the case of maximal ones. Moreover, we obtain effective stability conditions to exclude unacceptable self-vibrating regimes for nonlinear closed-loop control systems with the given parabolic dynamics and three-positional Neumann boundary controllers.

The paper is organized as follows. In Section 2 we formulate the feedback robust control problem of our study and present the main properties of the parabolic dynamics used in the sequel. Section 3 is devoted to solving first-order ODE approximation problems under maximal perturbations that allows us to justify a suboptimal structure of boundary controls in the parabolic system. In Section 4 we optimize parameters of this structure along the parabolic dynamics. Section 5 deals with computing a feedback boundary controller that ensures the best system behavior under maximal perturbations and keeps transients within the required state constraint region for any admissible disturbances on a sufficiently large control interval. The concluding Section 6 provides stability conditions for the class of nonlinear closed-loop control systems under consideration.

2. PROBLEM FORMULATION AND BASIC REPRESENTATIONS

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with the closure $\mathrm{cl}\Omega$. Assume that the boundary Γ of Ω is a C^∞-manifold of dimension $n-1$ and that locally Ω lies on one side of Γ. Let a_0 and

a_{ij}, $i,j = 1,2,\ldots,n$, be given real-valued functions with the properties a_0, $a_{ij} \in C^\infty(\mathrm{cl}\Omega)$,

$$a_{ij}(x) = a_{ji}(x) \;\; \forall i,j = 1,\ldots,n, \;\; x \in \Omega;$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \beta \sum_{i=1}^{n} \xi_i^2, \;\; \beta > 0 \;\; \forall \xi \in \mathbf{R}^n, \; x \in \Omega. \tag{2.1}$$

Observe that the linear operator

$$A := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + a_0(x) \tag{2.2}$$

is self-adjoint and *uniformly strongly elliptic* on $L^2(\Omega)$ due to (2.1).

In this paper we study the following parabolic system with the Neumann boundary conditions:

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = w(t) \;\; \text{a.e. in } \; Q := (0,T) \times \Omega \\ y(0,x) = 0, \;\; x \in \Omega \\ (\alpha y + \dfrac{\partial y}{\partial \nu_A})|_\Sigma = u(t), \;\; \Sigma := (0,T] \times \Omega \end{cases} \tag{2.3}$$

where $\alpha > 0$ and the normal derivative $\dfrac{\partial}{\partial \nu_A}$ is defined by

$$\frac{\partial}{\partial \nu_A} := \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_j}\cos(\mathbf{n}, x_i).$$

In what follows we treat $w(\cdot)$ as an *uncertain disturbance* perturbing the system, and $u(\cdot)$ as a *control* that can be chosen to achieve a required system performance. It is well known that for each $(u,w) \in L^2(0,T) \times L^2(0,T)$ system (2.3) has a unique *generalized solution* in the sense of [7]. Moreover, it follows from [8] that this solution $y = y(t,x)$ is *continuous* on $\mathrm{cl}Q := [0,T] \times \mathrm{cl}\Omega$.

Given positive numbers $\bar{a}$, $\underline{a}$, $\bar{b}$, and $\underline{b}$, we define the sets of *admissible controls* $u(\cdot)$ and admissible *uncertain perturbations* $w(\cdot)$ by, respectively,

$$U_{ad} := \{u \in L^2(0,T) \mid u(t) \in [-\bar{a}, \underline{a}] \;\; \text{a.e. } \; t \in [0,T]\},$$
$$W_{ad} := \{w \in L^2(0,T) \mid w(t) \in [-\underline{b}, \bar{b}] \;\; \text{a.e. } \; t \in [0,T]\}.$$

Suppose that $x_0 \in \Omega$ is a given point at which one measures the system performance and that $\eta > 0$ is an assigned constant. We consider the following *minimax feedback control problem*:

(P) minimize $J(u) = \max\limits_{w(\cdot) \in W_{ad}} \displaystyle\int_0^T |u(y(t,x_0))|dt$

over $u(\cdot) \in U_{ad}$ subject to (2.3) with the *pointwise state constraints*

$$|y(t,x_0)| \leq \eta \;\; \forall t \in [0,T] \tag{2.4}$$

and the *feedback control law* formed by

$$u(t) = u(y(t, x_0)) \tag{2.5}$$

through the *Neumann boundary conditions* in (2.3).

We always assume that there exists at least one triplet $(u, w, y) \in U_{ad} \times W_{ad} \times C(\text{cl}Q)$ such that it is *feasible* to problem (P), i.e., satisfies all the constraints.

Note that we do not have any available information about uncertain perturbations $w(t)$ except the given boundary $\{-\underline{b},\ \bar{b}\}$ of their admissible values. The objective in (P) is to find a feedback control function $u = u(y) \in [-\bar{a}, \underline{a}]$ of the intermediate state $y = y(t, x_0)$ that keeps the system performance within the constraint region (2.4) for all admissible perturbations and minimizes the given cost functional in the case of worst perturbations. This is a *minimax robust control* problem for uncertain distributed parameter systems under hard state and control constraints. Problems of this kind are among the most difficult ones in the control theory, and we are not familiar with any effective methods to solve such problems in full generality. Let us describe an approach to solving (P) that takes into account certain specific features of parabolic systems and allows us to compute a feasible *suboptimal* (in some sense) feedback control.

Our approach employs the Fourier series *spectral representation* of solutions to the parabolic system (2.3). To this end we consider the *eigenvalue problem*

$$\begin{cases} -A\varphi + \lambda\varphi = 0 \\ (\alpha\varphi + \dfrac{\partial\varphi}{\partial\nu_A})|_\Sigma = 0 \end{cases} \tag{2.6}$$

involving eigenvalues λ and eigenfunctions φ. It is well known (see, e.g., [1]) that under the general assumptions made there exists a sequence of solutions $\{\lambda_k, \varphi_k\}_{k\in\mathbf{N}}$ to (2.6) such that

$$\begin{aligned} &\{\varphi_k\}_{k\in\mathbf{N}} \text{ is a complete orthonormal basis in } L^2(\Omega) \text{ and} \\ &\lambda_k = ck^{\frac{2}{n}} + o(k^{\frac{2}{n}}) \text{ for some } c > 0. \end{aligned}$$

Consider the numbers

$$\mu_k := \int_\Omega \varphi_k(x)dx \quad\text{and}\quad \nu_k := \int_\Gamma \varphi_k(\zeta)d\sigma_\zeta$$

where $d\sigma_\zeta$ denotes the surface measure. The following result [7, 8] provides the basic spectral representation of solutions to the parabolic system (2.3).

Proposition 1. *Let $(u, w) \in L^2(0,T) \times L^2(0,T)$. Then the corresponding solution $y(t, x)$ of system (2.3) is continuous on $\text{cl}Q$ and is represented in the form*

$$y(t,x) = \sum_{k=1}^{\infty}(\mu_k \int_0^t w(\theta)e^{\lambda_k\theta}d\theta + \nu_k \int_0^t u(\theta)e^{\lambda_k\theta}d\theta)e^{-\lambda_k t}\varphi_k(x) \tag{2.7}$$

where the series converges strongly in $L^2(\Omega)$ *for each* $t \in [0,T]$.

Employing the *maximum principle* for parabolic equations (cf. [4]), one gets *monotonicity properties* of transients in (2.3) with respect to both controls and perturbations that play a crucial role in what follows.

Proposition 2. *Let* $(u_i, w_i) \in L^2(0,T) \times L^2(0,T)$ *and let* $y_i(\cdot)$ *be the corresponding generalized solution to* (2.3) *for* $i = 1,\ 2$. *Then*

$$y_1(t,x) \geq y_2(t,x) \ \ \forall (t,x) \in Q$$

if $u_1(t) \geq u_2(t)$ *and* $w_1(t) \geq w_2(t)$ *for all* $t \in [0,T]$.

Remember that the control objective is to keep transients within the given state constraints under any admissible perturbations. Then Proposition 2 infers that *the bigger magnitude of a perturbation is, the more control of the opposite sign should be applied* to neutralize the perturbation and ensure the required state performance. This makes us to consider feedback control laws with the *compensation property*

$$u(y) \leq u(\tilde{y}) \ \text{ if } \ y \geq \tilde{y} \ \text{ and } \ y \cdot u(y) \leq 0 \ \ \forall y, \tilde{y} \in \mathbf{R}. \tag{2.8}$$

The latter property implies that

$$\int_0^T |u(y(t))|dt \geq \int_0^T |u(\tilde{y}(t))|dt \ \text{ if } \ y(t) \geq \tilde{y}(t) \geq 0 \ \text{ or } \ y(t) \leq \tilde{y}(t) \leq 0 \ \ \forall t \in [0,T],$$

i.e., the compensation of bigger (by magnitude) perturbations requires more cost with respect to the maximized cost functional in (P). This allows us to seek a *suboptimal control structure* in (P) by examining the control response to feasible perturbations of the *maximal magnitudes* $w(t) = \bar{b}$ and $w(t) = -\underline{b}$ for all $t \in [0,T]$.

3. APPROXIMATION PROBLEMS

In this section we develop multi-step approximation procedures to justify an acceptable structure of feasible suboptimal controls for problem (P).

Let $u = u(y)$ be a given feedback control law in (P). Then for any given perturbation $w = w(t)$ we have an open-loop control realization $u(t) = u(y(t,x_0))$ due to system (2.3). We consider only *feasible* pairs $(u,w) \in U_{ad} \times W_{ad}$ such that the corresponding transient $y(t,x_0)$ satisfies the state constraints (2.4). For any natural number $N = 1,2,\ldots$ we denote

$$y^N(t,x) := \sum_{k=1}^{N}(\mu_k \int_0^t w(\theta)e^{\lambda_k\theta}d\theta + \nu_k \int_0^t u(\theta)e^{\lambda_k\theta}d\theta)e^{-\lambda_k t}\varphi_k(x)$$

and conclude that for all $t \in [0,T]$

$$y^N(t,\cdot) \to y(t,\cdot) \ \text{ strongly in } L^2(\Omega) \text{ as } N \to \infty$$

due to Proposition 1. Moreover, considering $y^N(t,x)$ at the point of observation $x = x_0$, we get $y^N(t,x_0) = \sum_{k=1}^{N} y_k(t)$ with

$$\begin{cases} \dfrac{dy_k}{dt} = \lambda_k y_k + \mu_k w + \nu_k u \\ y_k(0) = 0, \quad k = 1, 2, \ldots, N. \end{cases}$$

Thus Proposition 1 allows us to approximate the original parabolic system (2.3) by systems of ordinary differential equations.

In what follows we assume that the eigenvalues in (2.6) satisfy the conditions

$$0 < \lambda_1 < \lambda_k, \; k = 2, 3, \ldots \tag{3.1}$$

that always hold, e.g., when $A = \Delta$ is the Laplacian. One can observe that under (3.1) the first term *asymptotically dominates* in the series (2.7) as $t \to \infty$. On this basis, we examine the case of $N = 1$ in the above ODE system to justify a *suboptimal* control structure for the original problem.

Taking into account the discussion after Proposition 2 as well as the *symmetry* of (P) relative to the origin, we consider the following *open-loop optimal control problem* with the admissible control set

$$\bar{U}_{ad} := \{u(\cdot) \in U_{ad} \mid -\bar{a} \le u(t) \le 0 \text{ a.e. } t \in [0,T]\}$$

in response to the upper level maximal perturbation $w(t) = \bar{b}$ on $[0,T]$:

($\bar{\text{P}}_1$) minimize $\bar{J}(u) = -\int_0^T u(t)dt$

over $u(\cdot) \in \bar{U}_{ad}$ subject to

$$\begin{cases} \dfrac{dy}{dt} = -\lambda_1 y + \varphi_1(x_0)(\mu_1 \bar{b} + \nu_1 u(t)) \text{ a.e. } t \in [0,T] \\ y(0) = 0 \end{cases} \tag{3.2}$$

and the state constraint

$$y(t) \le \eta \quad \forall t \in [0,T]. \tag{3.3}$$

The symmetric case of $w(t) = -\underline{b}$ in the lower boundary level can be considered similarly and actually can be reduced to ($\bar{\text{P}}_1$).

Note that the presence of state constraints relates ($\bar{\text{P}}_1$) to the class of most complicated optimal control problems for ODE systems. It is well known that in general their solutions involve *Borel measures* that make them fairly difficult for applications; see [3]. We avoid such difficulties by developing an *approximation procedure* in the vein of [10] to replace ($\bar{\text{P}}_1$) by a parametric family of standard optimal control problems with *no* state constraints. To solve approximation problems we employ the *Pontryagin maximum principle* [15] that provides *necessary and sufficient* conditions for optimality of approximating solutions. It occurs that optimal controls to approximation problems contain both *bang-bang and singular modes.* Passing to the limit, we obtain in this way an *exact solution* to the state-constrained problem ($\bar{\mathrm{P}}_1$) that *does not involve any measure.* The results obtained show that the state constraint (3.3) in ($\bar{\mathrm{P}}_1$) turns out to be a *regularization factor.* Such a surprising conclusion is due to the specific of problems like ($\bar{\mathrm{P}}_1$) reflecting the parabolic dynamics. The reader can find more details in [11, 14] where similar results are proved for approximation problems corresponding to Dirichlet boundary controls.

Theorem 3. *Let $\mu_1\varphi_1(x_0)\bar{b} > \lambda_1\eta$. Assume in addition that either*

$$\tau_1 := \frac{1}{\lambda_1}\ln\frac{\mu_1\varphi_1(x_0)\bar{b}}{\mu_1\varphi_1(x_0)\bar{b} - \lambda_1\eta} \geq T \tag{3.4}$$

or

$$\mu_1\varphi_1(x_0)\bar{b} - \bar{a}\nu_1\varphi_1(x_0) \leq \lambda_1\eta.$$

Then system (3.2), (3.3) *is controllable, i.e., there is $u(\cdot) \in \bar{U}_{ad}$ such that the corresponding trajectory of* (3.2) *satisfies the state constraint* (3.3). *Moreover, problem* ($\bar{\mathrm{P}}_1$) *admits an optimal control of the form*

$$\bar{u}_1(t) = \begin{cases} 0 & \text{if } t \in [0, \bar{\tau}_1) \\ \dfrac{\lambda_1\eta - \mu_1\varphi_1(x_0)\bar{b}}{\nu_1\varphi_1(x_0)} & \text{if } t \in [\bar{\tau}_1, T] \end{cases} \tag{3.5}$$

where $\bar{\tau}_1 = \min\{\tau_1, T\}$ with τ_1 computed in (3.4).

Note that in (3.5) we have only one switching from the original bang-bang level to an *intermediate singular mode.* For the symmetric problem ($\underline{\mathrm{P}}_1$) in which the system is operated under the lower level maximal perturbation $w(t) = -\underline{b}$, one can obtain the corresponding results from Theorem 3 by changing signs of the state and control variables.

According to the above discussions, the optimal controls derived for problems ($\bar{\mathrm{P}}_1$) and ($\underline{\mathrm{P}}_1$) can be viewed as *first-order suboptimal solutions* to the open-loop control problems arising from the original problem (P) under the maximal perturbations $w(t) = \bar{b}$ and $w(t) = -\underline{b}$. In the next section we admit this simple structure justified as a *suboptimal control structure* for the original problem under maximal perturbations and then optimize its parameters along the *parabolic dynamics over a large control interval.*

4. OPTIMAL CONTROL UNDER MAXIMAL PERTURBATIONS

Let us consider the following optimal control problem for the original parabolic system (2.3) under the upper level maximal perturbation $w(t) = \bar{b}$ on $[0, T]$:

($\bar{\mathrm{P}}$) minimize $\bar{J}(u) = -\int_0^T u(t)dt$

subject to system (2.3), state constraint (3.3), and boundary controls $u(\cdot) \in \bar{U}_{ad}$ of the form

$$u(t) = \begin{cases} 0 & \text{for } 0 \le t < \tau \\ -\bar{u} & \text{for } \tau \le t \le T. \end{cases} \tag{4.1}$$

We choose the Neumann boundary control structure (4.1) according to the results in Section 3 that justify its suboptimality under maximal perturbations. To solve ($\bar{\mathrm{P}}$) one should find optimal parameters $\bar{u} \in [0, \bar{a}]$ and $\tau \in [0, T]$ in (4.1) which keep the state constraint (3.3) along the parabolic dynamics (2.3) and minimize the given cost functional. In what follows we suppose that the control interval $[0, T]$ is *sufficiently large* and examine the *asymptotics* of optimal solutions as $T \to \infty$ based on assumption (3.1). It turns out that under this assumption optimal processes in ($\bar{\mathrm{P}}$) possess a kind of *turnpike property* that simplifies the solution while passing to the infinite horizon.

To formulate the main results we need to introduce the following numbers

$$\gamma := \sum_{k=1}^{\infty} \frac{\mu_k \varphi_k(x_0)}{\lambda_k} \quad \text{and} \quad \rho := \sum_{k=1}^{\infty} \frac{\nu_k \varphi_k(x_0)}{\lambda_k}$$

that are positive under the assumptions made in the next theorem.

Theorem 4. *In addition to the basic assumptions let us suppose that* $\nu_1 > 0$ *and*

$$0 < \gamma \bar{b} - \eta < \min\{\bar{a}\rho, \frac{\rho \mu_1 \bar{b}}{\nu_1}\}.$$

Consider the transcendental equation

$$\sum_{k=1}^{\infty} \frac{\varphi_k(x_0)}{\lambda_k} e^{-\lambda_k T} [\nu_k(\gamma \bar{b} - \eta) e^{\lambda_k \tau} - \rho \mu_k \bar{b}] = 0 \tag{4.2}$$

which has a unique solution $\tau = \bar{\tau}(T) \in (0, T)$ *for all* T *sufficiently large. Then any control* (4.1) *with*

$$\bar{u} = \frac{\gamma \bar{b} - \eta}{\rho} \tag{4.3}$$

is feasible to ($\bar{\mathrm{P}}$) *for all positive* $\tau \le \bar{\tau}(T)$ *being optimal to this problem when* $\tau = \bar{\tau}(T)$. *Moreover,* $\bar{\tau}(T) \downarrow \bar{\tau}$ *as* $T \to \infty$ *where the asymptotically optimal switching time* $\bar{\tau}$ *is computed by*

$$\bar{\tau} = \frac{1}{\lambda_1} \ln \frac{\rho \mu_1 \bar{b}}{\nu_1 (\gamma \bar{b} - \eta)}. \tag{4.4}$$

The proof of this theorem follows the scheme of [11, 14] for the case of Dirichlet boundary conditions. Let us observe that control (4.1) with parameters (4.3) and (4.4) is feasible for problem ($\bar{\mathrm{P}}$) on the interval $[0,T]$ with an arbitrary large T. Moreover, $\bar{\tau}$ is the *maximal* one among all switching times in (4.1) that keep the state constraint (3.3) on the whole infinite interval $[0,\infty)$. Therefore, this *asymptotically optimal control* with the infinite horizon is *suboptimal* for the given problem ($\bar{\mathrm{P}}$) on $[0,T]$ where T is sufficiently large.

Similar results hold for the symmetric optimal control problem ($\underline{\mathrm{P}}$) corresponding to the lower level maximal perturbation. This problem consists of minimizing the cost functional $\underline{J}(u) = \int_0^T u(t)dt$ subject to the parabolic system (2.3) with $w(t) = -\underline{b}$ on $[0,T]$, admissible boundary controls $0 \le u(t) \le \underline{a}$ of the form

$$u(t) = \begin{cases} 0 & \text{for } \ 0 \le t < \tau \\ \underline{u} & \text{for } \ \tau \le t \le T, \end{cases} \tag{4.5}$$

and the state constraint $-\eta \le y(t)$ for all $t \in [0,T]$.

Utilizing symmetric arguments, we justify the optimality of control (4.5) with the position

$$\underline{u} = \frac{\gamma\underline{b} - \eta}{\rho} \tag{4.6}$$

and the switching time $\tau = \underline{\tau}(T)$ satisfying (4.2) for $-\underline{b}$. One has $\underline{\tau}(T) \downarrow \underline{\tau}$ as $T \to \infty$ where the asymptotically optimal switching time is computed by

$$\underline{\tau} = \frac{1}{\lambda_1} \ln \frac{\rho\mu_1\underline{b}}{\nu_1(\gamma\underline{b} - \eta)}.$$

5. FEEDBACK CONTROL DESIGN FOR THE PARABOLIC SYSTEM

Let us go back to the original feedback control problem (P) and assume hereafter that its initial data satisfy all the assumptions in Theorem 4 as well as the symmetric ones for the lower level maximal perturbation $w(t) = -\underline{b}$. Based on the results above, we consider the following *three-positional feedback control law* in (2.5):

$$u(y) = \begin{cases} -\bar{u} & \text{if } \ y \ge \bar{\sigma} \\ 0 & \text{if } \ -\underline{\sigma} < y < \bar{\sigma} \\ \underline{u} & \text{if } \ y \le -\underline{\sigma} \end{cases} \tag{5.1}$$

that obviously satisfies the compensation property (2.8). We have established that structure (5.1) is *suboptimal* (optimal in the first order) with respect to the objective in (P) under the realization of the maximal boundary perturbations $w(\cdot) = \bar{b}$ and $w(\cdot) = -\underline{b}$. Furthermore, we computed optimal control parameters corresponding to the maximal perturbations with their asymptotics on the infinite horizon. Now our goal is to determine optimal parameters of the feedback control law (5.1) ensuring the desired behavior of the closed-loop system (2.3), (2.5), (5.1).

Let the positions $\bar{u}$ and $\underline{u}$ in (5.1) be computed by formulas (4.3) and (4.6), respectively. Under the assumptions made one obviously has $u(\cdot) \in U_{ad}$ for any control realization $u(t) =$

$u(y(t,x_0))$ corresponding to an arbitrary $w(\cdot) \in W_{ad}$. Moreover, these control positions ensure the transient *stabilization* as $t \to \infty$ within the required state interval $[-\eta, \eta]$ for any admissible perturbations. However, the state constraints (2.4) may be violated for some $t \in [0,T]$ if the *dead region* $[-\underline{\sigma}, \bar{\sigma}]$ is not properly designed. The next theorem determines optimal values of $\underline{\sigma}$ and $\bar{\sigma}$ such that the closed-loop control system exhibits the best possible behavior under the maximal perturbations and keeps transients within the given state constraints for any admissible perturbations on a large control interval $[0,T]$. The proof is based on the transient monotonicity with respect to both controls and perturbations; cf. [11, 14].

Theorem 5. *Under the assumptions made we consider the feedback control* (5.1) *with* $\bar{u}$ *and* $\underline{u}$ *computed in* (4.3) *and* (4.6), *respectively. Let*

$$\bar{\sigma}(T) := \bar{b}(\gamma - \sum_{k=1}^{\infty} \frac{\mu_k \varphi_k(x_0)}{\lambda_k} e^{-\lambda_k \bar{\tau}(T)}),$$
$$\underline{\sigma}(T) := \underline{b}(\gamma - \sum_{k=1}^{\infty} \frac{\mu_k \varphi_k(x_0)}{\lambda_k} e^{-\lambda_k \underline{\tau}(T)})$$

where $\bar{\tau}(T)$ *and* $\underline{\tau}(T)$ *are the corresponding unique solutions to* (4.2) *and its counterpart for* $-\underline{b}$. *Then the control law* (5.1) *is feasible for any perturbations* $w(\cdot) \in W_{ad}$ *and optimal in the case of maximal perturbations when* T *is sufficiently large. Moreover,* $\bar{\sigma}(T) \downarrow \bar{\sigma}$ *and* $\underline{\sigma}(T) \downarrow \underline{\sigma}$ *as* $T \to \infty$ *where the positive numbers*

$$\bar{\sigma} := \bar{b}(\gamma - \sum_{k=1}^{\infty} \frac{\mu_k \varphi_k(x_0)}{\lambda_k} [\frac{\nu_1(\gamma \bar{b} - \eta)}{\rho \mu_1 \bar{b}}]^{\frac{\lambda_k}{\lambda_1}}), \tag{5.2}$$
$$\underline{\sigma} := \underline{b}(\gamma - \sum_{k=1}^{\infty} \frac{\mu_k \varphi_k(x_0)}{\lambda_k} [\frac{\nu_1(\gamma \underline{b} - \eta)}{\rho \mu_1 \underline{b}}]^{\frac{\lambda_k}{\lambda_1}}). \tag{5.3}$$

form the maximal dead region $[-\underline{\sigma}, \bar{\sigma}]$ *under which feedback* (5.1) *keeps the state constraints* (2.4) *on the infinite horizon* $[0, \infty)$ *for any admissible perturbations.*

6. STABILITY OF THE FEEDBACK CONTROL SYSTEM

Let us consider the closed-loop control system

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = w(t) \text{ a.e. in } Q \\ y(0,x) = 0, \ x \in \Omega \\ (\alpha y + \dfrac{\partial y}{\partial \nu_A})|_{\Sigma} = u(y(t,x_0)) \end{cases} \tag{6.1}$$

where $u = u(y)$ is the three-positional feedback controller defined in (5.1). Note that although the parabolic equation in (6.1) is linear, the closed-loop system (6.1) is highly *nonlinear* with respect to the state y due to discontinuity of the feedback control law (5.1).

One of the most important characteristics of closed-loop dynamical systems is their *stability* in the sense of maintaining the initial stationary regime after terminating all the perturbations. Such a stability is an obligatory condition for a normal functioning of any automatic control system. We are going to consider the nonlinear control system (6.1) from this viewpoint.

Note that (6.1) is a distributed parameter system where controls acting in the Neumann boundary conditions are formed by the current intermediate state $y(t, x_0)$. This generates an *inertia* of the control system and essentially affects its stability. One can easily see that if $y(t, x_0)$ is strictly inside of the dead region $[-\underline{\sigma}, \bar{\sigma}]$ at the time t_0 of terminating all the perturbations, then system (6.1) maintains the stationary regime $y_0(x) \equiv 0$ as $t \to \infty$. This means the stability *in the small* of the initial state $y = 0$ that is *not sufficient* for a normal functioning of the nonlinear control system (6.1) since it does not exclude *self-vibrating regimes.*

Complications may arise when $y(t, x_0)$ reaches the boundary of the dead region while the latter is not sufficiently wide. Indeed, in such cases the transient trajectory moves back and forth between the dead region boundaries under switching control positions in (5.1) with no external perturbations $w(\cdot)$. The next theorem provides effective conditions that exclude such an auto-oscillation and thus ensures the required stability of the closed-loop control system (6.1). The proof of this theorem is based on a *variational approach* to stability that is possible due to monotonicity properties of the parabolic dynamics; see [11, 14] for more details.

Theorem 6. *The closed-loop system* (6.1), (5.1) *with arbitrary control parameters* $(\bar{u}, \underline{u}, \bar{\sigma}, \underline{\sigma})$ *is stable if*

$$\bar{\sigma} + \underline{\sigma} \geq \min\{\bar{u}, \underline{u}\}\left(\frac{\nu_1 \varphi_1(x_0)}{\lambda_1} - \rho\right) > 0. \tag{6.2}$$

Furthermore, let $\bar{b} \leq \underline{b}$ *and let* $(\bar{\sigma}_1, \underline{\sigma}_1)$ *be computed by*

$$\bar{\sigma}_1 = \bar{b}\gamma - \frac{\nu_1 \varphi_1(x_0)(\bar{b}\gamma - \eta)}{\rho} \quad \textit{and} \quad \underline{\sigma}_1 = \underline{b}\gamma - \frac{\nu_1 \varphi_1(x_0)(\underline{b}\gamma - \eta)}{\rho},$$

i.e., they are the first-term approximations of the asymptotically optimal dead region bounds in (5.2) *and* (5.3). *Then the stability condition* (6.2) *can be written as*

$$2\bar{\sigma}_1 + \underline{\sigma}_1 \geq \eta.$$

REFERENCES

1. S. Agmon: *Lectures on Elliptic Boundary Value Problems.* Van Nostrand, Princeton, 1965.

2. T. Basar and P. Bernard: *H_∞-Optimal Control and Related Minimax Design Problems.* Birkhäuser, Boston, 1991.

3. A. D. Ioffe and V. M. Tikhomirov: *Theory of Extremal Problems.* North-Holland, Amsterdam, 1979.

4. A. Friedman: *Partial Differential Equations of Parabolic Type.* Prentice-Hall, Englewood Cliffs, N.J., 1964.

5. B. van Keulen: *H_∞-Control for Distributed Parameter Systems: A State-Space Approach.* Birkhäuser, Boston, 1993.

6. N. N. Krasovskii and A. I. Subbotin: *Game-theoretical Control Problems.* Springer-Verlag, New York, 1988.

7. J. L. Lions: *Optimal Control of Systems Governed by Partial Differential Equations.* Springer-Verlag, Berlin, 1971.

8. U. Mackenroth: Convex parabolic boundary control problems with pointwise state constraints, *J. Math. Anal. Appl.* **87** (1982), 256–277.

9. B. S. Mordukhovich: Optimal control of ground water regime on two-way engineering reclamation systems, *Water Resources* **12** (1986), 244–253.

10. B. S. Mordukhovich: *Approximation Methods in Problems of Optimization and Control.* Nauka, Moscow, 1988.

11. B. S. Mordukhovich: Minimax design for a class of distributed control systems, *Autom. Remote Control* **50** (1989), 1333–1340.

12. B. S. Mordukhovich and K. Zhang: Feedback boundary control of constrained parabolic equations in uncertainty conditions. *Proc. 3rd Europ. Cont. Conf.*, pp. 129–134, Rome, Italy, 1995.

13. B. S. Mordukhovich and K. Zhang: Minimax control of parabolic systems with Dirichlet boundary conditions and state constraints, IIASA Working Paper WP-95-70, July 1995; to appear in *Appl. Math. Optim.*

14. B. S. Mordukhovich and K. Zhang: Feedback suboptimal control for constrained parabolic systems, Preprint, Wayne State University, 1996.

15. L. S. Pontryagin, B. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishenko: *The Mathematical Theory of Optimal Processes.* Wiley-Interscience, New York, 1962.